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Interior regularity of solutions to a complex Monge–Ampère equation

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Abstract. We give interior estimates for first derivatives of solutions to a type of complex Monge–Ampère equations in convex domains. We also show global estimates for first derivatives of solutions in arbitrary domains. These global estimates are then used to show interior regularity of solutions to the complex Monge–Ampère equations in hyperconvex domains having a bounded exhaustion function which is globally Lipschitz. Finally we give examples of domains which have such an exhaustion function and domains which do not.

1. Introduction

Assume that μ is a positive Borel measure on a domain $\Omega\subseteq\mathbf{C}^n$, $n\geq 2$, and φ some function on the boundary of Ω. Central to pluripotential theory is the study of the Dirichlet problem

$$\begin{cases} (dd^c u)^n = \mu & \text{in } \Omega, \\ u=\varphi & \text{on } \partial\Omega. \end{cases}$$

Here $d^c=i(\bar\partial-\partial)$ and note that if $u\in C^2(\Omega)$ then

$$(dd^c u)^n = 4^n n! \det\left(\frac{\partial^2 u}{\partial z_j \partial \bar z_k}\right) dV,$$

where $dV=\left(\frac{1}{2}i\right)^n dz_1\wedge d\bar z_1\wedge\ldots\wedge dz_n\wedge d\bar z_n$ is the volume form. It is possible to define $(dd^c u)^n$, the complex Monge–Ampère operator, for more general plurisubharmonic functions. How to define this operator on continuous plurisubharmonic functions was explained in [2]. It should be noted that Cegrell, see [9], recently has given a definition of $(dd^c u)^n$ which has the optimal domain of definition. In both cases $(dd^c u)^n$ is defined as a positive Borel measure on Ω. In this paper we shall always

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have $\mu = f\,dV$, where f is a function. We shall be considering the question of how regularity of f implies regularity of u. In general solutions to $(dd^c u)^n = 0$ can be irregular. One realizes this by thinking about a plurisubharmonic function which depends on $n-1$ variables only. However, if one demands that the boundary data be continuous then it can be proved in certain domains, as it was by Walsh in [15], that the solution is continuous. Put

$$\mathrm{PB}_\varphi(z) = \sup\Big\{ v(z)\,; v \in \mathcal{PSH}(\Omega) \text{ and } \limsup_{z\to z_0} v(z) \le \varphi(z_0) \text{ for all } z_0 \in \partial\Omega \Big\}.$$

It had been observed by Bremermann in [7] that if the problem

$$\begin{cases} (dd^c u)^n = 0 & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega \end{cases}$$

is solvable, then the solution is *the Perron–Bremermann envelope*

$$(\mathrm{PB}_\varphi)^*(z) = \limsup_{\zeta\to z} \mathrm{PB}_\varphi(\zeta).$$

The result Walsh obtained is the following.

Theorem 1.1. *Suppose that Ω is a bounded domain in $\mathbf{C}^n$ and $\varphi \in C(\partial\Omega)$. Assume that*

$$\liminf_{z\to z_0} \mathrm{PB}_\varphi(z) = \limsup_{z\to z_0} \mathrm{PB}_\varphi(z) = \varphi(z_0) \quad \textit{for all } z_0 \in \partial\Omega.$$

Then $\mathrm{PB}_\varphi \in C(\bar\Omega)$.

Higher order regularity is harder for the equation $(dd^c u)^n = 0$ as the example $u(z_1, z_2) = \max\{|z_1|^2 - \frac{1}{2}, |z_2|^2 - \frac{1}{2}, 0\}^2$ shows. This function is smooth on the boundary of the unit ball, meets $(dd^c u)^2 = 0$ but is not smooth. For more examples of lack of higher order regularity see Bedford's and Fornæss' paper [1]. The first result on higher order regularity was obtained in 1985 by Caffarelli, Kohn, Nirenberg and Spruck in [8]. The positivity of f is crucial in view of the example above and those given in [1].

Theorem 1.2. *Suppose that Ω is a bounded, strongly pseudoconvex domain in $\mathbf{C}^n$ with smooth boundary. Let $f \in C^\infty(\bar\Omega \times \mathbf{R})$ be a strictly positive function which is increasing in the second variable. Suppose that $\varphi \in C^\infty(\partial\Omega)$. Then the problem*

$$\begin{cases} \det\left(\dfrac{\partial^2 u}{\partial z_j \partial \bar z_k}\right) = f(z, u(z)) & \textit{in } \Omega, \\ u = \varphi & \textit{on } \partial\Omega, \\ u \in \mathcal{PSH}(\Omega) \cap C^2(\Omega) \cap C(\bar\Omega), \end{cases}$$

has a unique solution. Moreover $u \in C^\infty(\bar\Omega)$.

Remark 1.3. When we say that a function $g:\mathbf{R}\to\mathbf{R}$ is increasing we mean a function with the property that $x\le x'$ implies that $g(x)\le g(x')$. If $x<x'$ implies that $g(x)<g(x')$ we say that g is strictly increasing. Finally smooth will always mean C^∞-smooth.

Remark 1.4. Actually Caffarelli, Kohn, Nirenberg and Spruck proved a more general result than stated in Theorem 1.2. One can in fact allow the Monge–Ampère mass of u to depend on the gradient of u in a certain way. For details on this see [8].

A domain Ω in $\mathbf{C}^n$ is called *hyperconvex* if it admits a weak plurisubharmonic barrier at every boundary point, that is, for every $z_0\in\partial\Omega$ there exists $v\in\mathcal{PSH}(\Omega)$ such that $v<0$ and $\lim_{z\to z_0} v(z)=0$. Kerzman and Rosay showed in [13] that for bounded domains it is equivalent to say that there exists a smooth bounded strictly plurisubharmonic exhaustion function ϱ in Ω. This was improved upon by Błocki in [4] so that we can choose a smooth plurisubharmonic ϱ satisfying $\lim_{z\to z_0\in\partial\Omega}\varrho(z)=0$ and

$$\det\left(\frac{\partial^2\varrho}{\partial z_j\partial\bar z_k}\right)\ge 1.$$

If we do not demand that the solutions should be smooth we can get the following, which was proved by Błocki in [3].

Theorem 1.5. *Let Ω be a bounded, hyperconvex domain in $\mathbf{C}^n$. Assume that f is nonnegative, continuous and bounded in Ω. Suppose that φ is continuous on $\partial\Omega$ and that it can be continuously extended to a plurisubharmonic function on Ω. Then there exists a unique solution to the following problem*

$$\begin{cases} (dd^c u)^n = f(z) & \text{in } \Omega,\\ u=\varphi & \text{on } \partial\Omega,\\ u\in\mathcal{PSH}(\Omega)\cap C(\bar\Omega). \end{cases}$$

Błocki has also given a sufficient condition for smooth solutions in convex domains in [6]. This result has also been announced in [5].

Theorem 1.6. *Let Ω be a bounded, convex domain in $\mathbf{C}^n$. Assume that f is a strictly positive, smooth function in Ω such that*

$$\sup_{z\in\Omega}\left|\frac{\partial f^{1/n}}{\partial x_l}(z)\right|<\infty.$$

Then there exists a unique solution to the following problem

$$\begin{cases} \det\left(\dfrac{\partial^2 u}{\partial z_j\partial\bar z_k}\right)=f(z) & \text{in } \Omega,\\ \lim\limits_{z\to z_0} u(z)=0 & \text{for all } z_0\in\partial\Omega,\\ u\in\mathcal{PSH}(\Omega)\cap C^\infty(\Omega). \end{cases}$$

Note that a convex domain is hyperconvex since convex functions are plurisubharmonic and also that a hyperconvex domain is pseudoconvex since the function $\tilde{\varrho}(z) = -\log(-\varrho(z))$ is plurisubharmonic and $\lim_{z\to z_0} \tilde{\varrho}(z) = \infty$.

Definition 1.7. We say that a hyperconvex domain Ω satisfies *the nonprecipitousness condition*, or for short *the NP-condition*, if we can find a smooth plurisubharmonic function ϱ satisfying $\lim_{z\to z_0 \in \partial\Omega} \varrho(z) = 0$ and

$$\det\left(\frac{\partial^2 \varrho}{\partial z_j \partial \bar{z}_k}\right) \geq 1,$$

and the condition

$$\sup\left\{\left|\frac{\partial \varrho}{\partial x_j}(z)\right| ; z \in \Omega \text{ and } j = 1, \dots, 2n\right\} < \infty.$$

In Section 5 we shall prove Theorem 5.1, which is an extension of Theorem 1.6 to hyperconvex domains satisfying the NP-condition. In Section 2 we collect two comparison principles which will be used throughout the paper and in Section 3 we prove an interior estimate for first derivatives in convex domains which the author thinks is interesting in itself. In Section 4 we give a global estimate of first derivatives in arbitrary domains. This estimate is then used to prove Theorem 5.1 in Section 5. In Section 6 we give examples of hyperconvex domains which satisfy the NP-condition and hyperconvex domains which does not. Finally, I would like to mention that this paper is an expanded version of my licentiate thesis [12].

2. Comparison principles

We shall need the following two comparison principles, the first of which was proved by Bedford and Taylor in [2].

Lemma 2.1. *Suppose that Ω is a bounded domain in $\mathbf{C}^n$ and $v, w \in C(\bar{\Omega}) \cap \mathcal{PSH}(\Omega)$. Assume that $(dd^c v)^n \geq (dd^c w)^n$. Then*

$$\min_{z\in\bar{\Omega}}(w(z) - v(z)) = \min_{z\in\partial\Omega}(w(z) - v(z)).$$

The following lemma is sometimes useful.

Lemma 2.2. *Let Ω be a bounded domain in $\mathbf{C}^n$. Assume that $f \in C(\Omega \times \mathbf{R})$ is a nonnegative function which is increasing in the second variable. Suppose that $v, w \in C(\bar{\Omega}) \cap \mathcal{PSH}(\Omega)$. Then*

$$(dd^c w)^n \leq f(z, w(z)), \quad f(z, v(z)) \leq (dd^c v)^n$$

and $v \leq w$ on $\partial\Omega$ implies that $v \leq w$ in Ω.

