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Decomposition theorems for Q_p spaces

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Abstract. We study the Möbius invariant spaces Q_p and $Q_{p,0}$ of analytic functions. These scales of spaces include BMOA$=Q_1$, VMOA$=Q_{1,0}$ and the Dirichlet space$=Q_0$. Using the Bergman metric, we establish decomposition theorems for these spaces. We obtain also a fractional derivative characterization for both Q_p and $Q_{p,0}$.

1. Introduction

Let D be the unit disc in the complex plane, $dA(z)$ be the area measure and $g(z,w)=\log(|1-\bar{w}z|/|z-w|)$ be the Green's function of D with pole at $w\in D$. For $p\in(-1,\infty)$, the spaces Q_p and $Q_{p,0}$ are Möbius invariant function spaces consisting of all analytic functions f defined on D satisfying,

$$\|f\|_{Q_p}=\sup_{w\in D}\left(\int_D |f'(z)|^2 g(z,w)^p\,dA(z)\right)^{1/2}<\infty$$

and

$$\lim_{|w|\to 1_-}\int_D |f'(z)|^2 g(z,w)^p\,dA(z)=0,$$

respectively

The spaces Q_p and $Q_{p,0}$ have been much studied in recent years. We refer the reader to [A], [AL], [AXZ], [L], [NX], [X] and the references therein. It is proved in [AL] that Q_p is the Bloch space and $Q_{p,0}$ is the little Bloch space if $p\in(1,\infty)$. It is also proved in [NX] that for $-1<p<0$, the Q_p space contains only constant functions, Q_0 is the Dirichlet space, $Q_1=$BMOA and $Q_{1,0}=$VMOA. Therefore $0<p<1$ is the interesting range of p for the scales of spaces Q_p and $Q_{p,0}$.

Decompositions can be found in many sources in the literature. It is a useful tool in studying functions and function spaces. The related results for functions in holomorphic spaces are used in studying operators, such as Hankel and Toeplitz, and approximation by rational functions (see for example [P] and [Z]). Decomposition theories for Bloch space, little Bloch space, Dirichlet space, BMOA and VMOA are established in [R] and [RS]. We also refer the reader to [P], [Z], [RW] and the references therein. The main purpose of this paper is to establish similar decomposition theorems for Q_p and $Q_{p,0}$ when $0<p<1$.

Let μ be a nonnegative measure on D. For any arc I on ∂D, denote the Carleson square based on I by $S(I)=\{z\in D:1-|I|<|z|<1 \text{ and } z/|z|\in I\}$. Here and later $|I|$ denotes the length—the normalized arc length—of I. We say that μ is a bounded p-Carleson measure if $\mu(S(I))\le C|I|^p$ for any arc I on ∂D, and μ is a compact p-Carleson measure if $\mu(S(I))=o(|I|^p)$. The square root of the best constant C in the above inequality is denoted by $\|\mu\|_p$. Clearly, 1-Carleson measures are the classical Carleson measures.

Denote the unit point mass supported on z by δ_z. Delaying the definition of the lattice to the next section, we state the main results of this paper.

Theorem 1. *Suppose that $0<p\le 1$. There exists an $\eta_0>0$, such that for any η-lattice $\{z_j\}_{j=0}^\infty$ in D with $0<\eta<\eta_0$, the following are true:*

(a) *If $f\in Q_p$, then*

$$f(z)=\sum_{j=0}^{\infty}\lambda_j\frac{(1-|z_j|^2)^{b-p/2}}{(1-\bar z_j z)^b},\quad b>\frac{1+p}{2}, \tag{1}$$

and

$$\left\|\sum_{j=0}^{\infty}|\lambda_j|^2\delta_{z_j}\right\|_p\le C\|f\|_{Q_p}. \tag{2}$$

(b) *If $\{\lambda_j\}_{j=0}^\infty$ is such that the measure $\sum_{j=0}^\infty|\lambda_j|^2\delta_{z_j}$ is a bounded p-Carleson measure, then f, defined by (1), is in Q_p and*

$$\|f\|_{Q_p}\le C\left\|\sum_{j=0}^{\infty}|\lambda_j|^2\delta_{z_j}\right\|_p.$$

Theorem 2. *Suppose that $0<p\le 1$. There exists an $\eta_0>0$, such that for any η-lattice $\{z_j\}_{j=0}^\infty$ in D with $0<\eta<\eta_0$, the following are true:*

(a) *If $f\in Q_{p,0}$, then*

$$f(z)=\sum_{j=0}^{\infty}\lambda_j\frac{(1-|z_j|^2)^{b-p/2}}{(1-\bar z_j z)^b},\quad b>\frac{1+p}{2}, \tag{3}$$

and the measure $\sum_{j=0}^\infty|\lambda_j|^2\delta_{z_j}$ is a compact p-Carleson measure.

(b) *If $\{\lambda_j\}_{j=0}^\infty$ is such that the measure $\sum_{j=0}^\infty|\lambda_j|^2\delta_{z_j}$ is a compact p-Carleson measure, then f, defined by (3), is in $Q_{p,0}$.*

We end this section with the following remarks. For $p=1$, Theorems 1 and 2 have been proved in [RS] and [RW]. For the real version of Q_p space, a decomposition theorem with wavelet base has been established recently in [EJPX]. Similar results for generalized Q_p and $Q_{p,0}$ spaces are gathered at the end of this paper.

2. Preliminaries

The following characterizations of Q_p and $Q_{p,0}$ in terms of bounded p-Carleson measures and compact p-Carleson measures can be found in [ASX]. For the case $p=1$, these results are well known (see, for example, [G, Chapter VI]).

Theorem A. *Suppose that $0<p<1$ and that f is analytic on D.*

(i) *The function f is in Q_p if and only if the measure $|f'(z)|^2(1-|z|^2)^p dA(z)$ is a bounded p-Carleson measure.*

(ii) *The function f is in $Q_{p,0}$ if and only if the measure $|f'(z)|^2(1-|z|^2)^p dA(z)$ is a compact p-Carleson measure.*

Corollary B. *Suppose $0\le p_1<p_2\le 1$. Then*

$$Q_{p_1}\subset Q_{p_2}.$$

In fact, Corollary B is a consequence of Theorem A and the inequality

$$\int_{S(I)}|f'(z)|^2(1-|z|^2)^{p_2}\,dA(z)\le(2|I|)^{p_2-p_1}\int_{S(I)}|f'(z)|^2(1-|z|^2)^{p_1}\,dA(z)$$

for $I\subset\partial D$, which is obtained by using the estimate $1-|z|^2\le 2|I|$, if $z\in S(I)$.

