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On M-structure, the asymptotic-norming property and locally uniformly rotund renormings

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Abstract. Let $r, s \in]0,1]$, and let X be a Banach space satisfying the $M(r,s)$-inequality, that is,

$$\|x^{***}\| \geq r\|\pi_X x^{***}\| + s\|x^{***} - \pi_X x^{***}\| \quad \text{for } x^{***} \in X^{***},$$

where π_X is the canonical projection from X^{***} onto X^*. We show some examples of Banach spaces not containing c_0, having the point of continuity property and satisfying the above inequality for r not necessarily equal to one. On the other hand, we prove that a Banach space X satisfying the above inequality for $s=1$ admits an equivalent locally uniformly rotund norm whose dual norm is also locally uniformly rotund. If, in addition, X satisfies

$$\limsup_\alpha \|u^* + sx^*_\alpha\| \leq \limsup_\alpha \|v^* + x^*_\alpha\|$$

whenever $u^*, v^* \in X^*$ with $\|u^*\| \leq \|v^*\|$ and (x^*_α) is a bounded weak* null net in X^*, then X can be renormed to satisfy the $M(r,1)$ and the $M(1,s)$-inequality such that X^* has the weak* asymptotic-norming property I with respect to B_X.

1. Introduction

Following [4] and [10], a subspace X of a Banach space Y is said to be an *ideal* in Y if there exists a norm-one projection P on Y^* with $\operatorname{Ker} P = X^\perp$. If, moreover,

$$\|y^*\| \geq r\|Py^*\| + s\|y^* - Py^*\| \quad \text{for } y^* \in Y^*$$

holds for given $r, s \in]0,1]$, then we say that X is an *ideal satisfying the $M(r,s)$-inequality in Y*. When Y is the bidual of a Banach space X and the associated projection is the canonical projection, we will say, for simplicity, that X *satisfies the $M(r,s)$-inequality* (see [1]). For $r=s=1$, we obtain the classical notion of M-ideal (see [12]).

A Banach space X has the *point of continuity property* if for every nonempty closed and bounded subset of X, the identity map has some point of weak-to-norm continuity. We prove that if X is a nonreflexive Banach space satisfying the $M(1,s)$-inequality, then X fails to have the point of continuity property, but there are Banach spaces having the point of continuity property, and satisfying the $M(r,s)$-inequality for r not necessarily equal to one, as can be seen below.

In a Banach space X, we denote the closed unit ball by B_X and the unit sphere by S_X. The canonical projection is denoted by π_X. The Banach space of all bounded linear operators on X and its subspace of compact operators will be denoted by $\mathcal{L}(X)$ and $\mathcal{K}(X)$, respectively.

A Banach space X is a *U^*-space* or *has property U^** if $\|x^{***}-\pi_X x^{***}\|<\|x^{***}\|$ whenever $x^{***}\in X^{***}$ with $\pi_X x^{***}\neq 0$ (see [1] and [2]). Observe that the $M(r,1)$-inequality implies property U^*.

Following [12], given $r,s\in]0,1]$, we will say that X has *property $M^*(r,s)$* if whenever $u^*,v^*\in X^*$ with $\|u^*\|\leq\|v^*\|$ and (x^*_α) is a bounded weak* null net in X^*, then

$$\limsup_\alpha \|ru^*+sx^*_\alpha\|\leq \limsup_\alpha \|v^*+x^*_\alpha\|.$$

It is clear that property $M^*(1,1)$ is equivalent to property (M^*) (see [12]).

