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H^1-boundedness of Riesz transforms and imaginary powers of the Laplacian on Riemannian manifolds

Michel Marias and Emmanuel Russ

Abstract. We prove that the linearized Riesz transforms and the imaginary powers of the Laplacian are H^1-bounded on complete Riemannian manifolds satisfying the doubling property and the Poincaré inequality, where H^1 denotes the Hardy space on M.

1. Introduction and statement of the results

Let M be a complete, noncompact Riemannian manifold. We denote by d the geodesic distance on M, by dx the Riemannian measure, by ∇ the Riemannian gradient and by Δ the Laplace–Beltrami operator. For all $x \in M$ and all $r>0$, let $B(x,r)$ be the open geodesic ball of radius r centered at x and $V(x,r)$ its volume.

Say that M satisfies the doubling property if there exists a positive constant C such that

$$V(x,2r) \le CV(x,r) \quad \text{for all } x \in M \text{ and } r>0. \tag{1.1}$$

If (1.1) holds, one easily sees that there exist $C, D>0$ such that for all $x \in M$, all $r>0$ and all $\theta>1$,

$$V(x,\theta r) \le C\theta^D V(x,r). \tag{1.2}$$

Say that the uniform L^2-Poincaré inequality holds on M if there exists a positive constant C such that, for all $x \in M$ and $r>0$,

$$\int_{B(x,r)} |f(x)-f_{B(x,r)}|^2\,dx \le Cr^2 \int_{B(x,2r)} \|\nabla f(x)\|^2\,dx \tag{1.3}$$

for all $f \in C^\infty(B(x,2r))$, where

$$f_{B(x,r)} = \frac{1}{V(x,r)} \int_{B(x,r)} f(y)\,dy.$$

It is well known (see [18]) that the conjunction of (1.1) and (1.3) implies the so-called Neumann–Poincaré inequality: there exists $C>0$ such that, for all $x\in M$ and $r>0$,

$$\int_{B(x,r)} |f(x)-f_{B(x,r)}|^2\,dx \le Cr^2 \int_{B(x,r)} \|\nabla f(x)\|^2\,dx \tag{1.4}$$

for all $f\in C^\infty(B(x,r))$.

Since M satisfies the doubling property (1.1), it is a space of homogeneous type. One may therefore consider the Hardy space $H^1(M)$ as defined in [9]. We briefly recall how $H^1(M)$ is defined. Say that a complex-valued function a on M is an atom if it is supported in a ball $B(y_0,r)$ and satisfies

$$\|a\|_2 \le \frac{1}{V(y_0,r)^{1/2}} \quad\text{and}\quad \int_M a(x)\,dx = 0. \tag{1.5}$$

A function f on M belongs to $H^1(M)$ if there exist $(\lambda_n)_{n\in\mathbf{N}}\in l^1$ and a sequence of atoms $(a_n)_{n\in\mathbf{N}}$ such that

$$f = \sum_{n\in\mathbf{N}} \lambda_n a_n, \tag{1.6}$$

where the series converges in $L^1(M)$. The norm $\|f\|_{H^1(M)}$ is the infimum of $\sum_{n\in\mathbf{N}} |\lambda_n|$ over all such decompositions.

A function u on M is said to be harmonic if $\Delta u=0$ on M. For $d=1,2,\dots$, denote by $\mathcal{H}_d(M)$ the space of harmonic functions on M of growth at most d. This means that $u\in\mathcal{H}_d(M)$ if u is harmonic and there exist $C>0$ and $p_0\in M$ so that $|u(x)|\le C(1+d(x,p_0))^d$ for all $x\in M$. Notice that the celebrated conjecture of Yau, which states that $\mathcal{H}_d(M)$ is finite dimensional, is solved by Li and Tam for $d=1$, [19], in the case when M has nonnegative Ricci curvature, and by Colding and Minicozzi for all $d\ge 1$ on manifolds satisfying the doubling volume property and the Neumann–Poincaré inequality, [10], Theorem 0.7.

The Riesz transform on M is the operator $R=\nabla\Delta^{-1/2}$. For $u\in\mathcal{H}_1(M)$, we define, as in [22], the linearized Riesz transform R_u by

$$R_u(f)(x) = \langle R(f)(x), \nabla u(x)\rangle = \langle \nabla\Delta^{-1/2} f(x), \nabla u(x)\rangle \tag{1.7}$$

for all $f\in C_0^\infty(M)$ and all $x\in M$, where $\langle\cdot,\cdot\rangle$ is the Riemannian inner product on the tangent space of M at x.

Our first result deals with the $H^1(M)$-boundedness of R_u.

Theorem 1. *Let M be a complete noncompact Riemannian manifold satisfying the doubling property* (1.1) *and the Poincaré inequality* (1.3). *Then for any $u\in\mathcal{H}_1(M)$, R_u extends to a bounded operator on $H^1(M)$.*

Our next result is about imaginary powers of the Laplace–Beltrami operator. For all $\beta\in\mathbf{R}$, the operator $\Delta^{i\beta}$ is defined via spectral theory, it is L^2-bounded and one has

$$\|\Delta^{i\beta}\|_{2\to 2}=1.$$

The following statement holds.

Theorem 2. *Assume that M is a complete noncompact Riemannian manifold satisfying the doubling property* (1.1) *and the Poincaré inequality* (1.3). *Then, there exists $C>0$ such that, for all $\beta\in\mathbf{R}$, $\Delta^{i\beta}$ is $H^1(M)$-bounded and*

$$\|\Delta^{i\beta}\|_{H^1\to H^1}\leq C\big(1+\sqrt{|\beta|}\,e^{\pi|\beta|/2}\big).$$

We first say a few words about the geometric context of these two results. Assumptions (1.1) and (1.3) are satisfied when M has nonnegative Ricci curvature. Indeed, by the Bishop comparison theorem (see [5]), M satisfies the doubling property. Also, in [6], P. Buser showed that these manifolds satisfy the Poincaré inequality. Recall that both (1.1) and (1.3) remain valid if M is quasi-isometric to a manifold with nonnegative Ricci curvature, or is a cocompact covering manifold whose deck transformation group has polynomial growth. [12]. Note that there exist manifolds satisfying (1.1) and (1.3) and whose Ricci curvature is not nonnegative.