For $z,w\in D$, the Bergman distance between z and w is defined by

$$d(z,w)=\log\frac{1+\left|\dfrac{w-z}{1-\bar{w}z}\right|}{1-\left|\dfrac{w-z}{1-\bar{w}z}\right|}.$$

For $z_0\in D$, we call the disc $B(z_0,r)=\{z:d(z,z_0)<r\}$ an r-Bergman disc with center z_0. It is known that the Bergman disc $B(z_0,r)$ is a Euclidean disc with radius

$$\frac{e^r-1}{e^r+1}\frac{1-|z_0|^2}{1-\left(\dfrac{e^r-1}{e^r+1}\right)^2|z_0|^2}.$$

Therefore the area of the disc $B(z_0,r)$ is comparable to $r^2(1-|z_0|^2)^2$ for small r.

An η-lattice is a family of points $\{z_j\}_{j=0}^{\infty}$ in D such that $d(z_k, z_j) \geq \frac{1}{5}\eta$, if $k \neq j$, and the collection of discs $B_j = \{z : d(z, z_j) < \eta\}$ covers D.

Let $\{z_j\}_{j=0}^{\infty}$ be an η-lattice, and $\widetilde{B}_j$ and B_j be the $\frac{1}{10}\eta$-Bergman disc and the η-Bergman disc centered at z_j, respectively. It is standard that (see, for example, [CR]) there is a partition $\{D_j\}_{j=0}^{\infty}$ of D such that $z_j \in D_j$ and

$$\widetilde{B}_j \subseteq D_j \subseteq B_j.$$

In this paper, we shall always denote the $\frac{1}{2}$-Bergman disc centered at z_j by $\overline{B}_j$ for any given η-lattice $\{z_j\}_{j=0}^{\infty}$.

For fixed $b > 0$, let

$$k_w(z) = \frac{(1-|z|^2)^{b-1}}{(1-\bar{z}w)^{b+1}}.$$

The following lemmas can be found in [CR] or [R].

Lemma C. *There exists $C > 0$ such that, if $\eta = d(z, z_0) \leq 1$, then*

$$|k_w(z) - k_w(z_0)| \leq C\eta |k_w(z_0)|, \quad w \in D.$$

Lemma D. *Let $0 < \eta < \frac{1}{4}$ and $\{z_j\}_{j=0}^{\infty}$ be an η-lattice. There exists $C > 0$ such that for any analytic f on D and for all j,*

$$\int_{D_j} |f(z) - f(z_j)| \, dA(z) \leq C\eta^3 \int_{\overline{B}_j} |f(z)| \, dA(z).$$

Lemma E. *Let $0 < \eta < 1$ and $\{z_j\}_{j=0}^{\infty}$ be an η-lattice. There exists a positive integer $r = O(\eta^{-2})$ such that each point of D lies in at most r of the discs in $\{\overline{B}_j\}_{j=0}^{\infty}$. Furthermore, if $b > 0$ and f is analytic on D, then*

$$\sum_{j=0}^{\infty} \int_{\overline{B}_j} |f(z)|^2 (1-|z|^2)^{b-1} \, dA(z) \leq r \int_D |f(z)|^2 (1-|z|^2)^{b-1} \, dA(z).$$

Schur's lemma. *Suppose that $1 < q < \infty$, and that $q' = q/(q-1)$ (the conjugate number of q). Suppose further that $Q(z, w)$ is a positive function on $D \times D$. If there is a positive function g on D, such that*

$$\int_D Q(z, w) g^{q'}(w) \, dA(w) \leq C g^{q'}(z) \quad \text{and} \quad \int_D Q(w, z) g^q(w) \, dA(w) \leq C g^q(z)$$

hold for all $z\in D$, then the linear map given by

$$f\longmapsto\int_D Q(z,w)f(w)\,dA(w)$$

is a bounded map on $L^q(D)$.

Throughout this paper I denotes an arc on ∂D. The letter C denotes a positive constant which may vary at each occurrence but is independent of the essential variables and quantities. We also use the notation $A\asymp B$ to mean that A and B are comparable, i.e. $1/C\le A/B\le C$.

3. Proofs of the main results

For independent interests, we establish a key theorem first. For fixed $b>1$, define the linear operator T_σ by

$$T_\sigma\psi(w)=\int_D\frac{(1-|z|^2)^{b-1}}{|1-\bar zw|^{b+\sigma}}\psi(z)\,dA(z),\quad \sigma>0.$$

Theorem 3. *Suppose $0<p\le1$, $\sigma>\frac12(1-p)$ and ψ is a measurable function on D.*

(a) *If the measure $|\psi(z)|^2(1-|z|^2)^p\,dA(z)$ is a bounded p-Carleson measure, then $|T_\sigma\psi(z)|^2(1-|z|^2)^{2\sigma-2+p}\,dA(z)$ is also a bounded p-Carleson measure.*

(b) *If the measure $|\psi(z)|^2(1-|z|^2)^p\,dA(z)$ is a compact p-Carleson measure, then $|T_\sigma\psi(z)|^2(1-|z|^2)^{2\sigma-2+p}\,dA(z)$ is also a compact p-Carleson measure.*

Remark. For $p=1$, the result is proved in [RW]. For $\sigma=1$, the result is proved in [X].

Proof. For part (a), it is sufficient to show that the estimate

$$\int_{S(I)}|T_\sigma\psi(w)|^2(1-|w|^2)^{2\sigma-2+p}\,dA(w)\le C|I|^p$$

holds for any arc $I\subset\partial D$.