In [13], for the class of dual spaces, Zhibao Hu and Bor-Luh Lin introduced a property stronger than the asymptotic-norming property defined by James and Ho in [15], where it is proved that the asymptotic-norming property implies the Radon–Nikodým property. Let X be a Banach space and $\Phi\subset B_{X^*}$ a *one-norming set* of X, that is,

$$\|x\| = \sup_{x^*\in\Phi} x^*x \quad \text{for } x\in X.$$

A sequence $(x_n)_{n=1}^\infty$ in S_X is said to be *asymptotically normed* by Φ if for any $\varepsilon>0$, there are $x^*\in\Phi$ and $N\in\mathbf{N}$ such that $x^*x_n>1-\varepsilon$ for all $n\geq N$. For $\varkappa$ = I, II, III, a sequence $(x_n)_{n=1}^\infty$ in X is said to have the *property $\varkappa$* if

(I) $(x_n)_{n=1}^\infty$ is convergent;
(II) $(x_n)_{n=1}^\infty$ has a convergent subsequence;
(III) $\bigcap_{n=1}^\infty \overline{\text{co}}\,\{x_k : k\geq n\}\neq\emptyset$.

Let $\Phi \subset B_X$ be a one-norming set of X^*. Then X^* is said to have the *weak* asymptotic-norming property* $\varkappa$, $\varkappa$ = *I, II, III, with respect to* Φ (in short, Φ-ANP-$\varkappa$) if every sequence in S_{X^*} that is asymptotically normed by Φ has the property $\varkappa$. It is showed in [13] that if Φ is a one-norming set in B_X, then X^* has the Φ-ANP-$\varkappa$ if and only if X^* has the B_X-ANP-$\varkappa$, $\varkappa$ = I, II, III. The space X^* is said to have the *weak* asymptotic-norming property* $\varkappa$ (*weak*-ANP-*$\varkappa$) if X admits an equivalent norm for which X^* has the B_X-ANP-$\varkappa$.

Given a Banach space X, its norm is called *locally uniformly convex* or *locally uniformly rotund* (*LUR*) if for any $x \in S_X$ and $(x_n)_{n=1}^{\infty}$ in S_X with $\lim_{n\to\infty} \|x_n + x\| = 2$, then $\lim_{n\to\infty} \|x_n - x\| = 0$. The norm of X is said to be *strictly convex* or *rotund* if $x = y$ whenever $x, y \in S_X$ and $\|x + y\| = 2$.

M. Raja has showed in [19] that given a Banach space X, its dual X^* admits an equivalent dual LUR norm if and only if X can be renormed such that the weak and the weak* topologies agree on S_{X^*}. So, the weak*-ANP-$\varkappa$, $\varkappa$ = I, II, III, are equivalent (see [13]).

Given a Banach space X, its dual X^* has the *weak*-Kadec–Klee property* if for any sequence $(x_n^*)_{n=1}^{\infty}$ in X^* and $x^* \in X^*$, $\lim_{n\to\infty} \|x_n^* - x^*\| = 0$ whenever weak*-$\lim_{n\to\infty} x_n^* = x^*$ and $\|x_n^*\| = \|x^*\|$ for all $n \in \mathbf{N}$.

It is clear that if X is an M-ideal, then X is Hahn–Banach smooth, that is, $\|\pi_X x^{***}\| < \|x^{***}\|$ whenever $x^{***} \in X^{***}$ with $x^{***} \neq \pi_X x^{***}$. It is known [12] that if a Banach space X has property (M^*), then the weak* and the norm topologies agree on S_{X^*}. Using a different argument, the same result is obtained when $\mathcal{K}(X)$ is an M-ideal in $\mathcal{L}(X)$. We obtain the same results (see Theorem 3.2 below) supposing that X satisfies the $M(r,s)$-inequality for r and s with $(1-r)/s$ arbitrarily close to zero, $\mathcal{K}(X)$ is an ideal satisfying the $M(1,s)$-inequality and X has property $M^*(1,s)$ (s not necessarily equal to one). In particular, by the mentioned result of Raja, X^* admits an equivalent dual LUR norm (see Corollary 4.2).