Proof. Put $V(z)=v(z)-w(z)$. We want to show that $V\leq 0$ and do this by contradiction. Therefore assume that there exists $z_0\in\Omega$ such that $V(z_0)>0$. Define $\tilde{\Omega}=\{z\in\Omega; V(z)>0\}$. By assumption $\tilde{\Omega}$ is nonempty. Let Ω_0 be the component of $\tilde{\Omega}$ that contains z_0. In Ω_0 we have

$$(dd^c w)^n \leq f(z,w(z)) \leq f(z,v(z)) \leq (dd^c v)^n,$$

since f is increasing in the second variable. We have $v=w$ on the boundary of Ω_0 and an application of Lemma 2.1 tells us that $v=w$ in Ω_0, which contradicts our assumption. □

3. Interior estimates for first derivatives in convex domains

We now prove the following proposition, which is an extension of an estimate of Błocki, [6, Theorem 2.1].

Proposition 3.1. *Assume that Ω is a bounded convex domain in $\mathbf{C}^n$ and that K is a compact subset of Ω. Let $\varphi\colon\partial\Omega\to\mathbf{R}$ be a nonpositive function and $g\in C^\infty(\bar{\Omega}\times\mathbf{R})$ be a strictly positive function which is increasing in the second variable. Assume that $w\in C^\infty(\bar{\Omega})\cap\mathcal{PSH}(\Omega)$ is a solution of*

$$\begin{cases} \det\left(\dfrac{\partial^2 w}{\partial z_j \partial \bar{z}_k}\right) = g(z,w(z)) & \text{in } \Omega, \\ w(z)=\varphi(z) & \text{on } \partial\Omega. \end{cases}$$

Let D be the diameter of Ω,

$$C=\sup\left\{\left|\frac{\partial g^{1/n}}{\partial x_l}(z,t)\right| ; (z,t)\in\Omega\times\left[\inf_{z\in\Omega} w(z),0\right] \text{ and } l=1,\dots,2n\right\}$$

and

$$M=\sup\left\{\left|\min\left\{0,\frac{\partial w}{\partial \nu}(\zeta)\right\}\right| ; \zeta\in\partial\Omega,\ z\in K, \nu=\frac{\zeta-z\pm d_\Omega(z)e_l}{|\zeta-z\pm d_\Omega(z)e_l|} \text{ and } l=1,\dots,2n\right\},$$

where $e_1,\dots,e_{2n}$ is the standard basis in $\mathbf{R}^{2n}$. Then

$$\sup_{z\in K}\left|\frac{\partial w}{\partial x_l}(z)\right| \leq \frac{2\sup_{z\in K}|w(z)|+2\sup_{z\in\partial\Omega}|\varphi(z)|+2DM+CD^3}{\inf_{z\in K} d_\Omega(z)}+CD^2$$

for $l=1,\dots,2n$.

Proof. Take $z_0\in K$. After a translation we may assume that $z_0=0$. We can also assume that $g\in C^\infty(\mathbf{C}^n\times\mathbf{R})$. Put $\Omega_\varepsilon^l=\Omega-\varepsilon e_l$ and assume that $\varepsilon>0$ is such that $0\in\Omega_\varepsilon^l$. Now choose α_l such that $\Omega_\varepsilon^l\subseteq\alpha_l\Omega$. We can always choose $\alpha_l=(d_\Omega(0)+\varepsilon)/d_\Omega(0)$ and we see that the choice is independent of l. Thus we put $\alpha=(d_\Omega(0)+\varepsilon)/d_\Omega(0)$. Now suppose that W_l is a solution of

$$\begin{cases}\det\left(\dfrac{\partial^2 W_l}{\partial z_j\partial\bar z_k}\right)=g(z+\varepsilon e_l,W_l(z)) & \text{in } \Omega_\varepsilon^l,\\ W_l(z)=\varphi(z+\varepsilon e_l) & \text{on } \partial\Omega_\varepsilon^l.\end{cases}$$

The functions W_l are translations of w. We see that $W_l(z)=w(z+\varepsilon e_l)$. Superadditivity of the operator

$$\det\left(\frac{\partial^2 u}{\partial z_j\partial\bar z_k}(z)\right)^{1/n}$$

on plurisubharmonic functions u gives

$$\det\left(\frac{\partial^2(v_1+v_2)}{\partial z_j\partial\bar z_k}(z)\right)\ge(\psi_1^{1/n}(z)+\psi_2^{1/n}(z))^n$$

if

$$\det\left(\frac{\partial^2 v_l}{\partial z_j\partial\bar z_k}(z)\right)\ge\psi_l(z),\quad l=1,2.$$

Define $w_\alpha(z)=\alpha^2 w(\alpha^{-1}z)$. We have

$$\begin{aligned}\det\left(\frac{\partial^2 w_\alpha}{\partial z_j\partial\bar z_k}(z)\right)^{1/n}&=\det\left(\frac{\partial^2 w}{\partial z_j\partial\bar z_k}(\alpha^{-1}z)\right)^{1/n}\\&=g(\alpha^{-1}z,w(\alpha^{-1}z))^{1/n}\\&=g(\alpha^{-1}z,\alpha^{-2}w_\alpha(z))^{1/n}\\&\ge g(\alpha^{-1}z,w_\alpha(z))^{1/n}\\&\ge g(z,w_\alpha(z))^{1/n}-C|\alpha^{-1}z-z|\\&\ge g(z+\varepsilon e_l,w_\alpha(z))^{1/n}-C|\alpha^{-1}z-z|-C\varepsilon.\end{aligned}$$

Below we shall modify w_α to a plurisubharmonic function $\widetilde w_l$ which satisfies

$$\det\left(\frac{\partial^2\widetilde w_l}{\partial z_j\partial\bar z_k}(z)\right)\ge g(z+\varepsilon e_l,\widetilde w_l(z))\quad\text{in } \Omega_\varepsilon^l$$

and $\widetilde w_l\le W_l$ on $\partial\Omega_\varepsilon^l$. By Lemma 2.2 we can then conclude that $\widetilde w_l\le W_l$ in Ω_ε^l. Put $\nu_{\zeta,z,l}^+=\zeta-z+d_\Omega(z)e_l$ and $\tilde\nu_{\zeta,z,l}^+=\nu_{\zeta,z,l}^+/|\nu_{\zeta,z,l}^+|$. Let $\xi_t=\zeta+(1-t)\varepsilon(d_\Omega(0)+\varepsilon)^{-1}\tilde\nu_{\zeta,0,l}^+$ and

$$\widetilde N_{\alpha,l}=2D(1-\alpha^{-1})\sup\left\{\left|\min\left\{0,\frac{\partial w}{\partial\tilde\nu_{\zeta,0,l}^+}(\xi_t)\right\}\right|;\zeta\in\partial\Omega\text{ and }t\in[0,1]\right\}$$

and

$$\widetilde{M}_{\alpha,l}=(\alpha^2-1)\sup\{|w(\zeta)|\,;\zeta\in\bar{\Omega}\backslash\alpha^{-1}\Omega^l_\varepsilon\}.$$

Let $\widetilde{C}=C((1-\alpha^{-1})D+\varepsilon)$ and $\widetilde{w}_l(z)=w_\alpha(z)+\widetilde{C}(|z|^2-\alpha^2D^2)-\widetilde{N}_{\alpha,l}-\widetilde{M}_{\alpha,l}$. We have

$$\begin{aligned}\det\left(\frac{\partial^2\widetilde{w}_l}{\partial z_j\partial\bar{z}_k}(z)\right)&\geq\left(\det\left(\frac{\partial^2 w_\alpha}{\partial z_j\partial\bar{z}_k}(z)\right)^{1/n}+\widetilde{C}\right)^n\\&\geq(g(z+\varepsilon e_l,w_\alpha(z))^{1/n}-C|\alpha^{-1}z-z|-C\varepsilon+\widetilde{C})^n\\&\geq g(z+\varepsilon e_l,w_\alpha(z))\\&\geq g(z+\varepsilon e_l,\widetilde{w}_l(z)).\end{aligned}$$

We now must show that $W_l(z)\geq\widetilde{w}_l(z)$ for every $z\in\partial\Omega^l_\varepsilon$. First observe that if $z\in\partial\Omega^l_\varepsilon$ then there is $\zeta\in\partial\Omega$ such that $z=\zeta-\varepsilon e_l$. We have

$$\begin{aligned}W_l(z)-\widetilde{w}_l(z)&=W_l(z)-w_\alpha(z)-\widetilde{C}(|z|^2-\alpha^2D^2)+\widetilde{N}_{\alpha,l}+\widetilde{M}_{\alpha,l}\\&\geq W_l(z)-w_\alpha(z)+\widetilde{N}_{\alpha,l}+\widetilde{M}_{\alpha,l}\\&=w(\zeta)-\alpha^2w(\alpha^{-1}z)+\widetilde{N}_{\alpha,l}+\widetilde{M}_{\alpha,l}\\&=w(\zeta)-w(\alpha^{-1}(\zeta-\varepsilon e_l))-(\alpha^2-1)w(\alpha^{-1}z)+\widetilde{N}_{\alpha,l}+\widetilde{M}_{\alpha,l}\\&\geq w(\zeta)-w(\zeta-\varepsilon(d_\Omega(0)+\varepsilon)^{-1}\nu_{\zeta,0,l})+\widetilde{N}_{\alpha,l}\\&=\frac{\varepsilon|\nu_{\zeta,0,l}|}{d_\Omega(0)+\varepsilon}\frac{\partial w}{\partial\widetilde{\nu}^+_{\zeta,0,l}}(\xi)+\widetilde{N}_{\alpha,l}\geq0,\end{aligned}$$

where ξ, by the mean value theorem, is some point on the line segment

$$\{\xi_t=\zeta+(1-t)\varepsilon(d_\Omega(0)+\varepsilon)^{-1}\nu_{\zeta,0,l}\,;t\in[0,1]\}.$$