For any positive integer $n \le \log_2(1/|I|)$, let $2^n I$ be the arc on ∂D with the same center as I and the length $2^n|I|$. We have the following estimate

$$\begin{aligned}
&\int_{S(I)} |T_\sigma \psi(w)|^2 (1-|w|^2)^{2\sigma-2+p}\, dA(w) \\
&\le \int_{S(I)} \left(\int_D \frac{(1-|z|^2)^{b-1}}{|1-\bar{z}w|^{b+\sigma}} |\psi(z)|\, dA(z) \right)^2 (1-|w|^2)^{2\sigma-2+p}\, dA(w) \\
&= \int_{S(I)} \left(\left(\int_{S(2I)} + \int_{D\setminus S(2I)} \right) \frac{(1-|z|^2)^{b-1}}{|1-\bar{z}w|^{b+\sigma}} |\psi(z)|\, dA(z) \right)^2 (1-|w|^2)^{2\sigma-2+p} dA(w) \\
&\le 2 \int_{S(I)} \left(\int_{S(2I)} \frac{(1-|z|^2)^{b-1}(1-|w|^2)^{\sigma-1+p/2}}{|1-\bar{z}w|^{b+\sigma}} |\psi(z)|\, dA(z) \right)^2 dA(w) \\
&\quad +2 \int_{S(I)} \left(\int_{D\setminus S(2I)} \frac{(1-|z|^2)^{b-1}(1-|w|^2)^{\sigma-1+p/2}}{|1-\bar{z}w|^{b+\sigma}} |\psi(z)|\, dA(z) \right)^2 dA(w) \\
&= E_1 + E_2.
\end{aligned}$$

Consider the linear operator $B: L^2(D) \to L^2(D)$ defined by

$$h(w) \longmapsto \int_D K(w,z) h(z)\, dA(z),$$

where the kernel is given by

$$K(w,z) = \frac{(1-|z|^2)^{b-1-p/2}(1-|w|^2)^{\sigma-1+p/2}}{|1-\bar{z}w|^{b+\sigma}}.$$

It is easy to verify that for $\gamma = -\frac{1}{2}$, in fact γ can be any number in the interval $\left(-\min(\sigma+p, b) + \frac{1}{2}p, \min(\sigma+p, b) - \frac{1}{2}p - 1\right)$, and $g(z) = (1-|z|^2)^{\gamma/2}$ the estimates

$$\int_D K(w,z) g^2(w)\, dA(w) \le C g^2(z) \quad \text{and} \quad \int_D K(z,w) g^2(w)\, dA(w) \le C g^2(z)$$

hold for all $z \in D$. By Schur's lemma, we know that B is a bounded operator.

Letting

$$h(z) = |\psi(z)| (1-|z|^2)^{p/2} \chi_{S(2I)}(z), \quad z \in D,$$

we have clearly that $h \in L^2(D)$ and

$$\|h\|_{L^2}^2 = \int_{S(2I)} |\psi(z)|^2 (1-|z|^2)^p\, dA(z) \le C|I|^p.$$

Therefore we can estimate E_1 by

$$E_1 \le 2\int_D \left| \int_D K(w,z)h(z)\,dA(z)\right|^2 dA(w) = 2\|B(h)\|^2_{L^2} \le C\|h\|^2_{L^2} \le C|I|^p.$$

To estimate E_2, we note first that (see, for example, [G, p. 239]) for $n \ge 0$ the inequality

$$|1-\bar{z}w| \ge C2^n|I|$$

holds if $w\in S(I)$ and $z\in S(2^{n+1}I)\backslash S(2^nI)$. Direct computation yields also that for any fixed $a>1$, we have

$$\int_{S(2^nI)} (1-|w|^2)^{a-2}\,dA(w) \le C(2^n|I|)^a, \quad n\ge 0.$$

Hence, rewriting the set $D\backslash S(2I)$ as the disjoint union $\bigcup_{n=1}^{\infty} S(2^{n+1}I)\backslash S(2^nI)$, we can estimate E_2 by

$$\begin{aligned}
E_2 &= 2\int_{S(I)} \left(\sum_{n=1}^{\infty} \int_{S(2^{n+1}I)\backslash S(2^nI)} \frac{(1-|z|^2)^{b-1}}{|1-\bar{z}w|^{b+\sigma}} |\psi(z)|\,dA(z)\right)^2 (1-|w|^2)^{2\sigma-2+p}\,dA(w)\\
&\le C\int_{S(I)} \left(\sum_{n=1}^{\infty} \frac{1}{(2^n|I|)^{b+\sigma}} \int_{S(2^{n+1}I)} |\psi(z)|(1-|z|^2)^{b-1}\,dA(z)\right)^2 (1-|w|^2)^{2\sigma-2+p}\,dA(w)\\
&\le C|I|^{2\sigma+p}\left(\sum_{n=1}^{\infty} \frac{1}{(2^n|I|)^{b+\sigma}} \int_{S(2^{n+1}I)} |\psi(z)|(1-|z|^2)^{b-1}\,dA(z)\right)^2.
\end{aligned}$$

By Hölder inequality, we have

$$\begin{aligned}
\int_{S(2^{n+1}I)} |\psi(z)|(1-|z|^2)^{b-1}\,dA(z) &\le \left(\int_{S(2^{n+1}I)} |\psi(z)|^2(1-|z|^2)^p\,dA(z)\right)^{1/2}\\
&\quad\times \left(\int_{S(2^{n+1}I)} (1-|z|^2)^{2b-p-2}\,dA(z)\right)^{1/2}\\
&\le \left(\int_{S(2^{n+1}I)} |\psi(z)|^2(1-|z|^2)^p\,dA(z)\right)^{1/2}\\
&\quad\times (2^{n+1}|I|)^{b-p/2}.
\end{aligned}$$

Thus, we can continue the estimate of E_2 by

$$E_2 \le C|I|^p\left(\sum_{n=1}^{\infty} \frac{1}{2^{n\sigma}}\left(\frac{1}{(2^{n+1}|I|)^p}\int_{S(2^{n+1}I)} |\psi(z)|^2(1-|z|^2)^p\,dA(z)\right)^{1/2}\right)^2.$$

Since the measure $|\psi(z)|^2(1-|z|^2)^p\,dA(z)$ is a bounded p-Carleson measure, we have therefore

$$E_2 \le C|I|^p\biggl(\sum_{n=1}^{\infty}\frac{C}{2^{n\sigma}}\biggr)^2 \le C|I|^p.$$

This proves part (a).