In [12] it is proved that every M-ideal admits an equivalent LUR norm whose dual norm is LUR. It is also proved that if X is an M-ideal, then X admits an equivalent norm for which X is still an M-ideal and whose dual norm is strictly convex. In Theorem 4.5 below, we prove that if X is a nonreflexive Asplund space having property U^*, then there exists an equivalent LUR norm on X whose dual norm is LUR. On the other hand, it is proved in the same result that X admits an equivalent norm for which X has still property U^* and whose dual norm is strictly convex. If, moreover, X has property $M^*(1,s)$ (in particular, if $\mathcal{K}(X)$ is an ideal satisfying the $M(1,s)$-inequality in $\mathcal{L}(X)$), then X can be renormed to satisfy the $M(1,s)$-inequality and such that X^* has the B_X-ANP-I (see Theorem 4.7). As a consequence, we obtain that a Banach space X having property (M^*) admits an equivalent norm for which it is an M-ideal and X^* has the B_X-ANP-I.

2. The $M(r,s)$-inequality and the point of continuity property

First, we have the following result.

Proposition 2.1. ([1, Corollary 3.4]) *Let X be a nonreflexive Banach space satisfying the $M(1,s)$-inequality (in particular if X is an M-ideal). Then X contains an isomorphic copy of c_0, so X fails to have the point of continuity property.*

Now we present some Banach spaces which do not contain c_0, but satisfy the $M(r,s)$-inequality. Let J denote the James space, JT the James tree space (see, e.g., [7] and [8] for the definitions) and JT_∞ the space constructed by Ghoussoub and Maurey in [9]. Note that the space J has the Radon–Nikodým property (and so has the point of continuity property), since J is isometric to a separable dual space. It is well known [9], [17] that the predual B of the space JT has the point of continuity property but fails to have the Radon–Nikodým property. On the other hand, it is proved in [9] that the predual B_∞ fails to have the point of continuity property.

Examples 2.2. (1) ([1, Example 3.5]). *The space J can be renormed to satisfy the $M(r,1)$-inequality.*
(2) *The spaces B and B_∞ satisfy the $M\big(1/\sqrt{2}, 1/\sqrt{2}\big)$-inequality.*

Proof. Let $(e_{n,k})_{(n,k)\in T}$ be the usual basis in JT. A careful reading of [8, Chapter VIII] allows us to assert that there exists a sequence $(K_n)_{n=1}^\infty$ of finite rank operators on B with $\|K_n\|\le 1$ such that $\lim_{n\to\infty} K_n x = x$ and $\lim_{n\to\infty} K_n^* x^* = x^*$ for every $x\in B$, $x^*\in JT$, and writing $x^* = \sum_{n'=0}^\infty \sum_{k'=1}^{2^{n'}} t_{n',k'} e_{n',k'}$ for $x^*\in JT$, we have, for all n,

$$K_n^* x^* = \sum_{n'=1}^{n} \sum_{k'=1}^{2^{n'}} t_{n',k'} e_{n',k'}.$$

It is easy to show that, for all n and $x^*\in JT$,

$$\|K_n^* x^*\|^2 + \|x^* - K_n^* x^*\|^2 \le \|x^*\|^2.$$

Hence, for every n and $x,y\in B$ with $\|x\|, \|y\|\le 1$,

$$\|K_n x + (y - K_n y)\| \le \sqrt{2}.$$

So, by [3, Propositions 3.1 and 4.1], the result follows. The proof for the space B_∞ is similar. □

3. M-structure and the asymptotic-norming property

First, we introduce the following notation: given a Banach space X, for all $\varepsilon>0$, define

$$A_\varepsilon=\left\{(r,s):\frac{1-r}{s}<\varepsilon\right\}\quad\text{and}\quad A^\varepsilon=\left\{(r,s):\frac{1-s}{r}<\varepsilon\right\},$$

and consider the sets

$B=\{(r,s):X$ satisfies the $M(r,s)$-inequality$\}$;

$C=\{(r,s):X$ has property $M^*(r,s)\}$;

$D=\{(r,s):\mathcal{K}(X)$ is an ideal satisfying the $M(r,s)$-inequality in $\mathcal{L}(X)\}$.

