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Continuity of weak solutions of elliptic partial differential equations

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Abstract. The continuity of weak solutions of elliptic partial differential equations

$$\operatorname{div} \mathcal{A}(x, \nabla u) = 0$$

is considered under minimal structure assumptions. The main result guarantees the continuity at the point x_0 for weakly monotone weak solutions if the structure of $\mathcal{A}$ is controlled in a sequence of annuli $B(x_0, R_j)\setminus\overline{B}(x_0, r_j)$ with uniformly bounded ratio R_j/r_j such that $\lim_{j\to\infty} R_j=0$. As a consequence, we obtain a sufficient condition for the continuity of mappings of finite distortion.

1. Introduction

This note deals with the continuity of weakly monotone weak solutions of the elliptic partial differential equation

$$\operatorname{div} \mathcal{A}(x, \nabla u) = 0. \tag{1.1}$$

Here $\mathcal{A}: \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n$ satisfies

$$\alpha(x)|\xi|^p \le \mathcal{A}(x, \xi)\cdot\xi \le \beta(x)|\xi|^p$$

for some $1<p\le n$ and for some measurable functions $\alpha, \beta: \Omega \to \mathbf{R}_+$. We do not a priori make any integrability assumptions on β/α or α^{-1}. We do not even require in Section 3 that $\alpha>0$ a.e. in Ω. The lack of these assumptions is replaced by the assumption that u is weakly monotone. In fact, our main result (Theorem 3.1) shows that weakly monotone weak solutions of (1.1) are continuous at the point $x_0\in\Omega$ whenever there exists a sequence of annuli $A_j = B(x_0, R_j)\setminus\overline{B}(x_0, r_j)$ shrinking to the point x_0 such that $\varepsilon_{1,j}\le\alpha\le\beta\le\varepsilon_{2,j}$ in A_j for some constants $\varepsilon_{1,j}, \varepsilon_{2,j}>0$ with finite supremums $\sup_j(R_j/r_j)$ and $\sup_j(\varepsilon_{2,j}/\varepsilon_{1,j})$. The proof of this result is based on Harnack's inequality only. Theorem 3.1 is improved in Section 4 under the additional assumptions that $n-1<p<n$, the function β is bounded, and $\alpha>0$ a.e. in Ω.

The equation (1.1) is considered in [9], where the authors obtain among other things that weak solutions of (1.1) are weakly monotone in the sense of [12] if $\beta\in L^\infty(\Omega)$ and $\alpha>0$ a.e. in Ω. Thus by the known properties of weakly monotone functions [12, Theorem 1], any weak solution of (1.1) has a representative which is locally bounded and continuous outside a p-polar set, whenever $n-1<p<n$. For $p\geq n$ weakly monotone functions always have a continuous representative. There are examples of non-continuous weakly monotone functions for $p<n$. In the case $1<p\leq n-1$, the author is not aware of any general regularity results for weakly monotone functions.

Our ideas have a relation to the theory of mappings of finite distortion. This class of mappings has been intensively studied quite recently, see e.g. [1], [4], [5], [6], [7], [8] and the references therein. The continuity and monotonicity of mappings of finite distortion are studied in [5], where the authors prove continuity under the additional assumption that the distortion function is exponentially integrable. We comment on the continuity of mappings of finite distortion in Remark 3.2.

This paper is an improved version of the preprint [11]. In [11], Theorem 3.1 was proved by obtaining an annulus version of the well-known De Giorgi method.

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2. Preliminaries

We assume throughout that $\Omega\subset\mathbf{R}^n$ is an open set for $n\geq 2$, $x_0\in\Omega$, and u is a weak solution of (1.1) in Ω in the sense of Definition 2.1.

Our notation is fairly standard. The n-dimensional Lebesgue measure is denoted by $|\cdot|$ and $|\cdot|_1$ denotes the 1-dimensional Lebesgue measure. We write

$$\operatorname*{osc}_E v = \sup_E v - \inf_E v$$

for the oscillation of a real-valued function v defined on $E\subset\mathbf{R}^n$. The boundary of an open euclidean ball $B(x,r)\subset\mathbf{R}^n$ is denoted by $S(x,r)$.

Recall that the Sobolev space $u\in W^{1,p}_{\mathrm{loc}}(\Omega)$, $p>1$, consists of functions u in Ω which are locally L^p-integrable in Ω and whose distributional gradient ∇u is locally L^p-integrable in Ω.