For part (b) we note first that, by assumption, for any $\varepsilon>0$ there exists $\delta>0$ such that the estimate

$$\int_{S(I)}|\psi(z)|^2(1-|z|^2)^p\,dA(z)<\varepsilon|I|^p$$

holds when $|I|<\delta$. We have therefore

$$E_1 \le C\int_{S(2I)}|\psi(z)|^2(1-|z|^2)^p \le C\varepsilon|I|^p, \quad \text{if } |I|<\tfrac{1}{2}\delta.$$

Assume that $|I|<\frac{1}{2}\delta$. Let N be the largest integer satisfying $N<\log_2(\delta/2|I|)$. We have $2^{n+1}|I|<\delta$ if $n\le N$. Hence we have

$$\int_{S(2^{n+1}I)}|\psi(z)|^2(1-|z|^2)^p\,dA(z)\le\varepsilon(2^{n+1}|I|)^p \quad \text{for all } n\le N.$$

Since a compact p-Carleson measure is also a bounded p-Carleson measure, we have

$$\int_{S(2^{n+1}I)}|\psi(z)|^2(1-|z|^2)^p\,dA(z)\le C(2^{n+1}|I|)^p \quad \text{for all } n.$$

Thus, we can refine the previous estimate for E_2 by

$$\begin{aligned}
E_2 &\le C|I|^p\biggl(\sum_{n=1}^{\infty}\frac{1}{2^{n\sigma}}\biggl(\frac{1}{(2^{n+1}|I|)^p}\int_{S(2^{n+1}I)}|\psi(z)|^2(1-|z|^2)^p\,dA(z)\biggr)^{1/2}\biggr)^2\\
&\le C|I|^p\biggl(\sum_{n=1}^{N}\frac{\varepsilon^{1/2}}{2^{n\sigma}}+\sum_{n=N+1}^{\infty}\frac{C}{2^{n\sigma}}\biggr)^2\\
&\le C|I|^p\biggl(\varepsilon^{1/2}+\frac{1}{2^{N\sigma}}\biggr)^2\\
&\le C\biggl(\varepsilon^{1/2}+\biggl(\frac{4|I|}{\delta}\biggr)^{\sigma}\biggr)^2|I|^p.
\end{aligned}$$

The last inequality above is obtained by the fact that $N+1\ge\log_2(\delta/2|I|)$.

In summary, we have proved that the estimate

$$\int_{S(I)} |T_\sigma\psi(z)|^2(1-|z|^2)^{2\sigma-2+p}\,dA(z) \le E_1+E_2 \le C|I|^p\left(\varepsilon+\frac{|I|^{2\sigma}}{\delta^{2\sigma}}\right)$$

holds for all arcs I with $|I|<\frac{1}{2}\delta$. This is enough to conclude that the measure

$$|T_\sigma\psi(z)|^2(1-|z|^2)^{2\sigma-2+p}\,dA(z)$$

is a compact p-Carleson measure. □

For fixed $b>1$, consider the t-derivative with $t>0$:

$$f^{(t)}(z)=\frac{\Gamma(b+t)}{\pi\Gamma(b)}\int_D \frac{\bar{w}^{\lceil t-1\rceil}f'(w)}{(1-\bar{w}z)^{b+t}}(1-|w|^2)^{b-1}\,dA(w).$$

Here Γ is the gamma function and $\lceil t\rceil$ denotes the smallest integer which is larger than or equal to t.

Direct computation yields

$$(z^n)^{(t)}=\begin{cases} 0, & \text{if } n<\lceil t-1\rceil+1,\\ \dfrac{\Gamma(b+n+t-1-\lceil t-1\rceil)\Gamma(n+1)}{\Gamma(b+n)\Gamma(n-\lceil t-1\rceil)}z^{n-1-\lceil t-1\rceil}, & \text{if } n\ge\lceil t-1\rceil+1. \end{cases} \tag{4}$$

It is therefore easy to conclude that the t-derivative is just the usual tth order derivative if t is a positive integer.

Remark. The quantity in the right-hand side of the above formula is depending on b if t is not an integer.

Corollary 4. *Suppose that $0<p\le1$, $t>\frac{1}{2}(1-p)$ and f is analytic on D.*

(a) *The function f is in Q_p if and only if $|f^{(t)}(z)|^2(1-|z|^2)^{2t-2+p}\,dA(z)$ is a bounded p-Carleson measure.*

(b) *The function f is in $Q_{p,0}$ if and only if $|f^{(t)}(z)|^2(1-|z|^2)^{2t-2+p}\,dA(z)$ is a compact p-Carleson measure.*

Remark. For $t=2,3,\dots$, Corollary 4 is proved in [ANZ]. For $p=1$, the result is proved in [RW].

Proof. We show the "if" part of (a) first. Let

$$h(z)=\frac{\Gamma(b+1)z^{\lceil t-1\rceil}}{\pi\Gamma(b+t-1)}\int_D \frac{(1-|\zeta|^2)^{b+t-2}}{(1-\bar{\zeta}z)^{b+1}}f^{(t)}(\zeta)\,dA(\zeta).$$

By Theorem 3(a), we know that the measure $|h(z)|^2(1-|z|^2)^p\,dA(z)$ is a bounded p-Carleson measure.

Let $f(z)=\sum_{j=0}^{\infty}a_jz^j$, $f^{(t)}(z)=\sum_{j=0}^{\infty}a_j^{(t)}z^j$ and $m=\lceil t-1\rceil$. By formula (4), we have

$$a_j^{(t)}=a_{j+m+1}\frac{\Gamma(j+b+t)\Gamma(j+m+2)}{\Gamma(j+1)\Gamma(j+m+1+b)},\quad j\geq 0.$$

Direct computation yields

$$h(z)=\sum_{j=0}^{\infty}\frac{\Gamma(j+b+1)}{\Gamma(j+b+t)}a_j^{(t)}z^{j+m}=\sum_{j=0}^{\infty}\frac{\Gamma(j+b+1)\Gamma(j+m+2)}{\Gamma(j+m+b+1)\Gamma(j+1)}a_{j+m+1}z^{j+m}.$$

If $m=0$, it is easy to conclude from the above formula that $h(z)=f'(z)$. Therefore f is in Q_p.