The next result is due to Zhibao Hu and Bor-Luh Lin (see [13, Theorems 2.5, 3.1 and 3.3] and [14, Theorem 1]).

Theorem 3.1. *Let X be a Banach space. Then the following are true:*

(1) *X^* has the B_X-ANP-I if and only if X^* has the B_X-ANP-II and the norm of X^* is strictly convex;*

(2) *X^* has the B_X-ANP-II if and only if the weak* and the norm topologies agree on S_{X^*} if and only if B_{X^*} is weak* sequentially compact and X^* has the weak*-Kadec–Klee property;*

(3) *X^* has the B_X-ANP-III if and only if X is Hahn–Banach smooth.*

Theorem 3.2. *Let X be a Banach space. Then the following are true:*

(1) *if $B\cap A_\varepsilon\neq\emptyset$ for all $\varepsilon>0$ (in particular if X satisfies the $M(1,s)$-inequality), then X^* has the B_X-ANP-III;*

(2) *if X has property $M^*(1,s)$, then X^* has the B_X-ANP-II;*

(3) *if $\mathcal{K}(X)$ is an ideal satisfying the $M(1,s)$-inequality in $\mathcal{L}(X)$, then X^* has the B_X-ANP-II.*

Proof. (1) By [2, Lemma 4.2], X is Hahn–Banach smooth. So, by Theorem 3.1, X^* has the B_X-ANP-III.

(2) Let (x^*_α) in S_{X^*} be weak* converging to $x^*\in S_{X^*}$, and pick a weak* strongly exposed point x^*_0 of B_{X^*} (its existence is guaranteed by [4, Proposition 2.1]). So, by property $M^*(1,s)$, we have that

$$\limsup_\alpha \|x^*_0+s(x^*_\alpha-x^*)\|\le\limsup_\alpha\|x^*+(x^*_\alpha-x^*)\|=1.$$

Then, it follows that (x^*_α) converges to x^* in the norm topology. Hence, applying Theorem 3.1, X^* has the B_X-ANP-II.

(3) The proof of this part is similar to the one given in [12, Proposition VI.4.6]. We state it here for completeness.

Let (x^*_α) in S_{X^*} be weak* converging to $x^* \in S_{X^*}$. By [4, Theorem 2.5] and [1, Proposition 2.5], X^* has the Radon–Nikodým property. So, for fixed $\varepsilon > 0$, by [18, Lemma 2.18], there are $x_0 \in S_X$ and $t > 0$ such that the slice $S(x_0, B_{X^*}, t)$ has diameter less than ε. Let $x \in S_X$ with $x^*(x) > 1 - \frac{1}{4}t$. We can suppose, without loss of generality, that $x^*_\alpha(x) > 1 - \frac{1}{4}t$ for all α. Pick $y^* \in S_{X^*}$ such that $y^*(x_0) = 1$, so $y^* \in S(x_0, B_{X^*}, t)$. By [4, Lemma 2.2], there exists $U \in \mathcal{K}(X)$ such that

$$\|y^* \otimes x \pm s(I - U)\| \le 1 + \tfrac{1}{4}t,$$

where I is the identity operator of X. Then, as in [12, Proposition VI.4.6], for every α, we have that

$$\frac{x^*_\alpha(x) y^* \pm s(x^*_\alpha - U^* x^*_\alpha)}{1 + \frac{1}{4}t} \in S(x_0, B_{X^*}, t).$$

Finally, since $\operatorname{diam} S(x_0, B_{X^*}, t) < \varepsilon$ and U^* is weak*-to-norm continuous on B_{X^*}, it follows that, for every α, $s\|x^*_\alpha - x^*\| \le 3\varepsilon$. So, $\|x^*_\alpha - x^*\| \to 0$, and we apply Theorem 3.1. $\square$

The next result shows that condition $r = 1$ cannot be dropped to obtain Hahn–Banach smoothness.