First considered in $\mathbf{R}^n$, the issue of Riesz transforms on Riemannian manifolds has been raised in [32]. Note that Riesz transforms have been studied in various geometric contexts, such as Riemannian manifolds (see [20], [3]), Lie groups (see [21], [27], [1]), discrete groups (see [17]) and graphs (see [25], [26]). See [2] for an extended bibliography on the subject. Here, we concentrate on the case of Riemannian manifolds. Under very weak assumptions on M (namely, under (1.1) and an on-diagonal upper bound on the kernel of $e^{-t\Delta}$), it is proved in [11] that $|\nabla\Delta^{-1/2}|$ is L^p-bounded for all $1<p\leq 2$ and weak (1, 1). When M has nonnegative Ricci curvature, the Riesz transform is L^p-bounded for all $1<p<+\infty$ ([3]). Its H^1-L^1 boundedness is proved in [7] on Riemannian manifolds with nonnegative Ricci curvature, and in [26] under the assumptions of Theorem 1.

If one looks for an H^1-boundedness statement for Riesz transforms on manifolds, a new difficulty appears. Indeed, $\nabla\Delta^{-1/2}f$ is a vector-valued function, and it is not clear how to define a vector-valued H^1 space in this context. To overcome this difficulty, we take the scalar product of the Riesz transform with the gradient of a function in $\mathcal{H}_1(M)$. Thus, one obtains a scalar-valued operator, called the linearized

Riesz transform, which was first introduced in [22], where the H^1-boundedness of the linearized Riesz transform on Riemannian manifolds with nonnegative Ricci curvature is established.

In the Euclidean setting, the operators $\Delta^{i\beta}$ are H^1-bounded, (this is a consequence of the classical Calderón–Zygmund theory, see e.g. [31]). When M is a Riemannian manifold, a universal multiplier theorem of Stein ([30], Corollary 4, p. 121) shows that $\Delta^{i\beta}$ is $L^p(M)$-bounded for all $1<p<+\infty$. The H^1-L^1 boundedness of $\Delta^{i\beta}$ on Riemannian manifolds with nonnegative Ricci curvature is shown in [23]. For other geometric settings, see for example [8] and [29].

The proofs of Theorem 1 and Theorem 2 are similar. They go through a duality argument, which we quickly explain for R_u. In fact, to prove the H^1-boundedness of R_u, it is enough to show that there exists $C>0$ such that, for all atoms a and $\phi\in C_c(M)$,

$$\left|\int_M R_u a(x)\phi(x)\,dx\right| \le C\|\phi\|_{\mathrm{BMO}}. \tag{1.8}$$

(see Section 2.2 for the definition of BMO and further explanations).

To prove (1.8), one first introduces a truncated version $R_{u,\varepsilon}$, $\varepsilon>0$, of R_u and proves that, for all atoms a and all $\varepsilon>0$, $R_{u,\varepsilon}a\in L^1(M)$ and $R_{u,\varepsilon}a$ has integral 0 over M, which is a consequence of the harmonicity of u. Then, thanks to the L^2-boundedness of $R_{u,\varepsilon}$, weighted L^2-estimates for the gradient of the heat kernel (see Section 2.1) and some classical estimates for BMO functions, one proves (1.8) with $R_{u,\varepsilon}$ instead of R_u, with a constant $C>0$ independent of ε. Letting ε go to 0 yields (1.8).

The paper is organized as follows. In Section 2, we recall some known facts about the heat kernel of M (Subsection 2.1) and the BMO space on M (Subsection 2.2). In Section 3, we first prove that $R_{u,\varepsilon}a$ has integral 0 and then that (1.8) holds true. Finally, Theorem 2 is proved in Section 4.

Throughout this article the different constants will always be denoted by the same letter C. When their dependence or independence is significant, it will be clearly stated.

2. Preliminaries

2.1. Heat kernel estimates

In the sequel, we denote by p_t the heat kernel on M, i.e. the kernel of $e^{-t\Delta}$. Moreover, if y and y_0 are two fixed points in M, define, for all $x\in M$ and all $t>0$,

$$q_t(x)=p_t(x,y)-p_t(x,y_0).$$

When M satisfies (1.1) and (1.3), it is proved in [28] (see also [16]) that there exist $c_1, C_1, c_2, C_2>0$ such that, for all $x, y\in M$ and all $t>0$,

$$\frac{c_1}{V(x,\sqrt{t})}e^{-C_1 d^2(x,y)/t}\leq p_t(x,y)\leq \frac{C_2}{V(x,\sqrt{t})}e^{-c_2 d^2(x,y)/t}. \tag{2.1}$$

Moreover, the parabolic Harnack inequality holds on M (actually, it is proved in [28] that the conjunction of (1.1) and (1.3) is equivalent to the parabolic Harnack inequality on M, which is itself equivalent to (2.1)). As a consequence of this inequality, one easily obtains that p_t is Hölder continuous (see [26]).

Lemma 1. *There exist $c_3, C_3>0$ and $\gamma\in]0,1[$ such that, for all $x, y, y_0\in M$ and all $t>0$ satisfying $d(y,y_0)\leq\sqrt{t}$,*

$$|q_t(x)|\leq \frac{C_3}{V(x,\sqrt{t})}\left(\frac{d(y,y_0)}{\sqrt{t}}\right)^{\gamma}e^{-c_3 d^2(x,y)/t}.$$

As a consequence of Lemma 1, we have proved the following estimate in [26].