Assume that $m>0$ and denote the beta function by β. Recall that $\beta(x,y)=\Gamma(x)\Gamma(y)/\Gamma(x+y)$. We can rewrite $h(z)$ as

$$h(z)=\sum_{j=0}^{\infty}\frac{\beta(j+b+1,m)}{\beta(j+1,m)}(j+m+1)a_{j+m+1}z^{j+m}.$$

Let $P_m(z)=\sum_{j=0}^{m}ja_jz^{j-1}$ and

$$e(z)=\sum_{j=0}^{\infty}\left(1-\frac{\beta(j+b+1,m)}{\beta(j+1,m)}\right)a_{j+m+1}z^{j+m+1}.$$

We have therefore

$$f'(z)=h(z)+P_m(z)+e'(z).$$

Since P_m is bounded on D, it is clear that the measure $|P_m(z)|^2(1-|z|^2)^p\,dA(z)$ is a compact p-Carleson measure. To show that f is in Q_p, by Corollary B, it is sufficient to verify that e is in Q_0 (the Dirichlet space).

A standard estimate on beta functions yields

$$0<1-\frac{\beta(j+b+1,m)}{\beta(j+1,m)}\leq\frac{(b+1)m}{j+m+1},\quad j\geq 0.$$

Therefore

$$\int_D|e'(z)|^2\,dA(z)=\pi\sum_{j=0}^{\infty}(j+m+1)|a_{j+m+1}|^2\left|1-\frac{\beta(j+b+1,m)}{\beta(j+1,m)}\right|^2\asymp\sum_{j=0}^{\infty}\frac{|a_{j+m+1}|^2}{j+m+1}.$$

On the other hand, since the measure $|f^{(t)}(z)|^2(1-|z|^2)^{2t-2+p}\,dA(z)$ is a bounded p-Carleson measure, we have

$$\sum_{j=0}^{\infty}\frac{|a_j^{(t)}|^2}{(j+1)^{2t-1+p}}\asymp\int_D|f^{(t)}(z)|^2(1-|z|^2)^{2t-2+p}\,dA(z)<\infty.$$

This is equivalent to (since $|a_j^{(t)}|\asymp(j+1)^{-t}|a_{j+m+1}|$)

$$\sum_{j=0}^{\infty}(j+m+1)^{1-p}|a_{j+m+1}|^2<\infty.$$

Hence

$$\int_D|e'(z)|^2\,dA(z)\asymp\sum_{j=0}^{\infty}\frac{|a_{j+m+1}|^2}{j+m+1}\le\sum_{j=0}^{\infty}(j+m+1)^{1-p}|a_{j+m+1}|^2<\infty.$$

This proves that $e\in Q_0$.

To show the "only if" part of (a), we note that

$$|f^{(t)}(z)|\le\frac{\Gamma(b+t)}{\pi\Gamma(b)}\int_D\frac{(1-|w|^2)^{b-1}}{|1-\bar{w}z|^{b+t}}|f'(w)|\,dA(w).$$

Therefore the desired result follows from Theorem 3(a).

Part (b) of the Corollary can be proved similarly. □

Proof of Theorem 1. For the η-lattice $\{z_j\}_{j=0}^{\infty}$, without loss of generality, we assume that $\eta<1$ and $|z_j|>0$ for all j.

Assume that $f\in Q_p$. We have $f'\in L^2((1-|z|^2)^p\,dA)$. By the reproducing formula, we have

$$f'(w)=\frac{b}{\pi}\int_D\frac{f'(z)}{(1-\bar{z}w)^{b+1}}(1-|z|^2)^{b-1}\,dA(z),\quad b>0.$$

Recall that for the η-lattice $\{z_j\}_{j=0}^{\infty}$, there is a partition $\{D_j\}_{j=0}^{\infty}$ of D which satisfies the condition described in Section 2. Hence $f'(w)$ can be represented as

$$f'(w)=\frac{b}{\pi}\sum_{j=0}^{\infty}\int_{D_j}\frac{f'(z)}{(1-\bar{z}w)^{b+1}}(1-|z|^2)^{b-1}\,dA(z)\approx\frac{b}{\pi}\sum_{j=0}^{\infty}f'(z_j)|D_j|\frac{(1-|z_j|^2)^{b-1}}{(1-\bar{z}_jw)^{b+1}},$$

where $|D_j|=\int_{D_j}dA$ is the area of D_j. Therefore $f(w)$ can be approximated by

$$A(f)(w)=\frac{1}{\pi}\sum_{j=0}^{\infty}f'(z_j)|D_j|\frac{(1-|z_j|^2)^{b-1}}{\bar{z}_j(1-\bar{z}_jw)^b}.$$

The approximation operator A, defined as above, is clearly a linear operator. We now consider the Q_p norm of the error $f-A(f)$. Using the expression

$$f'(w)-A(f)'(w)=\frac{b}{\pi}\sum_{j=0}^{\infty}\int_{D_j}\frac{f'(z)}{(1-\bar{z}w)^{b+1}}(1-|z|^2)^{b-1}\,dA(z)$$
$$-\frac{b}{\pi}\sum_{j=0}^{\infty}f'(z_j)|D_j|\frac{(1-|z_j|^2)^{b-1}}{(1-\bar{z}_j w)^{b+1}},$$

we have (recall that $k_w(z)=(1-|z|^2)^{b-1}/(1-\bar{z}w)^{b+1}$)

$$|f'(w)-A(f)'(w)|\le\frac{b}{\pi}\sum_{j=0}^{\infty}\int_{D_j}|f'(z)|\,|k_w(z)-k_w(z_j)|\,dA(z)$$
$$+\frac{b}{\pi}\sum_{j=0}^{\infty}\int_{D_j}|f'(z)-f'(z_j)|\,|k_w(z_j)|\,dA(z)$$
$$=\sum\nolimits_1+\sum\nolimits_2.$$

By Lemma C, we have for every z_j,

$$|k_w(z)-k_w(z_j)|\le C\eta|k_w(z)|,\quad z\in D_j.$$

This implies that

$$\sum\nolimits_1\le C\eta\int_D|k_w(z)|\,|f'(z)|\,dA(z).$$

By Lemma D, we have for every z_j,

$$\int_{D_j}|f'(z)-f'(z_j)|\,dA(z)\le C\eta^3\int_{\overline{B}_j}|f'(z)|\,dA(z).$$