It follows that $\widetilde{w}_l\leq W_l$ where both are defined and we have

$$\begin{aligned}w(\varepsilon e_l)=W_l(0)\geq\widetilde{w}_l(0)&=w_\alpha(0)-\widetilde{C}\alpha^2D^2-\widetilde{N}_{\alpha,l}-\widetilde{M}_{\alpha,l}\\&=\alpha^2w(0)-\widetilde{C}\alpha^2D^2-\widetilde{N}_{\alpha,l}-\widetilde{M}_{\alpha,l}\end{aligned}$$

or, since

$$\alpha^2=\left(\frac{d_\Omega(0)+\varepsilon}{d_\Omega(0)}\right)^2=1+\frac{2\varepsilon d_\Omega(0)+\varepsilon^2}{d_\Omega(0)^2},$$

we get

$$\begin{aligned}
w(0)-w(\varepsilon e_l) \le{}& CD^2\left(\frac{D\varepsilon}{d_\Omega(0)+\varepsilon}+\varepsilon\right)\left(1+\frac{2\varepsilon d_\Omega(0)+\varepsilon^2}{d_\Omega(0)^2}\right)\\
&+\frac{2D\varepsilon}{d_\Omega(0)+\varepsilon}\sup\left\{\left|\min\left\{0,\frac{\partial w}{\partial\tilde\nu^+_{\zeta,0,l}}(\xi_t)\right\}\right|;\zeta\in\partial\Omega \text{ and } t\in[0,1]\right\}\\
&+\frac{2\varepsilon d_\Omega(0)+\varepsilon^2}{d_\Omega(0)^2}\sup\{|w(\zeta)|;\zeta\in\bar\Omega\setminus\alpha^{-1}\Omega^l_\varepsilon\}+\frac{2\varepsilon d_\Omega(0)+\varepsilon^2}{d_\Omega(0)^2}|w(0)|\\
\le{}& \frac{2\varepsilon d_\Omega(0)+\varepsilon^2}{d_\Omega(0)^2}\sup_{z\in K}|w(z)|+CD^2\left(\frac{D\varepsilon}{d_\Omega(0)+\varepsilon}+\varepsilon\right)\left(1+\frac{2\varepsilon d_\Omega(0)+\varepsilon^2}{d_\Omega(0)^2}\right)\\
&+\frac{2D\varepsilon}{d_\Omega(0)+\varepsilon}\sup\left\{\left|\min\left\{0,\frac{\partial w}{\partial\tilde\nu^+_{\zeta,0,l}}(\xi_t)\right\}\right|;\zeta\in\partial\Omega \text{ and } t\in[0,1]\right\}\\
&+\frac{2\varepsilon d_\Omega(0)+\varepsilon^2}{d_\Omega(0)^2}\sup\{|w(\zeta)|;\zeta\in\bar\Omega\setminus d_\Omega(0)(d_\Omega(0)+\varepsilon)^{-1}\Omega^l_\varepsilon\}.
\end{aligned}$$

Now let 0 and εe_l change roles, replace $\nu^+_{\zeta,z,l}$ by $\nu^-_{\zeta,z,l}=\zeta-z-d_\Omega(z)e_l$ and $\tilde\nu^+_{\zeta,z,l}$ by $\tilde\nu^-_{\zeta,z,l}=\nu^-_{\zeta,z,l}/|\nu^-_{\zeta,z,l}|$, and repeat the argument to get

$$\begin{aligned}
w(\varepsilon e_l)-w(0) \le{}& \frac{2\varepsilon d_\Omega(\varepsilon e_l)+\varepsilon^2}{d_\Omega(\varepsilon e_l)^2}\sup_{z\in K}|w(z)|\\
&+CD^2\left(\frac{D\varepsilon}{d_\Omega(\varepsilon e_l)+\varepsilon}+\varepsilon\right)\left(1+\frac{2\varepsilon d_\Omega(\varepsilon e_l)+\varepsilon^2}{d_\Omega(\varepsilon e_l)^2}\right)\\
&+\frac{2D\varepsilon}{d_\Omega(\varepsilon e_l)+\varepsilon}\sup\left\{\left|\min\left\{0,\frac{\partial w}{\partial\tilde\nu^-_{\zeta,\varepsilon e_l,l}}(\xi_t)\right\}\right|;\zeta\in\partial\Omega \text{ and } t\in[0,1]\right\}\\
&+\frac{2\varepsilon d_\Omega(\varepsilon e_l)+\varepsilon^2}{d_\Omega(\varepsilon e_l)^2}\sup\{|w(\zeta)|;\zeta\in\bar\Omega\setminus d_\Omega(\varepsilon e_l)(d_\Omega(\varepsilon e_l)+\varepsilon)^{-1}\Omega^l_\varepsilon\}.
\end{aligned}$$

Hence

$$\left|\frac{\partial w}{\partial x_l}(0)\right|\le\frac{2\sup_{z\in K}|w(z)|+2\sup_{z\in\partial\Omega}|\varphi(z)|+2DM+CD^3}{\inf_{z\in K}d_\Omega(z)}+CD^2$$

or since z_0 was an arbitrary point in K,

$$\sup_{z\in K}\left|\frac{\partial w}{\partial x_l}(z)\right|\le\frac{2\sup_{z\in K}|w(z)|+2\sup_{z\in\partial\Omega}|\varphi(z)|+2DM+CD^3}{\inf_{z\in K}d_\Omega(z)}+CD^2. \quad\square$$

If $w\equiv0$ on the boundary of Ω then we can get a better estimate. This is because the sole purpose of $\widetilde N_{\alpha,l}$ and $\widetilde M_{\alpha,l}$ in the above proof is to make sure that $W_l\ge\tilde w_l$ on $\partial\Omega^l_\varepsilon$. If $w\equiv0$ on $\partial\Omega$ then we can set $\widetilde N_{\alpha,l}=\widetilde M_{\alpha,l}=0$ and still be sure that $\tilde w_l\le0\equiv W_l$ on $\partial\Omega^l_\varepsilon$. We have thus proved the following corollary to the proof of Proposition 3.1.

Corollary 3.2. *Assume that Ω is a bounded convex domain in $\mathbf{C}^n$ and that K is a compact subset of Ω. Suppose that $g\in C^\infty(\bar\Omega\times\mathbf{R})$ is a strictly positive function which is increasing in the second variable. Assume that $w\in C^\infty(\bar\Omega)\cap\mathcal{PSH}(\Omega)$ is a solution of*

$$\begin{cases} \det\left(\dfrac{\partial^2 w}{\partial z_j\partial\bar z_k}\right)=g(z,w(z)) & \text{in } \Omega,\\ w(z)\equiv 0 & \text{on } \partial\Omega.\end{cases}$$

Let D be the diameter of Ω and

$$C=\sup\left\{\left|\frac{\partial g^{1/n}}{\partial x_l}(z,t)\right|;(z,t)\in\Omega\times\left[\inf_{z\in\Omega}w(z),0\right] \text{ and } l=1,\dots,2n\right\}.$$

Then

$$\sup_{z\in K}\left|\frac{\partial w}{\partial x_l}(z)\right|\le\frac{2\sup_{z\in K}|w(z)|+CD^3}{\inf_{z\in K}d_\Omega(z)}+CD^2 \quad \text{for } l=1,\dots,2n.$$

If we also assume that g is independent of the z-variable we get the following corollary.

Corollary 3.3. *Assume that Ω is a bounded convex domain in $\mathbf{C}^n$ and that K is a compact subset of Ω. Suppose that $g\in C^\infty(\mathbf{R})$ is a strictly positive function. Assume that $w\in C^\infty(\bar\Omega)\cap\mathcal{PSH}(\Omega)$ is a solution of*

$$\begin{cases} \det\left(\dfrac{\partial^2 w}{\partial z_j\partial\bar z_k}\right)=g(w(z)) & \text{in } \Omega,\\ w(z)\equiv 0 & \text{on } \partial\Omega.\end{cases}$$

Then

$$\sup_{z\in K}\left|\frac{\partial w}{\partial x_l}(z)\right|\le\frac{2\sup_{z\in K}|w(z)|}{\inf_{z\in K}d_\Omega(z)} \quad \text{for } l=1,\dots,2n.$$

4. Global estimates for first derivatives in arbitrary domains

We would like to remove the convexity condition in Proposition 3.1. This is possible to do. However then the estimate changes from an interior to a global estimate. One would think that Proposition 4.1 is much more useful than Proposition 3.1. This is not necessarily so, since sometimes it is trivial to estimate the constant M in Proposition 3.1 while an estimate of $\sup_{\zeta\in\partial\Omega}|\partial w(\zeta)/\partial x_l|$ might be harder. It should be noted that the estimate in Proposition 4.1 is very close to an estimate that was obtained by Bedford and Taylor in [2] by more or less the same method. A similar estimate was also given by Caffarelli, Kohn, Nirenberg and Spruck in [8] using different methods.

Proposition 4.1. *Assume that Ω is a bounded domain in $\mathbf{C}^n$ and that $g\in C^\infty(\bar\Omega\times\mathbf{R})$ is a strictly positive function which is increasing in the second variable. Suppose that $\varphi\in C(\partial\Omega)$ and that $w\in C^\infty(\bar\Omega)\cap\mathcal{PSH}(\Omega)$ is a solution of*

$$\begin{cases} \det\left(\dfrac{\partial^2 w}{\partial z_j\partial\bar z_k}\right)=g(z,w(z)) & \text{in } \Omega,\\ w(z)=\varphi(z) & \text{on } \partial\Omega.\end{cases}$$

Let D be the diameter of Ω and

$$C=\sup\left\{\left|\frac{\partial g^{1/n}}{\partial x_l}(z,t)\right|;(z,t)\in\Omega\times\Big[\inf_{z\in\Omega}w(z),\sup_{\zeta\in\partial\Omega}\varphi(\zeta)\Big] \text{ and } l=1,\dots,2n\right\}.$$

Then

$$\left|\frac{\partial w}{\partial x_l}(z_0)\right|\le\sup_{\zeta\in\partial\Omega}\left|\frac{\partial w}{\partial x_l}(\zeta)\right|+CD^2 \quad \text{for all } z_0\in\Omega \text{ and } l=1,\dots,2n.$$

Proof. Take $z_0\in K$. After a translation we may assume that $z_0=0$. We can also assume that $g\in C^\infty(\mathbf{C}^n\times\mathbf{R})$. Let $e_1,\dots,e_{2n}$ be the standard basis in $\mathbf{R}^{2n}\cong\mathbf{C}^n$. Put $\Omega^l_\varepsilon=\Omega-\varepsilon e_l$ and assume that $\varepsilon>0$ is such that $0\in\Omega^l_\varepsilon$. As in the proof of Proposition 3.1 let W_l be the solution of

$$\begin{cases} \det\left(\dfrac{\partial^2 W_l}{\partial z_j\partial\bar z_k}\right)=g(z+\varepsilon e_l,W_l(z)) & \text{in } \Omega^l_\varepsilon,\\ W_l(z)=\varphi(z+\varepsilon e_l) & \text{on } \partial\Omega^l_\varepsilon.\end{cases}$$