Weak solutions of elliptic partial differential equations

Let $1<p<\infty$ and let $\alpha,\beta\colon\Omega\to\mathbf{R}_+$ be measurable functions such that

$$0\le\alpha(x)\le\beta(x)<\infty \quad \text{for a.e. } x\in\Omega.$$

Suppose further that $\mathcal{A}\colon\mathbf{R}^n\times\mathbf{R}^n\to\mathbf{R}^n$ satisfies the following assumptions:

(A1) the mapping $x\mapsto\mathcal{A}(x,\xi)$ is measurable for all $\xi\in\mathbf{R}^n$;

(A2) the mapping $\xi\mapsto\mathcal{A}(x,\xi)$ is continuous for a.e. $x\in\mathbf{R}^n$;

(A3) $\mathcal{A}(x,\xi)\cdot\xi\ge\alpha(x)|\xi|^p$ for all $\xi\in\mathbf{R}^n$ and a.e. $x\in\mathbf{R}^n$;

(A4) $|\mathcal{A}(x,\xi)|\le\beta(x)|\xi|^{p-1}$ for all $\xi\in\mathbf{R}^n$ and a.e. $x\in\mathbf{R}^n$.

The reason why assumptions (A1) and (A2) are required is that they ensure the measurability of the composed function $x\mapsto\mathcal{A}(x,v(x))$ for all measurable functions v. Assumptions (A3) and (A4) describe the elliptic structure of $\mathcal{A}$.

Definition 2.1. We call u a *weak solution of* (1.1) in Ω if $u\in W^{1,p}_{\mathrm{loc}}(\Omega)$ and

$$\int_\Omega \mathcal{A}(x,\nabla u(x))\cdot\nabla\phi(x)\,dx=0 \quad \text{for all } \phi\in C_0^\infty(\Omega).$$

Weakly monotone functions

Weakly monotone functions were introduced in [12] as follows.

Definition 2.2. A function $u\in W^{1,p}_{\mathrm{loc}}(\Omega)$ is *weakly monotone*, if for every relatively compact subdomain Ω' of Ω and for every pair of constants $m\le M$ such that

$$(m-u)^+\in W_0^{1,p}(\Omega')$$

and

$$(u-M)^+\in W_0^{1,p}(\Omega')$$

we have

$$m\le u\le M \quad \text{a.e. in } \Omega'.$$

If β is bounded in Ω and $\alpha>0$ a.e. in Ω, then a simple approximation argument shows that one can as well test with all functions $\phi\in W^{1,p}(\Omega)$ whose support is compactly contained in Ω ([3, Lemma 3.11]). In this case we have the following result [9, Lemma 2.7].

Lemma 2.3. *Weak solutions of* (1.1) *in* Ω *are weakly monotone if* $\beta\in L^\infty(\Omega)$ *and* $\alpha>0$ *a.e. in* Ω.

The uniformly bounded case $\beta\in L^{\infty}(\Omega)$, $\alpha^{-1}\in L^{\infty}(\Omega)$, is studied in the celebrated paper [13] by Serrin. In this case non-negative solutions of (1.1) satisfy Harnack's inequality and the (local Hölder) continuity follows by the standard iteration argument.

3. Harnack's inequality

In this section we prove our general sufficient condition for the continuity of weakly monotone weak solutions of (1.1). The proof of the result is based on a modification of Harnack's inequality. We consider an arbitrary exponent $1<p<\infty$ in this section.

Theorem 3.1. *Let $A_j=B(x_0,R_j)\setminus\overline{B}(x_0,r_j)$ be a sequence of open annuli with the properties*

(i) $\lim_{j\to\infty} R_j=0$,

(ii) $\sup_{j\in\mathbf{N}}(R_j/r_j)=:s<\infty$,

(iii) $\varepsilon_{1,j}\le\alpha\le\beta\le\varepsilon_{2,j}$ *in* A_j *for some positive constants* $\varepsilon_{1,j}$ *and* $\varepsilon_{2,j}$ *with* $\sup_{j\in\mathbf{N}}(\varepsilon_{2,j}/\varepsilon_{1,j})=:t<\infty$.

Then each weakly monotone weak solution u of (1.1) in $B(x_0,R_1)$ has a representative, which is continuous at x_0.