Proposition 3.3. *The space J admits an equivalent norm for which it still satisfies the $M(r, 1)$-inequality, but fails to be Hahn–Banach smooth.*

Proof. Let $r \in]0, 1[$ be such that J satisfies the $M(r, 1)$-inequality, and let $\tilde{J} = \mathbf{R} \times J$ be the equivalent renorming of J with the norm

$$\|(\alpha, x)\| = \max\{|\alpha| + (1 - r)\|x\|, \|x\|\}, \quad \alpha \in \mathbf{R}, \ x \in J,$$

where $\|\cdot\|$ denotes the norm on J for which J satisfies the $M(r, 1)$-inequality. We prove that $\tilde{J}$ satisfies the $M(r, 1)$-inequality and fails to be Hahn–Banach smooth. In fact, it is easy to show that $\tilde{J}^{***} = \mathbf{R} \times J^{***}$ with the norm

$$\|(\alpha, \varphi)\| = \max\{r|\alpha| + \|\varphi\|, |\alpha|\}, \quad \alpha \in \mathbf{R}, \ \varphi \in J^{***}.$$

Then, for every $(\alpha, \varphi) \in \tilde{J}^{***}$, we have

$$\begin{aligned} r\|(\alpha, \pi_J(\varphi))\| + \|(0, \varphi - \pi_J(\varphi))\| &= r \max\{r|\alpha| + \|\pi_J(\varphi)\|, \|\pi_J(\varphi)\|\} + \|\varphi - \pi_J(\varphi)\| \\ &\le \max\{r|\alpha| + \|\varphi\|, \|\varphi\|\} = \|(\alpha, \varphi)\|. \end{aligned}$$

Finally, it is straightforward to show that, for every $(\alpha, \varphi) \in \tilde{J}^{***}$,

$$P_{\tilde{J}^\perp}(\alpha, \varphi) = \{0\} \times [B_{J^\perp}(\varphi, \max\{\operatorname{dist}(\varphi, J^\perp), (1 - r)|\alpha|\})].$$

So, it follows that $\tilde{J}$ fails to be Hahn–Banach smooth. $\square$

Remark 3.4. Observe that, since the space J does not contain a copy of c_0, J cannot be renormed to satisfy the $M(1, s)$-inequality for any $s \in]0, 1]$.

4. M-structure and LUR renormings

It is well known [12, Theorem III.4.6] that every M-ideal admits an equivalent LUR norm whose dual norm is also LUR. In the mentioned result, it is proved that there exists an equivalent norm on every M-ideal for which its dual norm is strictly convex and it is still an M-ideal. The next result is essentially proved in [12, Theorem VI.4.17].

Lemma 4.1. *Let X be a Banach space and $r,s\in]0,1]$ with $r+\frac{1}{2}s>1$. If $\mathcal{K}(X)$ is an ideal satisfying the $M(r,s)$-inequality in $\mathcal{L}(X)$, then X has property $M^*(r,s)$.*

Proof. By [4, Theorem 3.1], there is a net (K_α) in $B_{\mathcal{K}(X)}$ with $\lim_\alpha \|K_\alpha x-x\|=0$ for all $x\in X$ and $\lim_\alpha \|K_\alpha^* x^*-x^*\|=0$ for all $x^*\in X^*$, such that

$$\limsup_\alpha \|rSK_\alpha+sT(I-K_\alpha)\|\leq 1 \quad \text{for } S,T\in B_{\mathcal{L}(X)}.$$

So,

$$\limsup_\alpha \|rS+s(I-K_\alpha)\|\leq 1 \quad \text{for } S\in B_{\mathcal{K}(X)}.$$

Now the proof follows as the one given in [3, Proposition 4.1]. □

Since a Banach space X is Hahn–Banach smooth if and only if the weak and the weak* topologies agree on S_{X^*} (see [12, Lemma III.2.14], by Theorems 3.1 and 3.2, the above lemma, [3, Proposition 3.1] and [19, Proposition 2] we have the following result.