Study $w(z)-W_l(z)$ on $\partial(\Omega\cap\Omega^l_\varepsilon)$. We have

$$\begin{aligned} w(z)-W_l(z)&=w(z)-w(z+\varepsilon e_l)\\ &\ge-\varepsilon\sup\left\{\left|\frac{\partial w}{\partial x_l}(\xi_t)\right|;\xi_t=z+t\varepsilon e_l,\ z\in\partial(\Omega\cap\Omega^l_\varepsilon)\text{ and } t\in[0,1]\right\}.\end{aligned}$$

Therefore, if we define

$$\widetilde W_l(z)=W_l(z)-\varepsilon\sup\left\{\left|\frac{\partial w}{\partial x_l}(\xi_t)\right|;\xi_t=z+t\varepsilon e_l,\ z\in\partial(\Omega\cap\Omega^l_\varepsilon)\text{ and } t\in[0,1]\right\}$$

we have $w\ge\widetilde W_l$ on $\partial(\Omega\cap\Omega^l_\varepsilon)$. If we modify $\widetilde W_l$ to $\tilde w_l(z)=\widetilde W_l(z)+C\varepsilon(|z|^2-D^2)$ we still have $w\ge\tilde w_l$. We have

$$\begin{aligned}\det\left(\frac{\partial^2\widetilde W_l}{\partial z_j\partial\bar z_k}(z)\right)^{1/n}&=\det\left(\frac{\partial^2 W_l}{\partial z_j\partial\bar z_k}(z)\right)^{1/n}\\ &=g(z+\varepsilon e_l,W_l(z))^{1/n}-g(z,W_l(z))^{1/n}+g(z,W_l(z))^{1/n}\\ &\ge g(z,W_l(z))^{1/n}-C\varepsilon\\ &\ge g(z,\widetilde W_l(z))^{1/n}-C\varepsilon\end{aligned}$$

and

$$\det\left(\frac{\partial^2\widetilde{w}_l}{\partial z_j\partial\bar z_k}(z)\right)\geq\left(\det\left(\frac{\partial^2\widetilde{W}_l}{\partial z_j\partial\bar z_k}(z)\right)^{1/n}+C\varepsilon\right)^n$$
$$\geq(g(z,\widetilde{W}_l(z))^{1/n}-C\varepsilon+C\varepsilon)^n=g(z,\widetilde{W}_l(z))\geq g(z,\widetilde{w}_l(z)).$$

By Lemma 2.2 we see that $\widetilde{w}_l\leq w$ where both are defined and because of this

$$\begin{aligned}w(0)&\geq\widetilde{w}_l(0)=\widetilde{W}_l(0)-C\varepsilon D^2\\&=w(\varepsilon e_l)-C\varepsilon D^2-\varepsilon\sup\left\{\left|\frac{\partial w}{\partial x_l}(\xi_t)\right|;\xi_t=z+t\varepsilon e_l,\ z\in\partial(\Omega\cap\Omega_\varepsilon^l)\text{ and }t\in[0,1]\right\}\end{aligned}$$

and we have

$$w(\varepsilon e_l)-w(0)\leq C\varepsilon D^2+\varepsilon\sup\left\{\left|\frac{\partial w}{\partial x_l}(\xi_t)\right|;\xi_t=z+t\varepsilon e_l,\ z\in\partial(\Omega\cap\Omega_\varepsilon^l)\text{ and }t\in[0,1]\right\}.$$

Now let 0 and εe_l change roles and repeat the argument to get

$$w(0)-w(\varepsilon e_l)\leq C\varepsilon D^2+\varepsilon\sup\left\{\left|\frac{\partial w}{\partial x_l}(\xi_t)\right|;\xi_t=z+t\varepsilon e_l,\ z\in\partial(\Omega\cap\Omega_\varepsilon^l)\text{ and }t\in[0,1]\right\}.$$

Thus we can conclude that

$$\left|\frac{\partial w}{\partial x_l}(0)\right|\leq\sup_{\zeta\in\partial\Omega}\left|\frac{\partial w}{\partial x_l}(\zeta)\right|+CD^2.\quad\square$$

If we assume that g is independent of the z-variable we get the following corollary.

Corollary 4.2. *Assume that Ω is a bounded domain in $\mathbf{C}^n$ and that $g\in C^\infty(\mathbf{R})$ is a strictly positive function which is increasing. Suppose that $\varphi\in C(\partial\Omega)$ and that $w\in C^\infty(\bar\Omega)\cap\mathcal{PSH}(\Omega)$ is a solution of*

$$\begin{cases}\det\left(\dfrac{\partial^2 w}{\partial z_j\partial\bar z_k}\right)=g(w(z)) & \text{in }\Omega,\\ w(z)=\varphi(z) & \text{on }\partial\Omega.\end{cases}$$

Then

$$\left|\frac{\partial w}{\partial x_l}(z_0)\right|\leq\sup_{\zeta\in\partial\Omega}\left|\frac{\partial w}{\partial x_l}(\zeta)\right|\quad\text{for all }z_0\in\Omega\text{ and }l=1,\dots,2n.$$

5. Smooth solutions to the Dirichlet problem in hyperconvex domains satisfying the nonprecipitousness condition

Theorem 5.1. *Assume that Ω is a bounded hyperconvex domain in $\mathbf{C}^n$ and that $f\in C^\infty(\bar\Omega\times\mathbf{R})$ is a strictly positive function which is increasing in the second variable. If Ω satisfies the NP-condition, see Definition* 1.7, *then the problem*

$$\begin{cases} \det\left(\dfrac{\partial^2 u}{\partial z_j\partial\bar z_k}\right)=f(z,u(z)) & \text{in } \Omega,\\ \lim\limits_{z\to z_0} u(z)=0 & \text{for all } z_0\in\partial\Omega,\end{cases}$$

has a unique smooth strictly plurisubharmonic solution u, which moreover satisfies

$$\sup\left\{\left|\frac{\partial u}{\partial x_l}(z)\right|; z\in\Omega \text{ and } l=1,\dots,2n\right\}<\infty.$$

Conversely, if there is a smooth strictly plurisubharmonic solution u to the problem

$$\begin{cases} \det\left(\dfrac{\partial^2 u}{\partial z_j\partial\bar z_k}\right)=f(z,u(z)) & \text{in } \Omega,\\ \lim\limits_{z\to z_0} u(z)=0 & \text{for all } z_0\in\partial\Omega,\end{cases}$$

which satisfies

$$\sup\left\{\left|\frac{\partial u}{\partial x_l}(z)\right|; z\in\Omega \text{ and } l=1,\dots,2n\right\}<\infty,$$

then Ω satisfies the NP-condition.

Before we prove this theorem we state two propositions which we shall use in proving Theorem 5.1. In [14] Schulz established the following result.

Proposition 5.2. *Assume that Ω is a bounded pseudoconvex domain in $\mathbf{C}^n$ and that $g\in C^\infty(\bar\Omega\times\mathbf{R})$ is an increasing function in the second variable which satisfies $g(z,t)>0$ for all $z\in\bar\Omega$ and $t\in\mathbf{R}$. Suppose that $w\in C^\infty(\bar\Omega)\cap\mathcal{PSH}(\Omega)$ is a solution of*

$$\begin{cases} \det\left(\dfrac{\partial^2 w}{\partial z_j\partial\bar z_k}\right)=g(z,w(z)) & \text{in } \Omega,\\ w(z)\equiv 0 & \text{on } \partial\Omega.\end{cases}$$

Then for any $\varepsilon>0$ there is a constant C which depends only on n, ε, $\|w(z)\|_{C^1(\Omega)}$, $(\inf_{z\in\Omega} g(z,w(z)))^{-1}$, $\|g(z,w(z))\|_{C^1(\Omega)}$ and the diameter of Ω such that

$$\left|\frac{\partial^2 w}{\partial z_j\partial\bar z_k}(z)\right|\le\frac{C}{(-w(z))^{(2+\varepsilon)}}\quad \text{for } j,k=1,\dots,n.$$

We shall also use the following result which was proven by Błocki in [6].

Proposition 5.3. *Let w be a C^4 plurisubharmonic function in an open set Ω in $\mathbf{C}^n$ and $\psi(z)=\det(\partial^2 w(z)/\partial z_j\partial\bar z_k)$. Assume that for some nonnegative K_0, K_1, b, B_0 and B_1 we have*

$$\|w\|_{C^1(\Omega)}\le K_0,\quad \sup_{z\in\Omega}\Delta w(z)\le K_1,\quad b\le\psi(z)\le B_0\quad\text{and}\quad \|\psi^{1/n}(z)\|_{C^1(\Omega)}\le B_1.$$

Then for any $\Omega'\Subset\Omega$ there are two constants α and C where $\alpha\in(0,1)$ depends only on n, K_0, K_1, b, B_0 and B_1, and C depends, besides those quantities, on $\inf_{z\in\Omega'} d_\Omega(z)$, such that

$$\sup\left\{\left\|\frac{\partial^2 w}{\partial z_j\partial\bar z_k}\right\|_{C^\alpha(\Omega')};j,k=1,\dots,n\right\}\le C.$$