Proof. Define u pointwise by

$$u(z)=\liminf_{r\to 0}\frac{1}{|B(z,r)|}\int_{B(z,r)}u\,dx\quad\text{for all } z\in\Omega \tag{3.1}$$

and let

$$\begin{aligned}
A_j'&=B\big(x_0,R_j-\tfrac14(R_j-r_j)\big)\setminus\overline{B}\big(x_0,r_j+\tfrac14(R_j-r_j)\big),\\
M_j&=\sup_{B(x_0,\frac12(R_j+r_j))}u,\\
m_j&=\inf_{B(x_0,\frac12(R_j+r_j))}u.
\end{aligned}$$

We are free to assume that $R_{j+1}<r_j$ for all j. First, notice that u is bounded in $B\big(x_0,\frac12(R_j+r_j)\big)$ by the weak monotonicity. In fact, since u is a solution of (1.1) in A_j in the sense of [13], u is continuous in A_j. Hence $(u-\sup_{A_j'}u)^+$ belongs to $W_0^{1,p}\big(B\big(x_0,\frac12(R_j+r_j)\big)\big)$, and Definition 2.2 implies that $u\le\sup_{A_j'}u$ a.e. in $B\big(x_0,\frac12(R_j+r_j)\big)$. The pointwise definition (3.1) guarantees that $u\le\sup_{A_j'}u$ everywhere in $B\big(x_0,\frac12(R_j+r_j)\big)$. Since the lower bound can be treated similarly, we conclude that the numbers M_j and m_j are finite. Consequently, $v_j=u-m_j$

is a non-negative solution of (1.1) in $B(x_0, A_{j+1})$. By [13], v_j satisfies Harnack's inequality

$$\sup_B v_j \le C \inf_B v_j$$

for all balls B such that $2B \subset A_{j+1}$. We may cover $S\big(x_0, \frac{1}{2}(R_{j+1}+r_{j+1})\big)$ by balls with radius $\frac{1}{4}(R_{j+1}-r_{j+1})$ such that the number of balls in the covering only depends on s and n. A repeated application of Harnack's inequality yields

$$\sup_{S(x_0, \frac{1}{2}(R_{j+1}+r_{j+1}))} v_j \le C \inf_{S(x_0, \frac{1}{2}(R_{j+1}+r_{j+1}))} v_j.$$

Here C depends only on n, p, s, and t. Using the weak monotonicity as above, we infer

$$\sup_{B(x_0, \frac{1}{2}(R_{j+1}+r_{j+1}))} v_j \le C \inf_{B(x_0, \frac{1}{2}(R_{j+1}+r_{j+1}))} v_j$$

with a constant C depending only on n, p, s, and t. For shortness, we rewrite this inequality in the form

$$M_{j+1} - m_j \le C(m_{j+1} - m_j). \tag{3.2}$$

We now proceed in essentially the standard way. Let $\lambda = (C-1)/C$, where $C > 1$ is the constant of (3.2). Since $0 < \lambda < 1$, it suffices to show that

$$M_{j+1} - m_{j+1} \le \lambda(M_j - m_j) \quad \text{for all } j.$$

Suppose that $m_{j+1} - m_j \le C^{-1}(M_j - m_j)$. Then by (3.2),

$$M_{j+1} - m_{j+1} = M_{j+1} - m_j + m_j - m_{j+1} \le (C-1)(m_{j+1} - m_j) \le \lambda(M_j - m_j).$$

The case $m_{j+1} - m_j > C^{-1}(M_j - m_j)$ is obvious, since then

$$\begin{aligned} M_{j+1} - m_{j+1} &\le M_j - m_j - (m_{j+1} - m_j) \\ &< M_j - m_j - C^{-1}(M_j - m_j) = \lambda(M_j - m_j). \quad \square \end{aligned}$$

Remark 3.2. The proof of Theorem 3.1 only requires the weak monotonicity on $B(x_0, R_1)$ and Harnack's inequality in annuli A_j for balls B such that $2B \subset A_j$. The constant in Harnack's inequality should be independent of j. We next discuss a special case, which has some interest from the point of view of mappings of finite distortion, see [5]. By definition, a mapping $f\colon \Omega \to \mathbf{R}^n$ is of *finite distortion* in Ω if the components satisfy $f_k \in W^{1,1}_{\text{loc}}(\Omega)$ and there is a measurable function $K(x) \ge 1$ such that for a.e. $x \in \Omega$

$$|Df(x)|^n \le K(x) J(x, f).$$

Here $Df(x)$ is the a.e. defined derivative matrix and $J(x,f)=\det Df(x)$ is the Jacobian determinant. If moreover $K:=\operatorname{ess\,sup}_{x\in\Omega} K(x)<\infty$ and $f_k\in W^{1,n}_{\mathrm{loc}}(\Omega)$, the mapping f is called K-*quasiregular* in Ω.