Applying Lemma C again, we get

$$\int_{D_j}|f'(z)-f'(z_j)|\,|k_w(z_j)|\,dA(z)\le C\eta^3\int_{\overline{B}_j}|f'(z)|\,|k_w(z_j)|\,dA(z)$$
$$\le C\eta^3\int_{\overline{B}_j}|f'(z)|(|k_w(z)-k_w(z_j)|+|k_w(z)|)\,dA(z)$$
$$\le C\eta^3\int_{\overline{B}_j}|f'(z)|(C\eta|k_w(z)|+|k_w(z)|)\,dA(z)$$
$$\le C\eta^3\int_{\overline{B}_j}|k_w(z)|\,|f'(z)|\,dA(z).$$

Therefore, by Lemma E, we obtain that

$$\sum\nolimits_2 \le C\eta^3 \sum_{j=0}^{\infty} \int_{\bar{B}_j} |k_w(z)|\,|f'(z)|\,dA(z) \le C\eta \int_D |k_w(z)|\,|f'(z)|\,dA(z).$$

In summary, we have

$$|f'(w)-A(f)'(w)| \le C\eta \int_D |k_w(z)|\,|f'(z)|\,dA(z), \quad w\in D.$$

Applying Theorem 3(a) to the above estimate, we get

$$\|f-A(f)\|_{Q_p} \le C\eta \|f\|_{Q_p}, \quad f\in Q_p.$$

It is clear that we can choose a smaller η (say $\eta=1/3C$) so that the operator A is invertible and that $A^{-1}=\sum_{n=0}^{\infty}(\mathrm{Id}-A)^n$ is bounded. Here Id is the identity operator on Q_p.

We have constructed an approximation operator A with bounded inverse. For any $f(z)\in Q_p$, we can write

$$f(z)=AA^{-1}(f)(z)=\frac{1}{\pi}\sum_{j=0}^{\infty} A^{-1}(f)'(z_j)|D_j|\frac{(1-|z_j|^2)^{b-1}}{\bar{z}_j(1-\bar{z}_j z)^b}.$$

Picking

$$\lambda_j=\frac{A^{-1}(f)'(z_j)|D_j|(1-|z_j|^2)^{p/2-1}}{\pi\bar{z}_j},$$

we have the desired decomposition

$$f(z)=\sum_{j=0}^{\infty}\lambda_j\frac{(1-|z_j|^2)^{b-p/2}}{(1-\bar{z}_j z)^b}.$$

It remains to show that the measure $\sum_{j=0}^{\infty}|\lambda_j|^2\delta_{z_j}$ is a bounded p-Carleson measure. In fact for an analytic function g on $\widetilde{B}_j$, by the mean value theorem, we have

$$|g(z_j)|^2(1-|z_j|^2)^p \le \frac{C}{|\widetilde{B}_j|}\int_{\widetilde{B}_j} |g(z)|^2(1-|z|^2)^p\,dA(z).$$

Therefore, we obtain that

$$\begin{aligned}\sum_{z_j\in S(I)}|\lambda_j|^2 &\le C\sum_{z_j\in S(I)}|A^{-1}(f)'(z_j)|^2(1-|z_j|^2)^{p-2}|D_j|^2\\
&\le C\sum_{z_j\in S(I)}\frac{|D_j|^2}{|\widetilde{B}_j|(1-|z_j|^2)^2}\int_{\widetilde{B}_j}|A^{-1}(f)'(z)|^2(1-|z|^2)^p\,dA(z)\\
&\le C\int_{S(2I)}|A^{-1}(f)'(z)|^2(1-|z|^2)^p\,dA(z)\\
&\le C\|A^{-1}(f)\|_{Q_p}|I|^p\\
&\le C\|f\|_{Q_p}|I|^p.\end{aligned}$$

This completes the proof of (a).

For part (b), it suffices to show that the measure $|f'(z)|^2(1-|z|^2)^p\,dA(z)$ is a bounded p-Carleson measure. By the given formula we have

$$f'(w)=b\sum_{j=0}^{\infty}\lambda_j\bar{z}_j\frac{(1-|z_j|^2)^{b-p/2}}{(1-\bar{z}_jw)^{b+1}}.$$

Applying the substitution $z=(z_j-\zeta)/(1-\bar{z}_j\zeta)$ in the following integration, we know that there is a positive constant

$$C_j=\frac{\pi\left(\dfrac{e^\eta-1}{e^\eta+1}\right)^2}{\left(1-\left(\dfrac{e^\eta-1}{e^\eta+1}\right)^2|z_j|^2\right)}\frac{2b}{1-\left(\dfrac{4e^\eta}{(e^\eta+1)^2}\right)^b}$$

such that

$$\int_{\widetilde{B}_j}\frac{(1-|z|^2)^{b-1}}{(1-\bar{z}w)^{b+1}}\,dA(z)=\frac{|\widetilde{B}_j|}{C_j}\frac{(1-|z_j|^2)^{b-1}}{(1-\bar{z}_jw)^{b+1}}.$$

Therefore we have that

$$\begin{aligned}f'(w)&=b\sum_{j=0}^{\infty}\lambda_j\bar{z}_jC_j\frac{(1-|z_j|^2)^{1-p/2}}{|\widetilde{B}_j|}\int_D\chi_{\widetilde{B}_j}(z)\frac{(1-|z|^2)^{b-1}}{(1-\bar{z}w)^{b+1}}\,dA(z)\\
&=b\int_D\frac{(1-|z|^2)^{b-1}}{(1-\bar{z}w)^{b+1}}\sum_{j=0}^{\infty}\lambda_j\bar{z}_jC_j\left(\frac{(1-|z_j|^2)^{1-p/2}}{|\widetilde{B}_j|}\chi_{\widetilde{B}_j}(z)\right)dA(z).\end{aligned}$$

By Theorem 3(a), we only need to show that the measure

$$\left|\sum_{j=0}^{\infty}\lambda_j\bar{z}_jC_j\frac{(1-|z_j|^2)^{1-p/2}}{|\widetilde{B}_j|}\chi_{\widetilde{B}_j}(z)\right|^2(1-|z|^2)^p\,dA(z)$$

is a bounded p-Carleson measure.