Proof of Theorem 5.1. Assume that Ω satisfies the NP-condition and let ϱ be an exhaustion function satisfying the conditions in Definition 1.7. We shall now construct a sequence $(\Omega_m)_{m=1}^\infty$ of hyperconvex sets with smooth boundary such that $\Omega_m\Subset\Omega_{m+1}$ and $\bigcup_{m=1}^\infty\Omega_m=\Omega$. By Sard's theorem there is a sequence $a_m>0$ such that $\lim_{m\to\infty}a_m=0$ and the sets $\Omega_m=\{z\in\Omega;\varrho_m(z)=\varrho(z)+a_m<0\}$ have smooth boundary and $\Omega_m\Subset\Omega_{m+1}$. Let $u_m, m\in\mathbf{N}$, be the solution of

$$\begin{cases}\det\left(\dfrac{\partial^2 u_m}{\partial z_j\partial\bar z_k}\right)=f(z,u_m(z)) & \text{in } \Omega_m,\\ \lim\limits_{z\to z_0}u_m(z)=0 & \text{for all } z_0\in\partial\Omega_m,\end{cases}$$

which by Theorem 1.2 is smooth. By Lemma 2.2 we have that $u_{m+m'+1}\le u_{m+m'}$ in Ω_m for $m\in\mathbf{N}$ and $m'\in\mathbf{Z}_+$. Now let $B_R(z_0)$ be a ball that contains Ω and put $K=\sup_{z\in\bar\Omega}f(z,0)$. For $v=K^{1/n}(|z-z_0|^2-R^2)$ we have

$$\det\left(\frac{\partial^2 v}{\partial z_j\partial\bar z_k}\right)=K,$$

and Lemma 2.1 assures that $v\le u_j$ on Ω_j. Suppose that $\Omega'\Subset\Omega''\Subset\Omega$. There is an $N\in\mathbf{N}$ so that if $m\ge N$ then $\Omega''\Subset\Omega_m$. Put $u(z)=\lim_{m\to\infty}u_m(z)$ for $z\in\Omega$. This makes sense since $\bigcup_{m=1}^\infty\Omega_m=\Omega$, and since $\lim_{z\to z_0}v(z)=0$ for all $z_0\in\partial\Omega$ we get $\lim_{z\to z_0}u(z)=0$ for all $z_0\in\partial\Omega$. Our task is to show that the sequence $(u_m)_{m=1}^\infty$ is uniformly bounded in $C^{2,\alpha}(\Omega')$ for some $\alpha>0$. We have

$$\sup\{|u_m(z)|;z\in\Omega'\text{ and }m\in\mathbf{N}\}\le\sup_{z\in\Omega'}|v(z)|<\infty.$$

If we let D be the diameter of Ω and

$$C=\sup\left\{\left|\frac{\partial f^{1/n}}{\partial x_l}(z,t)\right|;(z,t)\in\Omega\times\left[\inf_{z\in\Omega}v(z),0\right]\text{ and }l=1,\dots,2n\right\},$$

then by Proposition 4.1 we have

$$\sup_{z\in\Omega'}\left|\frac{\partial u_m}{\partial x_l}(z)\right| \le \sup_{\zeta\in\partial\Omega_j}\left|\frac{\partial u_m}{\partial x_l}(\zeta)\right| + CD^2 \quad \text{for } l=1,\dots,2n.$$

Let

$$C_1 = \sup\left\{|f^{1/n}(z,t)|\,;(z,t)\in\Omega\times\left[\inf_{z\in\Omega} v(z),0\right] \text{ and } l=1,\dots,2n\right\}.$$

Then

$$\det\left(\frac{\partial^2 C_1\varrho_m}{\partial z_j\partial\bar z_k}(z)\right), = C_1^n \det\left(\frac{\partial^2 \varrho_m}{\partial z_j\partial\bar z_k}(z)\right) \ge C_1^n \ge f(z,u_m(z)) = \det\left(\frac{\partial^2 u_m}{\partial z_j\partial\bar z_k}(z)\right),$$

and hence $C_1\varrho_m \le u \le 0$ in Ω_m. If we let $e_1,\dots,e_{2n}$ be the standard basis in $\mathbf{R}^{2n}$ it follows that

$$\left|\frac{\partial u_m}{\partial x_l}\right| \le C_1\left|\frac{\partial\varrho}{\partial x_l}\right|$$

for points $z_0\in\partial\Omega_m$ such that

$$\{z_0+te_l\,;t\in\mathbf{R}\}\cap\Omega_m\cap B_\varepsilon(z_0)\ne\emptyset \quad \text{for all } \varepsilon>0.$$

We need to pay special attention to boundary points z_0 where we have

$$\{z_0+te_l\,;t\in\mathbf{R}\}\cap\Omega_m\cap B_\varepsilon(z_0)=\emptyset \quad \text{for some } \varepsilon>0.$$

The function $u_m(z_0+te_l)$ is nonnegative as a function of t and zero when $t=0$. This shows that $\partial u_m(z_0)/\partial x_l=0$ and as a result the inequality

$$\left|\frac{\partial u_m}{\partial x_l}\right| \le C_1\left|\frac{\partial\varrho}{\partial x_l}\right|$$

holds for all boundary points. Since Ω is assumed to satisfy the NP-condition we have

$$\sup\left\{\left|\frac{\partial u_m}{\partial x_l}(z)\right|;z\in\Omega' \text{ and } m\in\mathbf{N}\right\} \le C_1\sup_{z\in\Omega}\left|\frac{\partial\varrho}{\partial x_l}(z)\right| < \infty.$$

Using Proposition 5.2 we get, for a fixed $\varepsilon>0$,

$$\left|\frac{\partial^2 u_m}{\partial z_k\partial\bar z_l}(z)\right| |u_m(z)|^{2+\varepsilon} \\ \le \widetilde{C}\Big(n,\varepsilon,\Big(\inf_{z\in\Omega_m} f(z,u_m(z))\Big)^{-1},\|f(z,u_m(z))\|_{C^1(\Omega_m)},\|u_m\|_{C^1(\Omega_m)},D\Big)$$

for $k,l=1,\dots,n$. We have proven that $(u_m)_{m=1}^{\infty}$ is uniformly bounded in $C^1(\Omega)$. It follows that $\widetilde{C}<\infty$. Since $\Omega'\Subset\Omega''\Subset\Omega_N$ we have

$$\left|\frac{\partial^2 u_m}{\partial z_k \partial \bar{z}_l}(z)\right| \le \frac{\widetilde{C}}{|u_m(z)|^{2+\varepsilon}} \le \frac{\widetilde{C}}{a_N^{2+\varepsilon}}.$$

We can now use Proposition 5.3. All the constants in Proposition 5.3 are under control since $\inf_{z\in\Omega'} d_{\Omega_m}(z)\ge\inf_{z\in\Omega'} d_{\Omega''}(z)>0$, so we see that $(u_m)_{m=1}^{\infty}$ is uniformly bounded in $C^{2,\alpha}(\Omega')$ for some $\alpha\in(0,1)$. We can now use the Schauder theory described in Gilbarg and Trudinger's book [11] to establish that $u\in C^\infty(\Omega')$, and since Ω' was arbitrary we get $u\in C^\infty(\Omega)$. We also see that in arbitrary compact Ω' we have

$$\sup_{z\in\Omega'}\left|\frac{\partial u}{\partial x_l}(z)\right| \le C_1 \sup_{z\in\Omega}\left|\frac{\partial \varrho}{\partial x_l}(z)\right| < \infty,$$

and again since Ω' was arbitrary

$$\sup_{z\in\Omega}\left|\frac{\partial u}{\partial x_l}(z)\right| < \infty.$$

Uniqueness follows from Lemma 2.2.

Conversely, if we have a smooth plurisubharmonic solution u to the problem

$$\begin{cases} \det\left(\dfrac{\partial^2 u}{\partial z_j \partial \bar{z}_k}\right) = f(z,u(z)) & \text{in } \Omega, \\ \lim\limits_{z\to z_0} u(z)=0 & \text{for all } z_0\in\partial\Omega, \end{cases}$$

which satisfies

$$\sup\left\{\left|\frac{\partial u}{\partial x_j}(z)\right| ; z\in\Omega \text{ and } j=1,\dots,2n\right\} < \infty$$

we see that this solution satisfies all the conditions in Definition 1.7, at least after multiplication by a constant, and hence Ω satisfies the NP-condition. □

Corollary 5.4. *Let Ω be a bounded hyperconvex domain in $\mathbf{C}^n$ and $f\in C^\infty(\bar{\Omega})$ be a strictly positive function. If Ω satisfies the NP-condition, see Definition* 1.7, *then the problem*

$$\begin{cases} \det\left(\dfrac{\partial^2 u}{\partial z_j \partial \bar{z}_k}\right) = f(z) & \textit{in } \Omega, \\ \lim\limits_{z\to z_0} u(z)=0 & \textit{for all } z_0\in\partial\Omega, \end{cases}$$

has a unique smooth strictly plurisubharmonic solution u, which satisfies

$$\sup\left\{\left|\frac{\partial u}{\partial x_l}(z)\right|; z\in\Omega \text{ and } l=1,\dots,2n\right\}<\infty.$$

Conversely, if there is a smooth plurisubharmonic solution u to the problem

$$\begin{cases} \det\left(\dfrac{\partial^2 u}{\partial z_j\partial\bar z_k}\right)=f(z) & \text{in } \Omega,\\ \lim_{z\to z_0} u(z)=0 & \text{for all } z_0\in\partial\Omega,\end{cases}$$

satisfying

$$\sup\left\{\left|\frac{\partial u}{\partial x_l}(z)\right|; z\in\Omega \text{ and } l=1,\dots,2n\right\}<\infty,$$

then Ω satisfies the NP-condition.

It would be desirable to describe the hyperconvex domains Ω for which it is possible to solve the problem

$$\begin{cases} \det\left(\dfrac{\partial^2 \varrho}{\partial z_j\partial\bar z_k}\right)\geq 1 & \text{in } \Omega,\\ \lim_{z\to z_0} \varrho(z)=0 & \text{for all } z_0\in\partial\Omega,\\ \sup_{z\in\Omega}\left|\dfrac{\partial\varrho}{\partial x_l}(z)\right|<\infty. & \end{cases}$$

Błocki's result on defining functions for hyperconvex domains shows that

$$\begin{cases} \det\left(\dfrac{\partial^2 \varrho}{\partial z_j\partial\bar z_k}\right)\geq 1 & \text{in } \Omega,\\ \lim_{z\to z_0} \varrho(z)=0 & \text{for all } z_0\in\partial\Omega\end{cases}$$

is a natural condition. If we combine this with the condition that the defining function should satisfy

$$\sup_{z\in\Omega}\left|\frac{\partial\varrho}{\partial x_l}(z)\right|<\infty$$

we get an interesting class of domains which deserves study.