Let $1<p<n$ and let $f:\Omega\to\mathbf{R}^n$ be a mapping whose coordinate functions f_k, $k=1,\dots,n$, belong to $W^{1,p}_{\mathrm{loc}}(\Omega)$ and are weakly monotone. Let $x_0\in\Omega$ and assume that $R_j>r_j>0$ are radii with the properties

(i) $\lim_{j\to\infty} R_j=0$;

(ii) $\sup_{j\in\mathbf{N}}(R_j/r_j)<\infty$;

(iii) f is K-quasiregular in each annulus $A_j=B(x_0,R_j)\setminus\overline{B}(x_0,r_j)$ with K independent of j.

Then f is continuous at x_0. The claim follows from Theorem 3.1, since the coordinate functions f_k satisfy Harnack's inequality in annuli A_j with a constant depending only on n and K, see [3, pp. 269–271] and [13].

4. The case $n-1<p<n$

In this section we prove a sphere version of Theorem 3.1 under the additional assumptions that $n-1<p<n$ and $\beta\in L^\infty(\Omega)$. The idea of the proof resembles the one in Theorem 3.1, but we need more complicated arguments to obtain the sequence of Harnack's inequalities.

We assume throughout this section that $n-1<p<n$, the function β is bounded, and $\alpha>0$ a.e. in Ω. Let

$$\varepsilon_2=\operatorname*{ess\,sup}_{x\in\Omega}\beta(x).$$

In this case each solution of (1.1) in Ω is weakly monotone (Lemma 2.3) and we are able to use the following special properties of weakly monotone functions obtained in the proof of [12, Theorem 1].

Lemma 4.1. *Let $p>n-1$ and let $v\in W^{1,p}_{\mathrm{loc}}(\Omega)$ be a weakly monotone function in Ω defined pointwise by*

$$v(z)=\liminf_{r\to0}\frac{1}{|B(z,r)|}\int_{B(z,r)} v\,dx \quad \textit{for all } z\in\Omega.$$

Then for any $B(x_0,R)\Subset\Omega$ there is a set $E\subset(0,R)$ of linear measure zero such that v is continuous in $S(x_0,r)$ and

$$\operatorname*{osc}_{B(x_0,r)} v=\operatorname*{osc}_{S(x_0,r)} v \quad \textit{for all } r\in(0,R)\setminus E.$$

Moreover, $\operatorname{osc}_{S(x_0,r)} v$ is non-decreasing in $(0,R)\setminus E$.

Proof. The pointwise definition of v guarantees that v is p-quasicontinuous in Ω [2, pp. 161–162] and that

$$\operatorname*{ess\,osc}_{B(x_0,r)} v = \operatorname*{osc}_{B(x_0,r)} v \quad \text{for all balls } B(x_0,r) \Subset \Omega.$$

Hence the assertions hold by the proof of [12. Theorem 1]. □

Lemma 4.2. *Let u be a non-negative solution of (1.1) in Ω and assume that there are numbers $\varepsilon_1>0$ and $0<\delta<1$ together with radii R_j and measurable subsets A_j of $]0,R_j[$ such that*

(i) $\lim_{j\to\infty} R_j=0$;

(ii) $|A_j|_1\geq\delta R_j$;

(iii) $\alpha\geq\varepsilon_1$ *in* $\bigcup_{r\in A_j} S(x_0,r)$.

Then there is a sequence of radii $r_j\in]0,R_j[$ such that

$$\sup_{B(x_0,r_j)} u\leq C \inf_{B(x_0,r_j)} u \quad \textit{for all } j=1,2,\dots.$$

The constant C depends only on n, p, δ, and the ratio $\varepsilon_2/\varepsilon_1$.