In fact, it is clear that $\{C_j\}_{j=0}^{\infty}$ is bounded. Since $\{\widetilde{B}_j\}_{j=0}^{\infty}$ is a set of disjoint Bergman discs, we have

$$\begin{aligned}
&\int_{S(I)}\left|\sum_{j=0}^{\infty}\lambda_j\bar{z}_jC_j\frac{(1-|z_j|^2)^{1-p/2}}{|\widetilde{B}_j|}\chi_{\widetilde{B}_j}(z)\right|^2(1-|z|^2)^p\,dA(z)\\
&\qquad\le C\int_{S(I)}\sum_{j=0}^{\infty}|\lambda_j|^2\frac{(1-|z_j|^2)^{2-p}}{|\widetilde{B}_j|^2}\chi_{\widetilde{B}_j}(z)(1-|z|^2)^p\,dA(z)\\
&\qquad= C\sum_{j=0}^{\infty}\int_{S(I)\cap\widetilde{B}_j}|\lambda_j|^2\frac{(1-|z_j|^2)^{2-p}}{|\widetilde{B}_j|^2}(1-|z|^2)^p\,dA(z)\\
&\qquad\le C\sum_{j:S(I)\cap\widetilde{B}_j\neq\emptyset}|\lambda_j|^2\\
&\qquad\le C\sum_{z_j\in S(2I)}|\lambda_j|^2\\
&\qquad\le C|I|^p.
\end{aligned}$$

The last inequality holds because the measure $\sum_{j=0}^{\infty}|\lambda_j|^2\delta_{z_j}$ is a bounded p-Carleson measure. $\square$

Proof of Theorem 2. It is proved implicitly in the proof of Theorem 1 that the approximation operator A satisfies

$$|A(f)'(w)|\le C\int_D|k_w(z)|\,|f'(z)|\,dA(z),\quad w\in D.$$

Hence by Theorem 3(b), we have that A maps $Q_{p,0}$ to $Q_{p,0}$. Following the proof of Theorem 1, we only need to show that A^{-1} maps $Q_{p,0}$ to $Q_{p,0}$ as well.

In fact, for any $\varepsilon>0$, we can find a positive integer $N>0$ such that

$$\left\|\sum_{n=N+1}^{\infty}(\mathrm{Id}-A)^n\right\|<\varepsilon,$$

that is, for any $f\in Q_p$ and any arc $I\subset\partial D$,

$$\begin{aligned}
\int_{S(I)}\left|\left(\sum_{n=N+1}^{\infty}(\mathrm{Id}-A)^n(f)(z)\right)'\right|^2(1-|z|^2)^p\,dA(z)&\le|I|^p\left\|\sum_{n=N+1}^{\infty}(\mathrm{Id}-A)^nf\right\|_{Q_p}^2\\
&\le\varepsilon^2|I|^p\|f\|_{Q_p}^2.
\end{aligned}$$

On the other hand, the operators Id and A map $Q_{p,0}$ to $Q_{p,0}$ and so does the operator $\sum_{n=0}^{N}(\mathrm{Id}-A)^n$. Therefore there exists a $\delta>0$, such that for any arc I with $|I|<\delta$, we have the following estimate

$$\int_{S(I)}\left|\left(\sum_{n=0}^{N}(\mathrm{Id}-A)^n(f)(z)\right)'\right|^2(1-|z|^2)^p\,dA(z)\le\varepsilon|I|^p.$$

Since $A^{-1}=\sum_{n=0}^{N}(\mathrm{Id}-A)^n+\sum_{n=N+1}^{\infty}(\mathrm{Id}-A)^n$, we have

$$\int_{S(I)}|A^{-1}(f)'(z)|^2(1-|z|^2)^p\,dA(z)\le(\|f\|_{Q_p}^2\varepsilon^2+\varepsilon)|I|^p\quad\text{for } I \text{ with } |I|<\delta.$$

This is enough. $\square$

4. Remarks

For $q>0$ and $p>-1$, let $C_{p,q}^{\alpha}$ be the space of analytic functions f in D satisfying

$$\int_{S(I)}|f'(z)|^q(1-|z|^2)^p\,dA(z)\le C|I|^{\alpha}$$

for any arc $I\subset\partial D$. The norm of f, denoted by $\|f\|_{C_{p,q}^{\alpha}}$, is the qth root of the best constant in the above inequality. Moreover we say that $f\in VC_{p,q}^{\alpha}$ if

$$\int_{S(I)}|f'(z)|^q(1-|z|^2)^p\,dA(z)=o(|I|^{\alpha}).$$

Clearly the spaces Q_p and $Q_{p,0}$ are special cases of $C_{p,q}^{\alpha}$ and $VC_{p,q}^{\alpha}$, respectively. A systematical study of these scales of spaces can be found in [Zh] and [Rä].

It is easy to check that for fixed q and α,

$$C_{p_1,q}^{\alpha}\subset C_{p_2,q}^{\alpha},\quad p_1<p_2.$$

The following results can be proved in a way similar to the proofs in Section 3.

Theorem 3′. *Suppose that $p\ge-1$, $\sigma>(q-p-1)/q$ and ψ is a measurable function on D.*

(a) *If $\int_{S(I)}|\psi(z)|^q(1-|z|^2)^p\,dA(z)\le C|I|^{\alpha}$ for any $I\subset\partial D$, then*

$$\int_{S(I)}|T_\sigma\psi(z)|^q(1-|z|^2)^{q(\sigma-1)+p}\,dA(z)\le C|I|^{\alpha}$$

for any $I\subset\partial D$.