Lemma 2. *For any $\alpha<2c_3$, there exists $C'_\alpha>0$ such that, for all $y, y_0\in M$ and all $t>0$ satisfying $d(y,y_0)\leq\sqrt{t}$,*

$$\int_M |\nabla_x q_t(x)|^2 \exp\left(\alpha\frac{d^2(x,y)}{t}\right)dx\leq \frac{1}{t}\left(\frac{d(y_0,y)}{\sqrt{t}}\right)^{2\gamma}\frac{C'_\alpha}{V(y,\sqrt{t})}.$$

Recall (see [11]) that, as a consequence of the upper estimate of p_t in (2.1), one also has the following estimate.

Lemma 3. *There exists $\delta>0$ such that, for all $y\in M$ and all $t>0$,*

$$\int_M |\nabla_x p_t(x,y)t|^2 e^{\delta d^2(x,y)/t}\,dx\leq\frac{C}{tV(y,\sqrt{t})}.$$

2.2. A few facts about BMO

Say that a locally square integrable function ϕ on M is in $\mathrm{BMO}(M)$ if

$$\|\phi\|^2_{\mathrm{BMO}}=\sup\frac{1}{V(B)}\int_B|\phi(x)-\phi_B|^2\,dx<+\infty, \tag{2.2}$$

where $V(B)$ is the volume of the ball B and the supremum is taken over all the balls of M. Since M satisfies the doubling property (1.1), M is a space of homogeneous type and the general theory of BMO developed in [9] holds. Using (2.2), one proves the classical inequality

$$|\phi_B - \phi_{2B}| \le C\|\phi\|_{\mathrm{BMO}},$$

which yields, as in [14], p. 142, that there exists $C>0$ such that, for all $\phi\in\mathrm{BMO}(M)$, all $k\geq 1$ and all balls $B\subset M$,

$$\frac{1}{V(2^kB)}\int_{2^kB}|\phi(x)-\phi_{2B}|^2\,dx \le Ck^2\|\phi\|_{\mathrm{BMO}}^2. \tag{2.3}$$

Define $\mathrm{VMO}(M)$ as the closure in $\mathrm{BMO}(M)$ of $C_c(M)$, the space of continuous functions on M with compact support. In the sequel, we also use the fact that the dual of $H^1(M)$ is $\mathrm{BMO}(M)$ ([9], Theorem B, p. 593) and that, as a consequence, $H^1(M)$ itself is the dual of $\mathrm{VMO}(M)$ ([9], Theorem 4.1). The duality implies the following characterization of $H^1(M)$: $f\in H^1(M)$, if $f\in L^1(M)$ and if there exists $C>0$ such that, for all functions $\phi\in C_c(M)$,

$$\left|\int_M f(x)\phi(x)\,dx\right| \le C\|\phi\|_{\mathrm{BMO}}.$$

Furthermore, in this situation,

$$\|f\|_{H^1(M)} \le KC,$$

where $K>0$ only depends on M.

3. H^1-boundedness of R_u

For all $\varepsilon>0$, define the truncated operator $R_{u,\varepsilon}$ by

$$R_{u,\varepsilon}f(x) = \frac{1}{\sqrt{\pi}}\int_\varepsilon^{1/\varepsilon}\langle\nabla_x e^{-t\Delta}f(x),\nabla u(x)\rangle\,\frac{dt}{\sqrt{t}},\quad f\in C_0^\infty(M).$$

The following holds.

Lemma 4. *For all $f\in C_0^\infty(M)$,*

$$\lim_{\varepsilon\to 0} R_{u,\varepsilon}f = R_u f \quad \textit{in } L^2(M).$$

Proof. The proof relies on H^∞ calculus for Δ (see [24] and [33]). Fix $\mu \in]0, \frac{1}{2}\pi[$, and set

$$\Gamma_\mu = \{z \in \mathbf{C} : |\arg z| < \mu\}.$$

For all $z \in \Gamma_\mu$ and all $\varepsilon > 0$, define

$$\psi_\varepsilon(z) = \frac{1}{\sqrt{\pi}} \int_\varepsilon^{1/\varepsilon} e^{-tz} z^{1/2} \frac{dt}{\sqrt{t}}.$$

For any function $g \in \mathcal{D}(\Delta^{1/2})$ and all $\varepsilon > 0$, define

$$u_\varepsilon = \frac{1}{\sqrt{\pi}} \int_\varepsilon^{1/\varepsilon} e^{-t\Delta} \Delta^{1/2} g \frac{dt}{\sqrt{t}}.$$

so that

$$u_\varepsilon = \psi_\varepsilon(\Delta) g.$$

Observe that

$$\lim_{\varepsilon \to 0} \psi_\varepsilon(z) = 1$$

uniformly on all compact subsets of Γ_μ. By H^∞ functional calculus for Δ, one therefore has

$$\lim_{\varepsilon \to 0} \|\Delta^{1/2} u_\varepsilon - \Delta^{1/2} g\|_2 = 0.$$

The L^2-boundedness of $\nabla \Delta^{-1/2}$ yields

$$\begin{aligned} \lim_{\varepsilon \to 0} \|\nabla u_\varepsilon - \nabla g\|_2 &= \lim_{\varepsilon \to 0} \|\nabla \Delta^{-1/2} \Delta^{1/2} u_\varepsilon - \nabla \Delta^{-1/2} \Delta^{1/2} g\|_2 \\ &\le C \lim_{\varepsilon \to 0} \|\Delta^{1/2} u_\varepsilon - \Delta^{1/2} g\|_2 = 0. \end{aligned}$$

Applying this with $g = \Delta^{-1/2} f$, one obtains

$$\lim_{\varepsilon \to 0} \left\| \frac{1}{\sqrt{\pi}} \nabla \int_\varepsilon^{1/\varepsilon} e^{-t\Delta} f \frac{dt}{\sqrt{t}} - \nabla \Delta^{-1/2} f \right\|_2 = 0,$$

which proves the claim. □

Observe that the kernel of $R_{u,\varepsilon}$ is given by

$$\frac{1}{\sqrt{\pi}} \int_\varepsilon^{1/\varepsilon} \langle \nabla_x p_t(x,y), \nabla u(x) \rangle \frac{dt}{\sqrt{t}}.$$

As in [26], the L^2-boundedness of $R_{u,\varepsilon}$, Lemma 3 and Lemma 2 show that, for all $\varepsilon > 0$ and all atoms a, $R_{u,\varepsilon} a \in L^1(M)$. The computations are analogous to those in [26], except that the integrals with respect to t are computed on $]\varepsilon, 1/\varepsilon[$ instead of $]0, +\infty[$.