Proof. It is enough to prove the assertion for arbitrary $v=u+1/k$, $k=1,2,\dots$, if only the choice of the radii r_j is independent of k. To do this, we first imitate a standard trick. Let R_j be such that $B(x_0,2R_j)\Subset\Omega$ and let $\psi_j\in W_0^{1,p}(B(x_0,2R_j))$ be a Lipschitz function with the properties $0\leq\psi_j\leq 1$, $\psi_j=1$ in $B(x_0,R_j)$, and $|\nabla\psi_j|\leq 2/R_j$. We choose $\eta_j=\psi_j^p v^{1-p}$ as a test function for v in Ω. Then

$$0\leq\int_\Omega \mathcal{A}(x,\nabla v)\cdot\nabla\eta_j\,dx=\int_\Omega \mathcal{A}(x,\nabla v)(p\psi_j^{p-1}v^{1-p}\nabla\psi_j-(p-1)v^{-p}\psi_j^p\nabla v)\,dx,$$

so that

$$(p-1)\int_\Omega v^{-p}\psi_j^p\mathcal{A}(x,\nabla v)\cdot\nabla v\,dx\leq p\int_\Omega \psi_j^{p-1}v^{1-p}\mathcal{A}(x,\nabla v)\cdot\nabla\psi_j\,dx.$$

By the structure assumptions (A3) and (A4), we obtain from Hölder's inequality that

$$\begin{aligned}(p-1)\int_\Omega \alpha v^{-p}\psi_j^p|\nabla v|^p\,dx&\leq p\int_\Omega \psi_j^{p-1}v^{1-p}|\mathcal{A}(x,\nabla v)|\,|\nabla\psi_j|\,dx\\&\leq p\varepsilon_2\int_\Omega \psi_j^{p-1}v^{1-p}|\nabla v|^{p-1}|\nabla\psi_j|\,dx\\&\leq p\varepsilon_2\Big(\int_\Omega|\nabla v|^p v^{-p}\psi_j^p\,dx\Big)^{(p-1)/p}\Big(\int_\Omega|\nabla\psi_j|^p\,dx\Big)^{1/p}.\end{aligned}$$

It follows that

$$\left(\int_{B(x_0,R_j)} \alpha|\nabla \log v|^p\,dx\right)\left(\int_{B(x_0,R_j)} |\nabla \log v|^p\,dx\right)^{(1-p)/p} \tag{4.1}$$
$$\leq \frac{p\varepsilon_2}{p-1}\left(\int_{B(x_0,2R_j)} |\nabla \psi_j|^p\,dx\right)^{1/p}.$$

Assume that $|\nabla \log v|$ takes positive values in a set of positive measure in $B(x_0,R_j)$. Then (4.1) implies $|\nabla \log v| \in L^p(B(x_0,R_j))$. This holds trivially if $|\nabla \log v| = 0$ a.e. in $B(x_0,R_j)$. Hence we may apply [9, Lemma 2.13] to conclude that $\log v$ is weakly monotone in $B(x_0,R_j)$.

Define v pointwise by

$$v(z) = \liminf_{r\to 0} \frac{1}{|B(z,r)|}\int_{B(z,r)} v\,dx \quad \text{for all } z\in\Omega.$$

Then by Lemma 4.1, there is a set $E\subset(0,R_1)$ of linear measure zero such that $\log v$ is continuous in $S(x_0,R)$ for all $R\in(0,R_1)\setminus E$, $\operatorname{osc}_{S(x_0,R)}\log v$ is non-decreasing in $(0,R_1)\setminus E$, and that

$$\operatorname*{osc}_{S(x_0,R)} \log v = \operatorname*{osc}_{B(x_0,R)} \log v \quad \text{for } R\in(0,R_1)\setminus E. \tag{4.2}$$

We are free to assume that E is independent of k and that Sobolev's inequality in spheres holds for all $R\in(0,R_1)\setminus E$, that is

$$\operatorname*{osc}_{S(x_0,R)} (\log v)^p \leq c(n,p)R^{p-(n-1)}\int_{S(x_0,R)} |\nabla \log v|^p\,dS. \tag{4.3}$$

Letting

$$S(A_j) = \bigcup_{r\in A_j} S(x_0,r)$$

it follows from (4.1) that

$$\int_{S(A_j)} |\nabla \log v|^p\,dx \leq \frac{p\varepsilon_2}{(p-1)\varepsilon_1}\int_{B(x_0,2R_j)} |\nabla\psi_j|^p\,dx \leq c(n,p)\frac{\varepsilon_2}{\varepsilon_1}R_j^{n-p}. \tag{4.4}$$