If one wants existence results for

$$\begin{cases} \det\left(\dfrac{\partial^2 u}{\partial z_j\partial\bar z_k}\right)=f(z,u(z)) & \text{in } \Omega,\\ \lim_{z\to z_0} u(z)=0 & \text{for all } z_0\in\partial\Omega.\end{cases}$$

in general hyperconvex domains one would need to obtain other estimates than those in Propositions 4.1, 5.2 and 5.3. To control the constants in these propositions we need global estimates of the first derivatives. Note that Propositions 5.2 and 5.3 are troublesome only if $f(z,t)$ really depends on the t-variable. If f is constant in the t-variable only Proposition 4.1 is troublesome. So in order to obtain an existence result we would need to get interior versions of Propositions 4.1, 5.2 and 5.3. If it were possible to somehow remove the convexity condition in Proposition 3.1 this would settle the matter when f only depends on the z-variable.

6. Which domains satisfy the nonprecipitousness condition?

In this section we shall give examples of domains satisfying and not satisfying the NP-condition. See Definition 1.7 for the definition of the NP-condition.

Proposition 6.1. *Assume that $\Omega_1, \dots, \Omega_N$ are hyperconvex domains satisfying the NP-condition. Then $\bigcap_{l=1}^N \Omega_l$ satisfies the NP-condition.*

Proof. As Ω_m satisfies the NP-condition we can write $\Omega_m = \{z \in \mathbf{C}^n; \varrho_m(z) < 0\}$, $m=1,\dots,n$, where ϱ_m is a smooth plurisubharmonic function satisfying

$$\det\left(\frac{\partial^2 \varrho_m}{\partial z_j \partial \bar{z}_k}(z)\right) \geq 1 \quad \text{for } z \in \Omega_m,$$

$\lim_{z\to z_0} \varrho_m(z) = 0$ for all $z_0 \in \partial\Omega_m$ and

$$\sup\left\{\left|\frac{\partial \varrho_m}{\partial x_l}(z)\right|; z \in \Omega_m \text{ and } l=1,\dots,2n\right\} < \infty.$$

Let $f\colon \{y\in\mathbf{R}^2; y_1 \leq 0,\ y_2 \leq 0\} \to \mathbf{R}$ be given by $f(y_1,y_2)=\max\{y_1,y_2\}$ and $\tilde{f}$ be a smooth convex strictly negative approximation of f satisfying $\tilde{f}(y_1,0)=\tilde{f}(0,y_2)=0$ for $y_1\leq 0$ and $y_2 \leq 0$. We can choose $\tilde{f}$ so that $f \leq \tilde{f}$ and $\partial\tilde{f}/\partial y_l > 0$ for $l=1,2$, and we shall do so.

We have $\Omega_1 \cap \Omega_2 = \{z\in\mathbf{C}^n; \max\{\varrho_1,\varrho_2\}<0\}$. Put $\varrho(z)=\tilde{f}(\varrho_1(z),\varrho_2(z))$. This is a smooth plurisubharmonic function. We see that $\lim_{z\to z_0}\varrho(z)=0$ for $z_0 \in \partial(\Omega_1\cap\Omega_2)$ and since $\max\{\varrho_1,\varrho_2\} \leq \varrho$ it follows that

$$\sup\left\{\left|\frac{\partial \varrho}{\partial x_l}(z)\right|; z \in \Omega_1\cap\Omega_2 \text{ and } l=1,\dots,2n\right\} < \infty.$$

We claim that

$$\det\left(\frac{\partial^2 \varrho}{\partial z_j \partial \bar{z}_k}\right) \geq C > 0 \quad \text{in } \Omega_1\cap\Omega_2.$$

One way of proving this claim is to use that for a Hermitian matrix B we have

$$(\det B)^{1/n} = \frac{1}{n} \inf_{A\in\mathcal{A}} \operatorname{Tr}(AB),$$

where $\mathcal{A}$ denotes the family of all Hermitian matrices A with $\det A=1$. For a proof of this equality see [10] or [4]. We have

$$\begin{aligned}\frac{\partial^2 \varrho}{\partial z_j \partial \bar{z}_k} &= \frac{\partial \tilde{f}}{\partial y_1}\frac{\partial^2 \varrho_1}{\partial z_j \partial \bar{z}_k} + \frac{\partial \tilde{f}}{\partial y_2}\frac{\partial^2 \varrho_2}{\partial z_j \partial \bar{z}_k} + \frac{\partial^2 \tilde{f}}{\partial y_1^2}\frac{\partial \varrho_1}{\partial z_j}\frac{\partial \varrho_1}{\partial \bar{z}_k} \\ &\quad + \frac{\partial^2 \tilde{f}}{\partial y_1 \partial y_2}\frac{\partial \varrho_2}{\partial z_j}\frac{\partial \varrho_1}{\partial \bar{z}_k} + \frac{\partial^2 \tilde{f}}{\partial y_1 \partial y_2}\frac{\partial \varrho_1}{\partial z_j}\frac{\partial \varrho_2}{\partial \bar{z}_k} + \frac{\partial^2 \tilde{f}}{\partial y_2^2}\frac{\partial \varrho_2}{\partial z_j}\frac{\partial \varrho_2}{\partial \bar{z}_k}.\end{aligned}$$

Therefore we have

$$\begin{aligned}\left(\frac{\partial^2 \varrho}{\partial z_j \partial \bar{z}_k}\right) &= \frac{\partial \tilde{f}}{\partial y_1}\left(\frac{\partial^2 \varrho_1}{\partial z_j \partial \bar{z}_k}\right) + \frac{\partial \tilde{f}}{\partial y_2}\left(\frac{\partial^2 \varrho_2}{\partial z_j \partial \bar{z}_k}\right) \\ &\quad + \begin{pmatrix} \frac{\partial \varrho_1}{\partial \bar{z}_1} & \frac{\partial \varrho_2}{\partial \bar{z}_1} \\ \vdots & \vdots \\ \frac{\partial \varrho_1}{\partial \bar{z}_n} & \frac{\partial \varrho_2}{\partial \bar{z}_n} \end{pmatrix} \begin{pmatrix} \frac{\partial^2 \tilde{f}}{\partial y_1^2} & \frac{\partial^2 \tilde{f}}{\partial y_1 \partial y_2} \\ \frac{\partial^2 \tilde{f}}{\partial y_2 \partial y_1} & \frac{\partial^2 \tilde{f}}{\partial y_2^2} \end{pmatrix} \begin{pmatrix} \frac{\partial \varrho_1}{\partial z_1} & \cdots & \frac{\partial \varrho_1}{\partial z_n} \\ \frac{\partial \varrho_2}{\partial z_1} & \cdots & \frac{\partial \varrho_2}{\partial z_n} \end{pmatrix}.\end{aligned}$$

In view of the convexity of $\tilde{f}$ we get

$$\begin{aligned} n \det\left(\frac{\partial^2 \varrho}{\partial z_j \partial \bar{z}_k}\right)^{1/n} &= \inf_{A\in\mathcal{A}} \operatorname{Tr}\left(A\left(\frac{\partial^2 \varrho}{\partial z_j \partial \bar{z}_k}\right)\right) \\ &\ge \frac{\partial \tilde{f}}{\partial y_1} \inf_{A\in\mathcal{A}} \operatorname{Tr}\left(A\left(\frac{\partial^2 \varrho_1}{\partial z_j \partial \bar{z}_k}\right)\right) + \frac{\partial \tilde{f}}{\partial y_2} \inf_{A\in\mathcal{A}} \operatorname{Tr}\left(A\left(\frac{\partial^2 \varrho_2}{\partial z_j \partial \bar{z}_k}\right)\right) \\ &\ge \frac{\partial \tilde{f}}{\partial y_1} + \frac{\partial \tilde{f}}{\partial y_2}. \end{aligned}$$

Multiplying ϱ by a constant if necessary we see that $\Omega_1 \cap \Omega_2$ satisfies the NP-condition. The proposition follows by induction. □

Proposition 6.2. *A strongly pseudoconvex domain $\Omega \Subset \mathbf{C}^n$ with smooth boundary satisfies the NP-condition.*

Proof. Take a smooth plurisubharmonic defining function ϱ for the strongly pseudoconvex domain. It is a standard fact that this defining function can be modified to a defining function which is strictly plurisubharmonic on the closure of the domain and hence the domain satisfies the NP-condition. The modification is done as follows. Let $\psi=\varrho+A\varrho^2$ for an $A\geq 0$. Then

$$\sum_{j,k=1}^{n}\frac{\partial^2\psi}{\partial z_j\partial\bar z_k}t_j\bar t_k=(1+2A\varrho)\sum_{j,k=1}^{n}\frac{\partial^2\varrho}{\partial z_j\partial\bar z_k}t_j\bar t_k+2A\left|\sum_{j=1}^{n}\frac{\partial\varrho}{\partial z_j}t_j\right|^2.$$

On $\partial\Omega$ we have

$$\sum_{j,k=1}^{n}\frac{\partial^2\psi}{\partial z_j\partial\bar z_k}t_j\bar t_k=\sum_{j,k=1}^{n}\frac{\partial^2\varrho}{\partial z_j\partial\bar z_k}t_j\bar t_k+2A\left|\sum_{j=1}^{n}\frac{\partial\varrho}{\partial z_j}t_j\right|^2$$

and for nonzero $t\in\{w\in\mathbf{C}^n;\sum_{j=1}^n\partial\varrho w_j/\partial z_j=0\}$ the first term is strictly positive and for other nonzero t the second term is strictly positive so if we choose A large enough then we can conclude that ψ is strictly plurisubharmonic on a neighborhood of $\partial\Omega$. Now choose an $\varepsilon>0$ such that ψ is strictly plurisubharmonic on the component of $\{z\in\mathbf{C}^n;|\psi(z)|<2\varepsilon\}$ which has nonempty intersection with $\partial\Omega$. Call this component U. Suppose that $\varphi\colon\mathbf{R}\to\mathbf{R}$ is a smooth convex function satisfying $\varphi(x)=x$ if $|x|<\frac{1}{2}\varepsilon$ and $\varphi(x)=-\varepsilon$ if $x\leq-2\varepsilon$. Let K be a compact subset of Ω such that $\{z\in\mathbf{C}^n;|\psi(z)|<2\varepsilon\}\setminus K=U$. Now put

$$\mu(z)=\begin{cases}\varphi\circ\psi(z), & \text{if } z\in(\Omega\cup U)\setminus K,\\ -\varepsilon, & \text{if } z\in K.\end{cases}$$

This is a plurisubharmonic defining function for Ω which is strictly plurisubharmonic near $\partial\Omega$. Let $\chi\in C_0^\infty(\Omega)$ be such that $\chi^{-1}(1)\supseteq\Omega\setminus U$. Put $\eta(z)=\mu(z)+\delta\chi(z)|z|^2$. For δ small enough this function gives us the exhaustion function we need. □