Corollary 4.2. *Let X be a Banach space, and consider the following statements*:

(a) *$B\cap A_\varepsilon\neq\emptyset$ for all $\varepsilon>0$ (in particular, X satisfies the $M(1,s)$-inequality)*;

(b) *$C\cap A_\varepsilon\neq\emptyset$ for all $\varepsilon>0$ (in particular, X has property $M^*(1,s)$)*;

(c) *$D\cap A_\varepsilon\neq\emptyset$ for all $\varepsilon>0$ (in particular, $\mathcal{K}(X)$ is an ideal satisfying the $M(1,s)$-inequality in $\mathcal{L}(X)$).*

If X satisfies one of the above statements, then X^* admits an equivalent dual locally uniformly rotund norm.

In what follows, we will suppose that X is a nonreflexive Banach space. We will use a different technique to obtain that if X is an Asplund space having property U^*, then X admits an equivalent LUR norm whose dual norm is also LUR. We begin with the next result, which is proved in [12, Proposition III.2.11] for M-ideals.

Proposition 4.3. *Let X be a Banach space satisfying the $M(r,s)$-inequality (respectively having property U^*), Y be a Banach space and $T:Y\to X$ be a weakly compact operator. Then*

$$|x^*|^* = \|x^*\| + \|T^*x^*\| \quad \text{for } x^* \in X^*$$

defines an equivalent dual norm on X^* for which $(X, |\cdot|)$ satisfies the $M(r,s)$-inequality (respectively has property U^*).

Proof. Following [12, Proposition III.2.11], we obtain that, for every $x^{***}\in X^{***}$,

$$|x^{***}|^{***} = \|x^{***}\| + \|T^{***}x^{***}\| \quad \text{and} \quad T^{***}x^{***} = T^*(\pi_X x^{***}).$$

So, if X satisfies the $M(r,s)$-inequality, then

$$\begin{aligned} |x^{***}|^{***} &= \|x^{***}\| + \|T^*(\pi_X x^{***})\| \\ &\geq r\|\pi_X x^{***}\| + s\|x^{***} - \pi_X x^{***}\| + r\|T^*(\pi_X x^{***})\| \\ &= r|\pi_X x^{***}|^{***} + s|x^{***} - \pi_X x^{***}|^{***}. \end{aligned}$$

Now suppose that X has property U^*, and let $x^{***}\in X^{***}$ with $\pi_X x^{***}\neq 0$. Then, we have

$$\begin{aligned} |x^{***}|^{***} &= \|x^{***}\| + \|T^*(\pi_X x^{***})\| \\ &> \|x^{***} - \pi_X x^{***}\| + \|T^*(\pi_X x^{***})\| \geq |x^{***} - \pi_X x^{***}|^{***}. \quad \square \end{aligned}$$

Lemma 4.4. *Let X be an Asplund space having property U^*. Then X admits a shrinking Markushevich basis.*

Proof. By [1, Theorem 4.4], X is weakly compactly generated, so [5, p. 237] X is weakly countably determined. Then, by [5, Proposition II.1.5 and Corollary VII.1.13] and [16, Lemma 4], X admits a shrinking Markushevich basis. □

Theorem 4.5. *If X is an Asplund space having property U^*, then*

(1) *X admits an equivalent LUR norm whose dual norm is also LUR;*

(2) *there exists an equivalent norm on X for which X still has property U^* and whose dual norm is strictly convex.*

Proof. (1) This follows from [1, Theorem 4.4] and [5, Theorem VII.1.14].