Using the fact that u is harmonic, we prove the following result.

Proposition 1. *For any atom a and any $\varepsilon>0$,*

$$\int_M R_{u,\varepsilon}a(x)\,dx=0. \tag{3.1}$$

Proof. The proof follows the lines of the one of the corresponding statement in [22], Section 4, and we give it for the sake of completeness. Let us assume that a is supported in $B(y_0,r)$. For $K\geq 2$, let $\phi_K(x)$ be a C^∞ function on M with $0\leq\phi_K\leq 1$, which is equal to 1 in $B(y_0,K-1)$ and to 0 outside $B(y_0,K+1)$, and satisfying $\|\nabla\phi_K\|_\infty\leq C$, where $C>0$ is an absolute constant. Define

$$I=\int_M |a(y)|\int_\varepsilon^{1/\varepsilon}\int_M |\langle\nabla_x p_t(x,y),\nabla u(x)\rangle|\,|\phi_K(x)|\,dx\,\frac{dt}{\sqrt{t}}\,dy.$$

By the boundedness of $|\nabla u|$ and the L^2-estimate of $\nabla_x p_t(x,y)$ given in Lemma 3, one has, for all $y\in M$ and all $t>0$,

$$\begin{aligned}
&\int_M |\langle\nabla_x p_t(x,y),\nabla u(x)\rangle|\,|\phi_K(x)|\,dx\\
&\quad\leq C\|\nabla u\|_\infty\left(\int_{B(y_0,K+1)}|\nabla_x p_t(x,y)|^2\,dx\right)^{1/2}V^{1/2}(y_0,K+1)\\
&\quad\leq C\|\nabla u\|_\infty\left(\int_{B(y_0,K+1)}e^{\delta d(x,y)^2/t}|\nabla_x p_t(x,y)|^2\,dx\right)^{1/2}V^{1/2}(y_0,K+1)\\
&\quad\leq C\|\nabla u\|_\infty\left(\frac{V(y_0,K+1)}{tV(y,\sqrt{t})}\right)^{1/2}.
\end{aligned}$$

This estimate and the fact that $\|a\|_1\leq 1$ yield

$$\begin{aligned}
I&\leq C\|\nabla u\|_\infty\int_{B(y_0,r)}|a(y)|\int_\varepsilon^{1/\varepsilon}\left(\frac{V(y_0,K+1)}{V(y,\sqrt{t})}\right)^{1/2}\frac{dt}{t}\,dy\\
&\leq C\max\left\{1,\left(\frac{K+1+r}{\sqrt{\varepsilon}}\right)^D\right\}\|\nabla u\|_\infty\int_{B(y_0,r)}|a(y)|\int_\varepsilon^{1/\varepsilon}\frac{dt}{t}\,dy<+\infty.
\end{aligned}\tag{3.2}$$

By the harmonicity of u and the Fubini theorem, applicable by (3.2), one obtains

$$\begin{aligned}
\int_M\phi_K(x)R_{u,\varepsilon}a(x)\,dx&=\frac{1}{\sqrt{\pi}}\int_M\phi_K(x)\int_\varepsilon^{1/\varepsilon}\int_M\langle\nabla_x p_t(x,y),\nabla u(x)\rangle a(y)\,dy\,\frac{dt}{\sqrt{t}}\,dx\\
&=\frac{1}{\sqrt{\pi}}\int_M a(y)\int_\varepsilon^{1/\varepsilon}\int_M\langle\nabla_x p_t(x,y),\nabla u(x)\rangle\phi_K(x)\,dx\,\frac{dt}{\sqrt{t}}\,dy\\
&=-\frac{1}{\sqrt{\pi}}\int_M a(y)\int_\varepsilon^{1/\varepsilon}\int_M\langle\nabla\phi_K(x),\nabla u(x)\rangle p_t(x,y)\,dx\,\frac{dt}{\sqrt{t}}\,dy.
\end{aligned}\tag{3.3}$$

Proposition 1 will therefore be a consequence of

$$\lim_{K\to+\infty}\int_M |a(y)|\int_\varepsilon^{1/\varepsilon}\int_M |\langle\nabla_x\phi_K(x),\nabla u(x)\rangle| p_t(x,y)\,dx\,\frac{dt}{\sqrt{t}}\,dy=0. \tag{3.4}$$

Indeed, if (3.4) is proved, then, since $R_{u,\varepsilon}a\in L^1(M)$, (3.3) and (3.4) yield

$$\int_M R_{u,\varepsilon}a(x)\,dx=\lim_{K\to+\infty}\int_M \phi_K(x)R_{u,\varepsilon}a(x)\,dx=0.$$

To prove (3.4), set

$$I_K=\int_M |a(y)|\int_\varepsilon^{1/\varepsilon}\int_M |\langle\nabla_x\phi_K(x),\nabla u(x)\rangle| p_t(x,y)\,dx\,\frac{dt}{\sqrt{t}}\,dy.$$