Example 6.3. Now we shall show that the bidisk $D^2=\{z\in\mathbf{C}^2;|z_1|<1 \text{ and } |z_2|<1\}$ in $\mathbf{C}^2$ does not satisfy the NP-condition. Define $E=\{x\in\mathbf{R}^2;x_1<0 \text{ and } x_2<0\}$ and $f(x_1,x_2)=-x_1\log(-x_1)-x_2\log(-x_2)+(x_1+x_2)\log(-x_1-x_2)$ on E. The function f is convex, homogeneous of degree 1 and zero on ∂E. Since f is homogeneous of degree 1 we have

$$\det\left(\frac{\partial^2 f}{\partial x_j\partial x_k}\right)\equiv 0.$$

This function is not quite the function we need for our construction since f is not globally bounded. We therefore modify it as

$$g(x_1,x_2)=\begin{cases}f(x_1,x_2), & \text{when } -1\leq x_1<0 \text{ and } -1\leq x_2<0,\\ f(x_1,-1), & \text{when } -1\leq x_1<0 \text{ and } x_2<-1,\\ f(-1,x_2), & \text{when } x_1<-1 \text{ and } -1\leq x_2<0,\\ f(-1,-1), & \text{when } x_1<-1 \text{ and } x_2<-1.\end{cases}$$

Obviously $g(x_1,x_2)=\max\{f(x_1,x_2),f(x_1,-1),f(-1,x_2),f(-1,-1)\}$ and therefore it is convex. Now define $u\colon D^2\to\mathbf{R}$ as $u(z_1,z_2)=g(\log|z_1|,\log|z_2|)$. At points where u is smooth we have

$$\det\left(\frac{\partial^2 u}{\partial z_j\partial \bar z_k}(z_1,z_2)\right)=\frac{1}{16|z_1|^2|z_2|^2}\det\left(\frac{\partial^2 g}{\partial x_j\partial x_k}(\log|z_1|,\log|z_2|)\right)$$

we see that

$$\det\left(\frac{\partial^2 u}{\partial z_j\partial \bar z_k}(z_1,z_2)\right)=0\quad\text{on } D^2\setminus\{z\in\mathbf{C}^2\,;\log|z_1|=\log|z_2|=-1\}.$$

The Monge–Ampère mass of u is concentrated on the set $\{z\in D^2;\log|z_1|=\log|z_2|=-1\}$. In order to use the comparison principles, i.e. Lemmas 2.1 and 2.2, we need to examine globally defined smooth regularizations of u. We need to conclude that when we apply the Monge–Ampère operator to these regularizations we get a function that is bounded as we approach boundary points. In order to achieve this we need to be a little careful when we regularize u.

Let $\varphi_w(z)$ be the Möbius transformation that is a bijection of $D=\{z\in\mathbf{C};|z|<1\}$ onto itself, send 0 onto w and leaves $w/|w|$ invariant. It is easy to see that

$$\varphi_w(z)=\frac{z+w}{z\bar w+1}.$$

Put $\varphi_w(z)=\varphi(z,w)$ and let $\zeta=\varphi(z,w)$. For every $w\in D$ there is a function ψ_w such that $\varphi(z,\psi_w(z,\zeta))=\zeta$. One sees this by solving

$$\frac{z+w}{z\bar w+1}=\zeta$$

for w. Taking real and imaginary parts we see that this equation has the same solution as the system

$$\begin{cases}(1-\operatorname{Re}(z\zeta))\operatorname{Re}(w)-\operatorname{Im}(z\zeta)\operatorname{Im}(w)=\operatorname{Re}(\zeta)-\operatorname{Re}(z),\\ -\operatorname{Im}(z\zeta)\operatorname{Re}(w)+(1+\operatorname{Re}(z\zeta))\operatorname{Im}(w)=\operatorname{Im}(\zeta)-\operatorname{Im}(z)\end{cases}$$

which is uniquely solvable since $1-(\operatorname{Re}(z\zeta))^2-(\operatorname{Im}(z\zeta))^2>0$ when $z,\zeta\in D$. Take a nonnegative $\chi\in C_0^\infty(D)$ such that $\int_D\chi\,d\lambda=1$. Write $\Phi(z,w)=(\varphi(z_1,w_1),\varphi(z_2,w_2))$ and define

$$u_\chi(z)=\int_{D^2}u(\Phi(z,w))\chi(w_1)\chi(w_2)\,d\lambda(w_1)\,d\lambda(w_2).$$

This function is globally defined. If we write $\Psi(w,z,\zeta)=(\psi_{w_1}(z_1,\zeta_1),\psi_{w_2}(z_2,\zeta_2))$ we get

$$u_\chi(z)=\int_{D^2}u(\zeta)\chi(\psi_{w_1}(z_1,\zeta_1))\chi(\psi_{w_2}(z_2,\zeta_2))\det\left(\frac{\partial\Psi}{\partial\zeta}(w,z,\zeta)\right)d\lambda(\zeta_1)\,d\lambda(\zeta_2).$$

Now we can differentiate with respect to z without any problems and u_χ is smooth. What remains to be shown to conclude that u_χ is plurisubharmonic is the sub-mean value inequality. Note that, for fixed $w=(w_1,w_2)$, the function $u(\Phi(z,w))$ is plurisubharmonic. Because of this and Fubini's theorem we get, for $z\in D^2$, $\zeta\in\mathbf{C}^2$ and $\tau\in\mathbf{R}$,

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} & u_\chi(z+\tau\zeta e^{i\theta})\,d\theta \\ &= \int_{D^2}\biggl(\frac{1}{2\pi}\int_0^{2\pi} u(\Phi(z+\tau\zeta e^{i\theta},w)\,d\theta\biggr)\chi(w_1)\chi(w_2)\,d\lambda(w_1)\,d\lambda(w_2) \\ &\ge \int_{D^2} u(\Phi(z,w))\chi(w_1)\chi(w_2)\,d\lambda(w_1)\,d\lambda(w_2) \\ &= u_\chi(z)\end{aligned}$$

for τ so small that $u_\chi(z+\tau\zeta e^{i\theta})$ is defined for all $\theta\in[0,2\pi]$ and we see that u_χ is plurisubharmonic.

We shall now estimate

$$\det\biggl(\frac{\partial^2 u_\chi}{\partial z_j\partial\bar z_k}\biggr).$$

First note that

$$\det\biggl(\frac{\partial^2 u_\chi}{\partial z_j\partial\bar z_k}\biggr)\le\frac{\partial^2 u_\chi}{\partial z_1\partial\bar z_1}\,\frac{\partial^2 u_\chi}{\partial z_2\partial\bar z_2}.$$

We have

$$\frac{\partial^2 u_\chi}{\partial z_1\partial\bar z_1}=\int_{D^2}\biggl(\frac{\partial^2}{\partial z_1\partial\bar z_1}u(\varphi(z_1,w_1),\varphi(z_2,w_2))\biggr)\chi(w_1)\chi(w_2)\,d\lambda(w_1)\,d\lambda(w_2)$$

and similarly for $\partial^2 u_\chi/\partial z_2\partial\bar z_2$. Using the chain rule we get

$$\begin{aligned}\frac{\partial^2}{\partial z_1\partial\bar z_1}u(\varphi(z_1,w_1),\varphi(z_2,w_2)) &= \biggl|\frac{\partial\varphi_{w_1}}{\partial z_1}(z_1)\biggr|^2\frac{\partial^2 u}{\partial z_1\partial\bar z_1}(\varphi(z_1,w_1),\varphi(z_2,w_2)) \\ &= \biggl|\frac{\partial\varphi_{w_1}}{\partial z_1}(z_1)\biggr|^2\frac{1}{4|z_1|^2}\frac{\partial^2 g}{\partial x_1^2}(\log|\varphi(z_1,w_1)|,\log|\varphi(z_2,w_2)|)\end{aligned}$$

at points where u is differentiable, which is almost everywhere. We do not need to worry about what happens when $|z_1|=0$ since $\partial^2 g/\partial x_1^2=0$ when $x_1<-1$ and since $x_1=\log|z_1|$ we see that the apparent singularity is nonexistent.

Now let us examine which sets $\varphi(z_1,\operatorname{supp}\chi)$ appear for different z_1. In order to make this analysis simple let us pick a χ such that $\operatorname{supp}\chi=\{w\in\mathbf{C};|w|\le\frac12\}$. Obviously $\varphi(0,\operatorname{supp}\chi)=\operatorname{supp}\chi$. If $z_1\neq 0$ then we can write

$$\varphi(z_1,w_1)=\frac{e^{2i\arg(w_1+1/\bar z_1)}}{z_1}+\frac{|z_1|^2-1}{|z_1|^2\bar w_1+\bar z_1}.$$

When we have rewritten $\varphi(z_1,w_1)$ in this form, it is possible to interpret the mapping as a composition of translations, dilations and inversions in the circle. Viewing $\varphi(z_1,w_1)$ in this way lets us conclude that, if we put

$$\alpha(z_1)=\frac{1}{|z_1|}+\frac{2(|z_1|^2-1)}{|z_1|(2+|z_1|)} \quad\text{and}\quad \beta(z_1)=\frac{1}{|z_1|}+\frac{2(|z_1|^2-1)}{|z_1|(2-|z_1|)},$$

then

$$\varphi(z_1,\operatorname{supp}\chi)\subset\{w_1\in\mathbf{C}\,;\beta(z_1)\le|w_1|<\alpha(z_1)\}.$$

We are now ready to begin our estimation of the integral

$$\int_{D^2}\left(\frac{\partial^2}{\partial z_1\partial\bar z_1}u(\varphi(z_1,w_1),\varphi(z_2,w_2))\right)\chi(w_1)\chi(w_2)\,d\lambda(w_1)\,d\lambda(w_2).$$