(2) Let $(x_i, f_i)_{i\in\Gamma}$ be a shrinking Markushevich basis obtained in the above lemma and let us consider the operator $S:X^*\to c_0(\Gamma)$ defined by

$$Sx^* = (x^*(x_i))_{i\in\Gamma} \quad \text{for } x^* \in X^*.$$

It is clear that S is injective and weak*-to-weak continuous. Let the norm $|\cdot|$ be defined on X^* by

$$|x^*| = \|x^*\| + \|Sx^*\| \quad \text{for } x^* \in X^*.$$

If we consider Day's norm on $c_0(\Gamma)$, which is LUR and thus strictly convex (see [6, pp. 95ff.]), then the norm $|\cdot|$ is a dual strictly convex norm on X^* (see [6, Theorem 1, p. 100] and [5, Theorem II.2.4]). On the other hand, the operator S is weakly compact and, if it is considered as an operator into $l_\infty(\Gamma)$, it is weak*-to-weak* continuous. Hence, S is the adjoint of a weakly compact operator from $l_1(\Gamma)$ into X. So, by the above proposition, $(X, |\cdot|)$ has property U^*. □

Corollary 4.6. *Let X be a Banach space, and consider the following statements:*

(a) *$B \cap A^\varepsilon \neq \emptyset$ for all $\varepsilon > 0$ (in particular, X satisfies the $M(r,1)$-inequality);*

(b) *$C \cap A^\varepsilon \neq \emptyset$ for all $\varepsilon > 0$ (in particular, X has property $M^*(r,1)$);*

(c) *$r > \frac{1}{2}$ and $\mathcal{K}(X)$ is an ideal satisfying the $M(r,1)$-inequality in $\mathcal{L}(X)$.*

If X satisfies one of the above statements, then the following are true:

(1) *X admits an equivalent LUR norm whose dual norm is also LUR;*

(2) *there exists an equivalent norm on X whose dual norm is strictly convex, and for which X satisfies the $M(r,s)$-inequality for all $(r,s) \in B$.*

Proof. By [3, Proposition 3.1], (b) $\Rightarrow$ (a), and, by Lemma 4.1, (c) $\Rightarrow$ (b). So, we suppose only the statement (a). By [2, Lemma 4.2], X has property U^*. On the other hand, there exists $(r,s) \in B$ such that $r+s>1$. So, by [1, Proposition 2.4], X is an Asplund space. Hence, applying Theorem 4.5 and Proposition 4.3, the proof is concluded. □

Theorem 4.7. *Let X be a Banach space having property U^*. If X has property $M^*(1,s)$ (in particular, $\mathcal{K}(X)$ is an ideal satisfying the $M(1,s)$-inequality in $\mathcal{L}(X)$), then X admits an equivalent norm $|\cdot|$ for which X has property U^* and X^* has the B_X-ANP-I. Moreover, $(X, |\cdot|)$ satisfies the $M(r,s)$-inequality for all $(r,s) \in B$.*

Proof. First, observe that X is an Asplund space (see Lemma 4.1, [3, Proposition 3.1] and [1, Proposition 2.5]). Consider the dual norm $|\cdot|$ defined in Theorem 4.5(2), which is strictly convex. So, by [11, Theorem 1], $B_{(X^*, |\cdot|)}$ is weak* sequentially compact. Now we prove that $(X^*, |\cdot|)$ has the weak*–Kadec–Klee property. In fact, let $(x_n^*)_{n=1}^\infty$ be a sequence in $S_{(X^*,|\cdot|)}$ and $x^* \in S_{(X^*,|\cdot|)}$ with weak*-$\lim_{n\to\infty} x_n^* = x^*$. By passing to a subsequence, we can suppose that $\|Sx^*\| \leq \lim_{n\to\infty} \|Sx_n^*\|$. Since $|x_n^*| = |x^*| = 1$ for all n, we have that

$$\lim_{n\to\infty} \|x_n^*\| = 1 - \lim_{n\to\infty} \|Sx_n^*\| \leq 1 - \|Sx^*\| = \|x^*\|.$$