By the boundedness of $|\nabla u|$ and $|\nabla\phi_K|$, the fact that ϕ_K is equal to 1 on $B(y_0,K-1)$ and the upper bound in (2.1), one has

$$\begin{aligned}I_K&\le C\int_M |a(y)|\int_\varepsilon^{1/\varepsilon}\int_{B(y_0,K+1)\setminus B(y_0,K-1)} p_t(x,y)\,dx\,\frac{dt}{\sqrt{t}}\,dy\\&\le C\int_M |a(y)|\int_\varepsilon^{1/\varepsilon} e^{-c(K-1-r)^2/t}\frac{V(y_0,K+1)}{V(y,\sqrt{t})}\,\frac{dt}{\sqrt{t}}\,dy\\&\le C\|a\|_1\int_\varepsilon^{1/\varepsilon} e^{-c(K-1-r)^2/t}\max\left\{1,\left(\frac{K+1+r}{\sqrt{t}}\right)^D\right\}\frac{dt}{\sqrt{t}},\end{aligned}$$

which goes to zero when K goes to $+\infty$ by the dominated convergence theorem. □

Proof of Theorem 1. Let a be an atom supported in $B=B(y_0,r)$ and consider $\phi\in C_c(M)$ such that $\|\phi\|_{\mathrm{BMO}}\le1$. Proposition 1 yields, for all $\varepsilon>0$,

$$\int_M R_{u,\varepsilon}a(x)\phi(x)\,dx=\int_M R_{u,\varepsilon}a(x)(\phi(x)-\phi_{2B})\,dx.$$

Decompose $\phi-\phi_{2B}$ as

$$\phi-\phi_{2B}=(\phi-\phi_{2B})\chi_{2B}+(\phi-\phi_{2B})\chi_{(2B)^c}=\phi_1+\phi_2,$$

and write

$$\int_M R_{u,\varepsilon}a(x)\phi(x)\,dx=\int_M R_{u,\varepsilon}a(x)\phi_1(x)\,dx+\int_M R_{u,\varepsilon}a(x)\phi_2(x)\,dx=E_1+E_2. \tag{3.5}$$

By Cauchy–Schwarz and (2.2),

$$\int_M |R_{u,\varepsilon}a(x)|\,|\phi_1(x)|\,dx \le \|R_{u,\varepsilon}a\|_2\|(\phi-\phi_{2B})\chi_{2B}\|_2 \le \|R_{u,\varepsilon}a\|_2 V(2B)^{1/2}.$$

We now deal with the term involving ϕ_2 in (3.5). Observe that, since a has mean value zero, one has

$$\int_B a(y)\langle \nabla_x p_t(x,y), \nabla u(x)\rangle\,dy = \int_B a(y)\langle \nabla_x q_t(x), \nabla u(x)\rangle\,dy,$$

with $q_t(x)=p_t(x,y)-p_t(x,y_0)$. Write

$$\begin{aligned}
E_2 &= \int_M R_{u,\varepsilon}a(x)\phi_2(x)\,dx \\
&= \sum_{k\ge1}\int_{2^{k+1}B\setminus 2^kB} R_{u,\varepsilon}a(x)\phi_2(x)\,dx \\
&= \frac{1}{\sqrt{\pi}}\sum_{k\ge1}\int_{2^{k+1}B\setminus 2^kB}\phi_2(x)\int_\varepsilon^{r^2}\int_B \langle\nabla_x p_t(x,y),\nabla u(x)\rangle a(y)\,dy\,\frac{dt}{\sqrt{t}}\,dx \\
&\quad+\frac{1}{\sqrt{\pi}}\sum_{k\ge1}\int_{2^{k+1}B\setminus 2^kB}\phi_2(x)\int_{r^2}^{1/\varepsilon}\int_B \langle\nabla_x q_t(x),\nabla u(x)\rangle a(y)\,dy\,\frac{dt}{\sqrt{t}}\,dx \\
&= \sum_{k\ge1} I_k + \sum_{k\ge1} J_k.
\end{aligned}$$

Fix $k\ge1$. When $y\in B(y_0,r)$ and $2^kr\le d(x,y_0)<2^{k+1}r$, one has $2^{k-1}r\le d(x,y)<2^{k+2}r$, which implies

$$\begin{aligned}
|I_k| &\le C\int_{2^{k+1}B\setminus 2^kB}|\phi_2(x)|\int_\varepsilon^{r^2}\int_B |\langle\nabla_x p_t(x,y),\nabla u(x)\rangle|\,|a(y)|\,dy\,\frac{dt}{\sqrt{t}}\,dx \\
&\le C\|\nabla u\|_\infty\int_B|a(y)|\int_\varepsilon^{r^2}\int_{2^{k-1}r\le d(x,y)<2^{k+2}r}|\nabla_x p_t(x,y)|\,|\phi_2(x)|\,dx\,\frac{dt}{\sqrt{t}}\,dy.
\end{aligned}$$

The Cauchy–Schwarz inequality, Lemma 3, (2.3) and the doubling property yield

$$\begin{aligned}
&\int_{2^{k-1}r\le d(x,y)<2^{k+2}r}|\nabla_x p_t(x,y)|\,|\phi_2(x)|\,dx \\
&\qquad\le\left(\int_{2^{k-1}r\le d(x,y)<2^{k+2}r}|\nabla_x p_t(x,y)|^2e^{\delta d^2(x,y)/t}\,dx\right)^{1/2} \\
&\qquad\quad\times\left(\int_{2^{k-1}r\le d(x,y)<2^{k+2}r}|\phi_2(x)|^2e^{-\delta d^2(x,y)/t}\,dx\right)^{1/2}
\end{aligned}$$

$$\leq \frac{C(k+2)}{\sqrt{tV(y,\sqrt{t})}} V^{1/2}(y, 2^{k+2}r) e^{-\delta 2^{2(k-1)} r^2/2t}$$
$$\leq \frac{C}{\sqrt{t}} (k+2) \left(\frac{2^{k+2}r}{\sqrt{t}} \right)^{D/2} e^{-\delta 2^{2(k-1)} r^2/2t}$$
$$\leq \frac{C'}{\sqrt{t}} (k+2) e^{-\beta 2^{2k} r^2/t}.$$

Therefore,

$$\int_\varepsilon^{r^2} \int_{2^{k-1}r \leq d(x,y) < 2^{k+2}r} |\nabla_x p_t(x,y)|\, |\phi_2(x)|\, dx \frac{dt}{\sqrt{t}} \leq Ck \int_0^{r^2} e^{-\beta 2^{2k} r^2/t} \frac{dt}{t}$$
$$\leq Ck \int_{2^{2k}}^{+\infty} e^{-\beta t} \frac{dt}{t} \leq Ck 2^{-2k},$$

which shows that

$$\sum_{k\geq 1} |I_k| \leq C,$$

since $\|a\|_1 \leq 1$.