(b) *If $\int_{S(I)}|\psi(z)|^q(1-|z|^2)^p\,dA(z)=o(|I|^{\alpha})$, then*

$$\int_{S(I)}|T_\sigma\psi(z)|^q(1-|z|^2)^{q(\sigma-1)+p}\,dA(z)=o(|I|^{\alpha}).$$

Theorem 1′. *Suppose that* $p\geq -1$ *and* $q>1$. *There exists an* $\eta_0>0$, *such that for any* η*-lattice* $\{z_j\}_{j=0}^{\infty}$ *in* D *with* $0<\eta<\eta_0$, *the following are true*:

(a) *If* $f\in C_{p,q}^{\alpha}$, *then*

$$f(z)=\sum_{j=0}^{\infty}\lambda_j\frac{(1-|z_j|^2)^{b+1-(2+p)/q}}{(1-\bar{z}_j z)^b},\quad b>\frac{1+p}{q}, \tag{5}$$

and

$$\sup_{\operatorname{arc} I}\left(\frac{1}{|I|^{\alpha}}\sum_{z_j\in S(I)}|\lambda_j|^q\right)^{1/q}\leq C\|f\|_{C_{p,q}^{\alpha}}. \tag{6}$$

(b) *If* $\{\lambda_j\}_{j=0}^{\infty}$ *satisfies* $\sum_{z_j\in S(I)}|\lambda_j|^q\leq C|I|^{\alpha}$ *for any arc* I, *then* f, *defined by* (5), *is in* $C_{p,q}^{\alpha}$ *and*

$$\|f\|_{C_{p,q}^{\alpha}}\leq C\sup_{\operatorname{arc} I}\left(\frac{1}{|I|^{\alpha}}\sum_{z_j\in S(I)}|\lambda_j|^q\right)^{1/q}.$$

Theorem 2′. *Suppose that* $p\geq -1$ *and* $q>1$. *There exists an* $\eta_0>0$, *such that for any* η*-lattice* $\{z_j\}_{j=0}^{\infty}$ *in* D *with* $0<\eta<\eta_0$, *the following are true*:

(a) *If* $f\in VC_{p,q}^{\alpha}$, *then*

$$f(z)=\sum_{j=0}^{\infty}\lambda_j\frac{(1-|z_j|^2)^{b+1-(2+p)/q}}{(1-\bar{z}_j z)^b},\quad b>\frac{1+p}{q}, \tag{7}$$

and

$$\sum_{z_j\in S(I)}|\lambda_j|^q=o(|I|^{\alpha}).$$

(b) *If* $\{\lambda_j\}_{j=0}^{\infty}$ *satisfies* $\sum_{z_j\in S(I)}|\lambda_j|^q=o(|I|^{\alpha})$, *then* f, *defined by* (7), *is in* $VC_{p,q}^{\alpha}$.

References

[A] AULASKARI, R., On Q_p functions, in *Complex Analysis and Related Topics* (*Cuernavaca, 1996*) (Ramírez de Arellano, E., Shapiro, M. V., Tovar, L. M. and Vasilevskiĭ, N. L., eds.), Oper. Theory Adv. Appl. **114**, pp. 21–29, Birkhäuser, Basel, 2000.

[AL] Aulaskari, R. and Lappan, P., Criteria for analytic functions to be Bloch and a harmonic or meromorphic function to be normal, in *Complex Analysis and its Applications* (*Hong Kong, 1993*) (Yang, C.-C., Wen, G. C., Li, K. Y. and Chiang, Y.-M., eds.), Pitman Res. Notes Math. Ser. **305**, pp. 136–146, Longman Sci. Tech., Harlow, 1994.

[ANZ] Aulaskari, R., Nowak, M. and Zhao, R., The nth derivative characterization of Möbius invariant Dirichlet space, *Bull. Austral. Math. Soc.* **58** (1998), 43–56.

[ASX] Aulaskari, R., Stegenga, D. A. and Xiao, J., Some subclasses of BMOA and their characterization in terms of Carleson measure, *Rocky Mountain J. Math.* **26** (1996), 485–506.

[AXZ] Aulaskari, R., Xiao, J. and Zhao, R., On subspaces and subsets of BMOA and UBC, *Analysis* **15** (1995), 101–121.

[CR] Coifman, R. R. and Rochberg, R., Representation theorems for holomorphic and harmonic functions in L^p, in *Representation Theorems for Hardy Spaces,* Astérisque **77**, pp. 11–66, Soc. Math. France, Paris, 1980.

[EJPX] Essén, M., Janson, S., Peng, L. and Xiao, J., Q spaces of several real variables, *Indiana Univ. Math. J.* **49** (2000), 575–615.

[G] Garnett, J. B., *Bounded Analytic Functions*, Academic Press, Orlando, Fla., 1981.

[L] Lappan, P., A survey of Q_p spaces, in *Complex Analysis and Related Topics* (*Cuernavaca, 1996*) (Ramírez de Arellano, E., Shapiro, M. V., Tovar, L. M. and Vasilevskiĭ, N. L., eds.), Oper. Theory Adv. Appl. **114**, pp. 147–154, Birkhäuser, Basel, 2000.

[NX] Nicolau, A. and Xiao, J., Bounded functions in Möbius invariant Dirichlet spaces, *J. Funct. Anal.* **150** (1997), 383–425.

[P] Power, S. C., *Hankel Operators on Hilbert Space,* Pitman Res. Notes Math. Ser. **64**, Pitman, Boston, Mass., 1982.

[Rä] Rättyä, J., On some complex function spaces and classes, *Ann. Acad. Sci. Fenn. Math. Dissertationes* **124** (2001).

[R] Rochberg, R., Decomposition theorems for Bergman spaces and their applications, in *Operators and Function Theory* (*Lancaster, 1984*) (Power, S. C., ed.), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci. **153**, pp. 225–277, Reidel, Dordrecht, 1985.

[RS] Rochberg, R. and Semmes, S., A decomposition theorem for BMO and applications, *J. Funct. Anal.* **67** (1986), 228–263.

[RW] Rochberg, R. and Wu, Z., A new characterization of Dirichlet type spaces and applications, *Illinois J. Math.* **37** (1993), 101–122.

[X] Xiao, J., *Holomorphic Q Classes*, Lecture Notes in Math. **1767**, Springer-Verlag, Berlin–Heidelberg, 2001.

[Zh] Zhao, R., On a general family of function spaces, *Ann. Acad. Sci. Fenn. Math. Dissertationes* **105** (1996).

[Z] ZHU, K., *Operator Theory in Function Spaces*, Marcel Dekker, New York, 1990.

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