Therefore, $\lim_{n\to\infty} \|x_n^*\| = \|x^*\|$. By Theorem 3.2, X^* has the B_X-ANP-II. Hence, by Theorem 3.1, we have that $\lim_{n\to\infty} \|x_n^* - x^*\| = 0$, so $\lim_{n\to\infty} |x_n^* - x^*| = 0$. Finally, again by Theorem 3.1, $(X^*, |\cdot|)$ has the B_X-ANP-I. □

Corollary 4.8. *If X has property (M^*) (in particular, if $\mathcal{K}(X)$ is an M-ideal in $\mathcal{L}(X)$), then X admits an equivalent norm for which X is an M-ideal and X^* has the B_X-ANP-I.*

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References

1. Cabello, J. C. and Nieto, E., On properties of M-ideals, *Rocky Mountain J. Math.* **28** (1998), 61–93.
2. Cabello, J. C. and Nieto, E., An ideal characterization of when a subspace of certain Banach spaces has the metric compact approximation property, *Studia Math.* **129** (1998), 185–196.
3. Cabello, J. C. and Nieto, E., On M-type structures and the fixed point property, *Houston J. Math.* **26** (2000), 549–560.
4. Cabello, J. C., Nieto, E. and Oja, E., On ideals of compact operators satisfying the $M(r,s)$-inequality, *J. Math. Anal. Appl.* **220** (1998), 334–348.
5. Deville, R., Godefroy, G. and Zizler, V., *Smoothness and Renormings in Banach Spaces*, Longman, Harlow, 1993.
6. Diestel, J., *Geometry of Banach Spaces—Selected Topics,* Lecture Notes in Math. **485**, Springer-Verlag, Berlin–Heidelberg, 1975.
7. van Dulst, D., *Reflexive and Superreflexive Banach Spaces,* Math. Centre Tracts **102**, Mathematisch Centrum, Amsterdam, 1978.
8. van Dulst, D., *Characterizations of Banach Spaces not Containing l^1,* CWI Tract **59**, Centrum voor Wiskunde en Informatica, Amsterdam, 1989.
9. Ghoussoub, N. and Maurey, B., G_δ-embeddings in Hilbert spaces I, *J. Funct. Anal.* **61** (1985), 72–97.
10. Godefroy, G., Kalton, N. J. and Saphar, P. D., Unconditional ideals in Banach spaces, *Studia Math.* **104** (1993), 13–59.
11. Hagler, J. and Sullivan, F., Smoothness and weak* sequential compactness, *Proc. Amer. Math. Soc.* **78** (1980), 497–503.
12. Harmand, P., Werner, D. and Werner, W., *M-ideals in Banach Spaces and Banach Algebras,* Lecture Notes in Math. **1547**, Springer-Verlag, Berlin–Heidelberg, 1993.
13. Hu, Z. and Lin, B.-L., On the asymptotic-norming property of Banach spaces, in *Function Spaces (Edwardsville, Ill., 1990)* (Jarosz, K., ed.), Lecture Notes in Pure and Appl. Math. **136**, pp. 281–294, Marcel Dekker, New York, 1992.

14. Hu, Z. and Lin, B.-L., Smoothness and the asymptotic-norming properties of Banach spaces, *Bull. Austral. Math. Soc.* **45** (1992), 285–296.
15. James, R. C. and Ho, A., The asymptotic-norming and the Radon–Nikodým properties for Banach spaces, *Ark. Mat.* **19** (1981), 53–70.
16. John, K. and Zizler, V., Smoothness and its equivalents in weakly compactly generated Banach spaces, *J. Funct. Anal.* **15** (1974), 1–11.
17. Lindenstrauss, L. and Stegall, C., Examples of separable Banach spaces which do not contain l_1 and whose duals are non-separable, *Studia Math.* **54** (1975), 81–105.
18. Phelps, R. R., *Convex Functions, Monotone Operators and Differentiability,* Lecture Notes in Math. **1364**, Springer-Verlag, Berlin–Heidelberg, 1989.
19. Raja, M., Locally uniformly rotund norms, *Mathematika* **46** (1999), 343–358.

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