The treatment of J_k is similar. One has

$$|J_k| \leq C \|\nabla u\|_\infty \int_B |a(y)| \int_{r^2}^{1/\varepsilon} \int_{2^{k-1}r \leq d(x,y) < 2^{k+2}r} |\nabla q_t(x)|\, |\phi_2(x)|\, dx \frac{dt}{\sqrt{t}}\, dy.$$

When $t \leq 2^{2k+4} r^2$, use Lemma 2, (2.3) and the doubling property to write, for an $\alpha < c_3$,

$$\int_{2^{k-1}r \leq d(x,y) < 2^{k+1}r} |\nabla q_t(x)|\, |\phi_2(x)|\, dx$$
$$\leq \left(\int_{2^{k-1}r \leq d(x,y) < 2^{k+2}r} |\nabla q_t(x)|^2 e^{2\alpha d^2(x,y)/t}\, dx \right)^{1/2}$$
$$\times \left(\int_{2^{k-1}r \leq d(x,y) < 2^{k+2}r} |\phi_2(x)|^2 e^{-2\alpha d^2(x,y)/t}\, dx \right)^{1/2}$$
$$\leq \frac{C(k+2)}{\sqrt{tV(y,\sqrt{t})}} \left(\frac{r}{\sqrt{t}} \right)^{\gamma} e^{-\alpha 2^{2(k-1)} r^2/t} V^{1/2}(y, 2^{k+2}r)$$
$$\leq \frac{C}{\sqrt{t}} (k+2) \left(\frac{r}{\sqrt{t}} \right)^{\gamma} \left(\frac{2^{k+2}r}{\sqrt{t}} \right)^{D/2} e^{-\alpha 2^{2(k-1)} r^2/t}$$
$$\leq \frac{C}{\sqrt{t}} (k+2) \left(\frac{r}{\sqrt{t}} \right)^{\gamma} e^{-\beta 2^{2k} r^2/t}.$$

When $t>2^{2k+4}r^2$, just write that $V(y,2^{k+2}r)\leq V(y,\sqrt{t})$ and the result still holds.

As a consequence,

$$\begin{aligned}\int_{r^2}^{1/\varepsilon}\int_{2^{k-1}\leq d(x,y)<2^{k+2}r}|\nabla q_t(x)|\,|\phi_2(x)|\,dx\,\frac{dt}{\sqrt{t}}&\leq C(k+2)\int_{r^2}^{+\infty}e^{-\beta 2^{2k}r^2/t}\left(\frac{r}{\sqrt{t}}\right)^{\gamma}\frac{dt}{t}\\&\leq C(k+2)2^{-k\gamma}\int_0^{2^{2k}}e^{-\beta v}v^{\gamma/2-1}\,dv\\&\leq C(k+2)2^{-k\gamma}\int_0^{+\infty}e^{-\beta v}v^{\gamma/2-1}\,dv.\end{aligned}$$

Thus,

$$\sum_{k\geq 1}|J_k|\leq C.$$

using the fact that $\|a\|_1\leq 1$ again.

Finally, we have proved that, for all functions $\phi\in C_c(M)$,

$$\left|\int_M R_{u,\varepsilon}a(x)\phi(x)\,dx\right|\leq\|R_{u,\varepsilon}a\|_2V(2B)^{1/2}\|\phi\|_{\mathrm{BMO}}+C\|\phi\|_{\mathrm{BMO}}\tag{3.6}$$

for all $\varepsilon>0$. Since $R_{u,\varepsilon}a$ converges to R_ua in $L^2(M)$ when ε goes to 0, (3.6) yields

$$\left|\int_M R_ua(x)\phi(x)\,dx\right|\leq\|R_ua\|_2V(2B)^{1/2}\|\phi\|_{\mathrm{BMO}}+C\|\phi\|_{\mathrm{BMO}}\leq C\|\phi\|_{\mathrm{BMO}}.$$

In the last inequality, we use the L^2-boundedness of R_u, (1.1) and the fact that $\|a\|_2\leq V(B)^{-1/2}$. Therefore, (1.8) and Theorem 1 are proved. □

4. Imaginary powers of the Laplace operator

We now prove Theorem 2. The arguments are analogous to those used in the proof of Theorem 1.

For all $\beta\in\mathbf{R}$, set $T_\beta=\Delta^{i\beta}$. For all $\varepsilon>0$, define

$$T_{\beta,\varepsilon}=\frac{1}{\Gamma(-i\beta)}\int_\varepsilon^{1/\varepsilon}t^{-i\beta-1}e^{-t\Delta}\,dt.$$

For all $f\in L^2(M)$, $T_{\beta,\varepsilon}f$ converges to $T_\beta f$ in $L^2(M)$ by H^∞ functional calculus.

Fix $\beta\in\mathbf{R}$. One first proves the following result.