By assumption (iii), there is $r_j\in\left[\frac{1}{2}\delta R_j,R_j\right]\setminus E$ such that

$$\left|\,]r_j,R_j[\,\cap A_j\right|_1 \geq \tfrac{1}{4}\delta R_j.$$

Hence integrating (4.3) over the set $[r_j, R_j]\cap A_j$ yields

$$\begin{aligned}\tfrac{1}{4}\delta R_j\Big(\operatorname*{osc}_{S(x_0,r_j)}\log v\Big)^p &\le \int_{[r_j,R_j]\cap A_j}\Big(\operatorname*{osc}_{S(x_0,t)}\log v\Big)^p\,dt\\ &\le \int_{[(\delta/2)R_j,R_j]\cap A_j}\Big(\operatorname*{osc}_{S(x_0,t)}\log v\Big)^p\,dt\\ &\le c(n,p)\int_{[(\delta/2)R_j,R_j]\cap A_j} t^{p+1-n}\Big(\int_{S(x_0,t)}|\nabla\log v|^p\,dS\Big)\,dt\\ &\le c(n,p)R_j^{\,p+1-n}\int_{S(A_j)}|\nabla\log v|^p\,dx.\end{aligned}$$

Here dS refers to integration with respect to the surface measure on $S(x_0,t)$. We arrive at

$$\Big(\operatorname*{osc}_{S(x_0,r_j)}\log v\Big)^p \le c(n,p)\frac{1}{\delta}R_j^{p-n}\int_{S(A_j)}|\nabla\log v|^p\,dx,$$

which together with (4.2) and (4.4) gives

$$\Big(\operatorname*{osc}_{B(x_0,r_j)}\log v\Big)^p = \Big(\operatorname*{osc}_{S(x_0,r_j)}\log v\Big)^p \le c(n,p)\frac{\varepsilon_2}{\delta\varepsilon_1}.$$

Hence

$$\log\Big(\frac{\sup_{B(x_0,r_j)}v}{\inf_{B(x_0,r_j)}v}\Big) = \operatorname*{osc}_{B(x_0,r_j)}\log v \le c(n,p)\Big(\frac{\varepsilon_2}{\delta\varepsilon_1}\Big)^{1/p}.$$

The assertion of the theorem follows by exponentiating the last inequality. □

The continuity of u at x_0 follows essentially the same way as in Theorem 3.1.

Theorem 4.3. *Let u be a solution of* (1.1) *in Ω such that α satisfies the assumptions of Lemma* 4.2 *at x_0. Then u is continuous at x_0.*

Proof. Using Lemma 4.2, we are able to proceed inductively and find a sequence $(r_j)_{j=1}^{\infty}$ decreasing to 0 such that

$$\sup_{B(x_0,r_{j+1})}u-\inf_{B(x_0,r_j)}u\le C\Big(\inf_{B(x_0,r_{j+1})}u-\inf_{B(x_0,r_j)}u\Big)$$

for all $j=1,2,\dots$. In fact, we may first choose any $r_1>0$ such that $B(x_0,2r_1)$ is compactly contained in Ω, and apply Lemma 4.2 to the non-negative solution $u-\inf_{B(x_0,r_1)}u$ of (1.1) in $B(x_0,r_1)$. By Lemma 4.2, there is a radius $r_2<r_1$ such that

$$\sup_{B(x_0,r_2)}u-\inf_{B(x_0,r_1)}u\le C\Big(\inf_{B(x_0,r_2)}u-\inf_{B(x_0,r_1)}u\Big).$$

We may continue this procedure step by step under the assumptions of Lemma 4.2. Now the continuity of u at x_0 follows in the same way as in Theorem 3.1. □

Remark 4.4. The local condition of Lemma 4.2 yields the following global condition in terms of the p-fine topology, see [3, Chapter 12]. Suppose that α is p-finely lower semicontinuous and satisfies $\alpha>0$ everywhere in Ω. Then the assumptions of Lemma 4.2 are satisfied for α at each point of Ω, see e.g. [10, Lemma 2.16] or [11]. Hence u is continuous in Ω by Theorem 4.3.

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