First it is obvious that $(4|z_1|^2)^{-1}$ is bounded when $|z_1|>e^{-1}$. We see that

$$\frac{\partial\varphi_{w_1}}{\partial z_1}(z_1)=\frac{w_1}{|w_1|}\frac{1-|w_1|^2}{(1+z_1|w_1|)^2}$$

and because of this we see that $|\partial\varphi(z_1,w_1)/\partial z_1|^2$ is bounded when $|w_1|\le\frac12$ and $|z_1|\le1$. Since also χ is bounded we have established the following inequality

$$\begin{aligned}\int_{D^2}&\left(\frac{\partial^2}{\partial z_1\partial\bar z_1}u(\varphi(z_1,w_1),\varphi(z_2,w_2))\right)\chi(w_1)\chi(w_2)\,d\lambda(w_1)\,d\lambda(w_2)\\&\le C\int_{\operatorname{supp}\chi\times\operatorname{supp}\chi}\frac{\partial^2 g}{\partial x_1^2}(\log|\varphi(z_1,w_1)|,\log|\varphi(z_2,w_2)|)\,d\lambda(w_1)\,d\lambda(w_2).\end{aligned}$$

Calculations give

$$\frac{\partial^2 g}{\partial x_1^2}(x_1,x_2)=\begin{cases}-x_2/x_1(x_1+x_2), & \text{when } -1<x_1<0 \text{ and } -1<x_2<0,\\ 1/x_1(x_1-1), & \text{when } -1<x_1<0 \text{ and } x_2<-1,\\ 0, & \text{when } x_1<-1 \text{ and } -1<x_2<0,\\ 0, & \text{when } x_1<-1 \text{ and } x_2<-1,\end{cases}$$

and the inequality

$$\frac{\partial^2 g}{\partial x_1^2}(x_1,x_2)\le\frac{|x_2|}{|x_1|\,|x_1+x_2|}\le\frac{|x_2|}{|x_1|^2}$$

follows. Hence we have

$$\begin{aligned}\int_{D^2}&\left(\frac{\partial^2}{\partial z_1\partial\bar z_1}u(\varphi(z_1,w_1),\varphi(z_2,w_2))\right)\chi(w_1)\chi(w_2)\,d\lambda(w_1)\,d\lambda(w_2)\\&\le C\int_{\operatorname{supp}\chi\times\operatorname{supp}\chi}\frac{|\log|\varphi(z_2,w_2)||}{(\log|\varphi(z_1,w_1)|)^2}\,d\lambda(w_1)\,d\lambda(w_2)\\&=C\int_{\operatorname{supp}\chi}|\log|\varphi(z_2,w_2)||\,d\lambda(w_2)\int_{\operatorname{supp}\chi}\frac{1}{(\log|\varphi(z_1,w_1)|)^2}\,d\lambda(w_1).\end{aligned}$$

In a similar fashion we get

$$\frac{\partial^2 u_\chi}{\partial z_2 \partial \bar{z}_2}(z_1, z_2) \le C \int_{\operatorname{supp}\chi} \big|\log|\varphi(z_1, w_1)|\big|\, d\lambda(w_1) \int_{\operatorname{supp}\chi} \frac{1}{(\log|\varphi(z_2, w_2)|)^2}\, d\lambda(w_2).$$

Setting $a = \varphi(z_1, w_1)$ we get

$$\begin{aligned}\int_{\operatorname{supp}\chi} \big|\log|\varphi(z_1, w_1)|\big|\, d\lambda(w_1) &\le C \int_{\varphi(z_1, \operatorname{supp}\chi)} \big|\log|a|\big|\, d\lambda(a)\\ &\le C \int_{\beta(z_1)}^{\alpha(z_1)} -r\log r\, dr = \left[\frac{r^2}{4} - \frac{r^2\log r}{2}\right]_{\beta(z_1)}^{\alpha(z_1)} = A(z_1).\end{aligned}$$

We also get

$$\begin{aligned}\int_{\operatorname{supp}\chi} \frac{1}{(\log|\varphi(z_1, w_1)|)^2}\, d\lambda(w_1) &\le C \int_{\varphi(z_1, \operatorname{supp}\chi)} \frac{1}{(\log|a|)^2}\, d\lambda(a)\\ &\le C \int_{\beta(z_1)}^{\alpha(z_1)} \frac{r}{(\log r)^2}\, dr\\ &= \left[-\frac{1}{\log r} + 2\log|\log r| + \xi(\log r)\right]_{\beta(z_1)}^{\alpha(z_1)} = B(z_1),\end{aligned}$$

where ξ is a smooth function satisfying $\xi(0)=0$. Now this implies that

$$\frac{\partial^2 u_\chi}{\partial z_1 \partial \bar{z}_1} \frac{\partial^2 u_\chi}{\partial z_2 \partial \bar{z}_2} \le C A(z_1) B(z_1)$$

near boundary points z such that $|z_1|=1$. The terms in $A(z_1)B(z_1)$ that are troublesome are

$$(\alpha(z_1)^2 - \beta(z_1)^2)(\log|\log\alpha(z_1)| - \log|\log\beta(z_1)|)$$

and

$$(\alpha(z_1)^2 - \beta(z_1)^2)\left(\frac{1}{\log\beta(z_1)} - \frac{1}{\log\alpha(z_1)}\right).$$

As we have $\alpha(z_1)^2 - \beta(z_1)^2 = (\alpha(z_1) + \beta(z_1))(\alpha(z_1) - \beta(z_1))$, $\alpha(z_1) - \beta(z_1) = O(|z_1| - 1)$, $\log|\log\alpha(z_1)| = \log\big|\log|z_1|\big| + O(1)$ and $\log|\log\beta(z_1)| = \log\big|\log|z_1|\big| + O(1)$, as $|z_1|$ tends to 1, one sees that the first term tends to zero as $|z_1|$ tends to 1. The second term is bounded since

$$\begin{aligned}&\lim_{|z_1|\to 1} (\alpha(z_1)^2 - \beta(z_1)^2)\left(\frac{1}{\log\beta(z_1)} - \frac{1}{\log\alpha(z_1)}\right)\\ &\quad = \lim_{|z_1|\to 1} \frac{K_1(|z_1|-1)^2 + O((|z_1|-1)^3)}{(\log|z_1| + K_2(|z_1|-1))(\log|z_1| + K_3(|z_1|-1)) + O((|z_1|-1)^2)} = K_4\end{aligned}$$

for some constants K_1, K_2, K_3 and K_4. Of course similar estimates with z_1 replaced by z_2 holds. Thus

$$\lim_{z\to z_0}\det\left(\frac{\partial^2 u_\chi}{\partial z_j\partial\bar z_k}\right)$$

is bounded when $z_0\in\partial D^2$.

Let us now examine the boundary behavior of the first derivatives of u_χ. We have

$$\begin{aligned}|u_\chi(z+\zeta)-u_\chi(z)| &= \int_{D^2}|u(\Phi(z+\zeta,w))-u(\Phi(z,w))|\chi(w_1)\chi(w_2)\,d\lambda(w_1)\,d\lambda(w_2)\\ &= \int_{D^2}|u'(\Phi(z,w))\Phi'(z,w)\zeta|\chi(w_1)\chi(w_2)\,d\lambda(w_1)\,d\lambda(w_2)+o(|\zeta|)\\ &\geq C\inf_{|w|\leq 1/2}(\|u'(\Phi(z,w))\|\,\|\Phi'(z,w)\|)|\zeta|+o(|\zeta|).\end{aligned}$$

Since we have $\lim_{z\to z_0}\inf_{|w|\leq 1/2}(\|u'(\Phi(z,w))\|\,\|\Phi'(z,w)\|)=\infty$, if $z_0\in\partial D^2$, we see that $\lim_{z\to z_0}\|u'_\chi(z)\|=\infty$.

Now let us compare u_χ with a smooth plurisubharmonic function $\varrho\colon D^2\to\mathbf{R}$ satisfying $\lim_{z\to z_0\in\partial D^2}\varrho(z)=0$ and

$$\det\left(\frac{\partial^2\varrho}{\partial z_j\partial\bar z_k}\right)\geq 1.$$

After multiplication with a constant if necessary we can assume that

$$\det\left(\frac{\partial^2 u_\chi}{\partial z_j\partial\bar z_k}\right)\leq 1.$$

Now Lemma 2.2 implies that $\varrho\leq u_\chi$ and we get

$$\lim_{z\to z_0}\|\varrho'(z)\|\geq\lim_{z\to z_0}\|u'_\chi(z)\|=\infty$$

and therefore the bidisk does not satisfy the NP-condition.

Observe that it is possible to use the fact that the bidisk does not satisfy the NP-condition to show that any hyperconvex domain $\Omega\subseteq\mathbf{C}^2$ such that $D^2\subseteq\Omega$ and $\partial\Omega\cap\partial D^2$ contains a nonempty relatively open set does not satisfy the NP-condition either. Assume that Ω satisfies the NP-condition. Then there is an exhaustion function u for Ω such that

$$\begin{cases}\det\left(\dfrac{\partial^2 u}{\partial z_j\partial\bar z_k}\right)=1 & \text{in } \Omega,\\ \lim\limits_{z\to z_0}u(z)=0 & \text{for all } z_0\in\partial\Omega,\\ \sup\limits_{z\in\Omega}\left|\dfrac{\partial u}{\partial x_l}(z)\right|<\infty.\end{cases}$$

We know that for D^2 there is an exhaustion function v such that

$$\begin{cases} \det\left(\dfrac{\partial^2 v}{\partial z_j \partial \bar{z}_k}\right) = 1 & \text{in } D^2, \\ \lim\limits_{z\to z_0} v(z) = 0 & \text{for all } z_0 \in \partial D^2, \\ \sup\limits_{z\in D^2} \left|\dfrac{\partial v}{\partial x_l}(z)\right| = \infty. & \end{cases}$$

Now by the comparison principle it is clear that $u \le v$ in D^2 and hence

$$\sup_{z\in\Omega} \left|\frac{\partial u}{\partial x_l}(z)\right| = \infty$$

which contradicts the assumption that Ω satisfies the NP-condition. In fact, it is enough to assume that $D^2 \subseteq \Omega$, $\partial\Omega \cap \partial D^2 = \partial\Omega \cap T_p(D^2)$ and that $\partial\Omega \cap T_p^{\mathbf{C}}(D^2)$ is relatively open. Here $T_p^{\mathbf{C}}(D^2) = \{z \in \mathbf{C}^2; \sum_{j=1}^2 (\partial v(p)/\partial z_j) z_j = 0\}$ and $p \in \partial D^2 \setminus \{z \in \mathbf{C}^2; |z_1| = |z_2| = 1\}$.

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