Proposition 2. *For all $\varepsilon>0$, $T_{\beta,\varepsilon}$ is $H^1(M)$-$L^1(M)$ bounded.*

Proof. Let a be an atom supported in a ball $B=B(y_0,r)$. Then, since $e^{-t\Delta}$ is a Markov semigroup, i.e.

$$\int_M p_t(x,y)\,dx=1 \tag{4.1}$$

for all $y\in M$ and all $t>0$ (see [15]), one has

$$\begin{aligned}\int_M |T_{\beta,\varepsilon}a(x)|\,dx &= \frac{1}{|\Gamma(-i\beta)|}\int_M\left|\int_\varepsilon^{1/\varepsilon} t^{-i\beta-1}\int_M p_t(x,y)a(y)\,dy\,dt\right|dx\\ &\leq C\int_M |a(y)|\int_\varepsilon^{1/\varepsilon}\int_M p_t(x,y)\,dx\,\frac{dt}{t}\,dy\\ &= C\|a\|_1\int_\varepsilon^{1/\varepsilon}\frac{1}{t}\,dt\\ &\leq C\int_\varepsilon^{1/\varepsilon}\frac{1}{t}\,dt\\ &= C(\varepsilon)\end{aligned} \tag{4.2}$$

by the fact that $\|a\|_1\leq 1$. □

We now state the following cancellation property (cf. Proposition 1).

Proposition 3. *For all $\varepsilon>0$ and all atoms a*

$$\int_M T_{\beta,\varepsilon}a(x)\,dx=0.$$

Proof. Let us assume that a is supported in a ball $B=B(y_0,r)$. From (4.2) we have that

$$\int_B |a(y)|\int_\varepsilon^{1/\varepsilon}\int_M p_t(x,y)\,dx\,\frac{dt}{t}\,dy<+\infty.$$

This allows us to apply Fubini and get

$$\begin{aligned}\int_M T_{\beta,\varepsilon}a(x)\,dx &= \frac{1}{\Gamma(-i\beta)}\int_M\int_B a(y)\int_\varepsilon^{1/\varepsilon} p_t(x,y)t^{-i\beta-1}\,dt\,dy\,dx\\ &= \frac{1}{\Gamma(-i\beta)}\int_\varepsilon^{1/\varepsilon} t^{-i\beta-1}\int_B a(y)\int_M p_t(x,y)\,dx\,dy\,dt\\ &= \frac{1}{\Gamma(-i\beta)}\int_\varepsilon^{1/\varepsilon} t^{-i\beta-1}\,dt\int_B a(y)\,dy\\ &=0,\end{aligned}$$

since (4.1) holds and a has mean value 0. □

Proof of Theorem 2. It is enough to show that there is a $C>0$, such that for all $\phi\in C_c(M)$ and all atoms a,

$$\left|\int_M T_\beta a(x)\phi(x)\,dx\right| \le \frac{C}{|\Gamma(-i\beta)|}\|\phi\|_{\mathrm{BMO}}.$$

Let a be an atom supported in $B=B(y_0,r)$ and $\phi\in C_c(M)$. Write

$$\phi-\phi_{2B}=(\phi-\phi_{2B})\chi_{2B}+(\phi-\phi_{2B})\chi_{(2B)^c}=\phi_1+\phi_2.$$

Then, for all $\varepsilon>0$,

$$\int_M T_{\beta,\varepsilon}a(x)\phi(x)\,dx=\int_{2B} T_{\beta,\varepsilon}a(x)\phi_1(x)\,dx+\int_{(2B)^c} T_{\beta,\varepsilon}a(x)\phi_2(x)\,dx=E_1+E_2.$$

The Cauchy–Schwarz inequality yields

$$\begin{aligned}|E_1| &\le \int_{2B} |T_{\beta,\varepsilon}a(x)\phi_1(x)|\,dx \le \|T_{\beta,\varepsilon}a\|_2\|\phi-\phi_{2B}\|_2 \\ &\le \|T_{\beta,\varepsilon}a\|_2 V^{1/2}(2B)\|\phi\|_{\mathrm{BMO}}.\end{aligned} \tag{4.3}$$

We now treat the term involving ϕ_2. We write

$$\begin{aligned}E_2 &= \int_{(2B)^c} T_{\beta,\varepsilon}a(x)\phi_2(x)\,dx \\ &= \frac{1}{\Gamma(-i\beta)}\sum_{k\ge1}\int_{2^{k+1}B\setminus 2^kB}\phi_2(x)\int_\varepsilon^{r^2} t^{-i\beta-1}\int_B p_t(x,y)a(y)\,dy\,dt\,dx \\ &\quad+\frac{1}{\Gamma(-i\beta)}\sum_{k\ge1}\int_{2^{k+1}B\setminus 2^kB}\phi_2(x)\int_{r^2}^{1/\varepsilon} t^{-i\beta-1}\int_B p_t(x,y)a(y)\,dy\,dt\,dx \\ &= \frac{1}{\Gamma(-i\beta)}\sum_{k\ge1}(I_k+J_k).\end{aligned}$$

By the estimates (2.1) of $p_t(x,y)$ we obtain that for all $k\ge1$, all $y\in B$ and all $0<t<r^2$,

$$\begin{aligned}\int_{2^{k+1}B\setminus 2^kB} p_t(x,y)|\phi_2(x)|\,dx &\le C\int_{2^{k-1}r\le d(x,y)\le 2^{k+2}r}\frac{e^{-cd(x,y)^2/t}}{V(y,\sqrt{t})}|\phi_2(x)|\,dx \\ &\le C\int_{d(x,y)\le 2^{k+2}r}\frac{e^{-c(2^{k-1}r)^2/t}}{V(y,\sqrt{t})}|\phi_2(x)|\,dx \\ &\le C(k+2)\frac{V(y,2^{k+2}r)}{V(y,\sqrt{t})}e^{-c2^{2k}r^2/t}\|\phi\|_{\mathrm{BMO}} \\ &\le C(k+2)e^{-c'2^{2k}r^2/t}\|\phi\|_{\mathrm{BMO}}.\end{aligned}$$

where the last line follows from the doubling property. Thus

$$
\begin{aligned}
|I_k| &\le \frac{C}{|\Gamma(-i\beta)|}(k+2)\|\phi\|_{\mathrm{BMO}} \int_B |a(y)|\,dy \int_\varepsilon^{r^2} \frac{1}{t} e^{-c2^{2k}r^2/t}\,dt \\
&\le \frac{C}{|\Gamma(-i\beta)|}(k+2)\|\phi\|_{\mathrm{BMO}} \int_\varepsilon^{r^2} \frac{1}{t} e^{-c2^{2k}r^2/t}\,dt \qquad (4.4)\\
&\le \frac{C}{|\Gamma(-i\beta)|}(k+2)2^{-2k}\|\phi\|_{\mathrm{BMO}},
\end{aligned}
$$

where $C>0$ only depends on M.

Let us now deal with J_k. Since a has mean value 0, one has

$$
\begin{aligned}
&\int_{2^{k+1}B\setminus 2^k B} \phi_2(x) \int_B p_t(x,y)a(y)\,dy\,dx \\
&\qquad = \int_{2^{k+1}B\setminus 2^k B} \phi_2(x) \int_B (p_t(x,y)-p_t(x,y_0))a(y)\,dy\,dx \\
&\qquad = \int_{2^{k+1}B\setminus 2^k B} \phi_2(x) \int_B q_t(x)a(y)\,dy\,dx.
\end{aligned}
$$

When $d(y,y_0)\le\sqrt{t}$, Lemma 1 yields

$$
|q_t(x)|\le C\left(\frac{r}{\sqrt{t}}\right)^\gamma \frac{1}{V(y_0,\sqrt{t})} e^{-cd^2(x,y_0)/t}.
$$

So, when $r<\sqrt{t}\le 2^{k+1}r$, one has

$$
\begin{aligned}
\int_{2^{k+1}B\setminus 2^k B} |q_t(x)|\,|\phi_2(x)|\,dx &\le C\left(\frac{r}{\sqrt{t}}\right)^\gamma \int_{2^{k+1}B\setminus 2^k B} |\phi_2(x)| \frac{e^{-c2^{2k}r^2/t}}{V(y_0,\sqrt{t})}\,dx \\
&\le C(k+1)\left(\frac{r}{\sqrt{t}}\right)^\gamma e^{-c2^{2k}r^2/t} \frac{V(2^{k+1}B)}{V(y_0,\sqrt{t})}\|\phi\|_{\mathrm{BMO}} \\
&\le C(k+1)\left(\frac{r}{\sqrt{t}}\right)^\gamma \left(\frac{2^{k+1}r}{\sqrt{t}}\right)^D e^{-c2^{2k}r^2/t}\|\phi\|_{\mathrm{BMO}} \\
&\le C'(k+1)\left(\frac{r}{\sqrt{t}}\right)^\gamma e^{-c'2^{2k}r^2/t}\|\phi\|_{\mathrm{BMO}}.
\end{aligned}
$$

When $\sqrt{t}\ge 2^{k+1}r$, just write that $V(2^{k+1}B)\le V(y_0,\sqrt{t})$ and the result still holds.

Therefore,

$$
\begin{aligned}
|J_k| &\le \frac{C}{|\Gamma(-i\beta)|}(k+1)\|\phi\|_{\mathrm{BMO}}\|a\|_1 \int_{r^2}^{+\infty} \left(\frac{r}{\sqrt{t}}\right)^\gamma e^{-c2^{2k}r^2/t}\,\frac{dt}{t} \\
&\le \frac{C}{|\Gamma(-i\beta)|}(k+1)2^{-k\gamma}\|\phi\|_{\mathrm{BMO}},
\end{aligned}
$$

which, combined with (4.4), gives,

$$\sum_{k\geq 1}|I_k|+|J_k|\leq \frac{C}{|\Gamma(-i\beta)|}\|\phi\|_{\mathrm{BMO}}\sum_{k\geq 1}(k2^{-2k}+k2^{-k\gamma})\leq \frac{c}{|\Gamma(-i\beta)|}\|\phi\|_{\mathrm{BMO}}. \tag{4.5}$$

From (4.3) and (4.5), it follows that, for all $\varepsilon>0$,

$$\left|\int_M T_{\beta,\varepsilon}a(x)\phi(x)\,dx\right|\leq C\|T_{\beta,\varepsilon}a\|_2V^{1/2}(2B)\|\phi\|_{\mathrm{BMO}}+\frac{C}{|\Gamma(-i\beta)|}\|\phi\|_{\mathrm{BMO}}.$$

Since $T_{\beta,\varepsilon}a$ converges to $T_\beta a$ in $L^2(M)$ when ε goes to 0, $\|T_\beta\|_{2\to 2}=1$ and $\|a\|_2\leq V(B)^{-1/2}$, one obtains

$$\left|\int_M T_\beta a(x)\phi(x)\,dx\right|\leq C\left(1+\frac{1}{|\Gamma(-i\beta)|}\right)\|\phi\|_{\mathrm{BMO}}.$$

which completes the proof of the H^1-boundedness of T_β.

Finally, recall that, for $\beta\in\mathbf{R}$,

$$|\Gamma(-i\beta)|=\sqrt{\frac{\pi}{\beta\sinh\pi\beta}}.$$

see [13]. Therefore,

$$\|\Delta^{i\beta}\|_{H^1\to H^1}\leq C\left(1+\sqrt{|\beta|}e^{\pi|\beta|/2}\right),$$

which completes the proof of Theorem 2. □

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Michel Marias
Aristotle University of Thessaloniki
GR-54006 Thessaloniki
Greece
email: marias@auth.gr

Emmanuel Russ
Université d'Aix-Marseille 3
FR-13397 Marseille Cedex 20
France
email: emmanuel.russ@univ.u-3mrs.fr