Ark. Mat., **41** (2003), 363–374

Banach spaces not containing l_1

Jorge Mujica

Dedicated to the memory of Klaus Floret (1941–2002).

Abstract. We show that if E is a complex Banach space which contains no subspace isomorphic to l_1, then each infinite dimensional subspace of E' contains a normalized sequence which converges to zero for the weak star topology.

The preceding result has been established by Hagler and Johnson [7] for real Banach spaces (see also [3, Chapter XII]). By adapting their proof we show that the result remains true in the case of complex Banach spaces. This result yields a very short proof of the Josefson–Nissenzweig theorem. We also give some applications to the study of the Schur property, the Dunford–Pettis property, and the separable quotient problem.

Before proving the main result we need some auxiliary lemmas. All unexplained terminology can be found in the book of Diestel [3]. For information on the Schur property or the Dunford–Pettis property we refer to Diestel's survey [2]. For information on the separable quotient problem we refer to the author's survey [10].

We recall that a sequence $(A_n, B_n)_{n=1}^{\infty}$ of pairs of disjoint nonempty subsets of a set S is said to be *independent* if for each sequence $(\theta_n)_{n=1}^{\infty} \in \{-1,1\}^{\mathbf{N}}$ and each $p \in \mathbf{N}$ we have that

$$\bigcap_{n=1}^{p} \theta_n A_n \neq \emptyset,$$

where

$$\theta_n A_n = \begin{cases} A_n, & \text{if } \theta_n = 1, \\ B_n, & \text{if } \theta_n = -1. \end{cases}$$

Then we have the following lemma, which is a variant of a result of Dor [4] (see also [3, Chapter XI]).

Lemma 1. *Let* $(f_n)_{n=1}^{\infty}$ *be a bounded sequence in* $l_\infty(S;\mathbf{C})$*. Let* C *and* D *be nonempty subsets of* $\mathbf{C}$ *such that*

$$\operatorname{Re}(z-w)\geq R \text{ and } |\operatorname{Im}(z-w)|\leq r \quad \text{for every } z\in C \text{ and } w\in D,$$

where $R>r>0$*. Let*

$$A_n=f_n^{-1}(C) \text{ and } B_n=f_n^{-1}(D) \quad \text{for every } n\in\mathbf{N}.$$

If the sequence $(A_n,B_n)_{n=1}^{\infty}$ *is independent, then* $(f_n)_{n=1}^{\infty}$ *is a basic sequence in* $l_\infty(S;\mathbf{C})$*, which is equivalent to the canonical Schauder basis of* l_1*.*

Proof. To prove the lemma we will show that

$$\left\|\sum_{j=1}^{n}\lambda_j f_j\right\|\geq\frac{R-r}{4}\sum_{j=1}^{n}|\lambda_j|$$

for every sequence $(\lambda_j)_{j=1}^{\infty}\subset\mathbf{C}$ and every $n\in\mathbf{N}$. Let $\alpha_j=\operatorname{Re}\lambda_j$ and $\beta_j=\operatorname{Im}\lambda_j$ for every j. Since $\left\|\sum_{j=1}^{n}\lambda_j f_j\right\|=\left\|\sum_{j=1}^{n}i\lambda_j f_j\right\|$, we may assume, without loss of generality, that

$$\sum_{j=1}^{n}|\alpha_j|\geq\sum_{j=1}^{n}|\beta_j|.$$

Let

$$P=\{j:1\leq j\leq n \text{ and } \alpha_j\geq 0\} \quad \text{and} \quad N=\{j:1\leq j\leq n \text{ and } \alpha_j<0\}.$$

Since the sequence $(A_j,B_j)_{j=1}^{\infty}$ is independent, we can find $s,t\in S$ such that

$$s\in\Big(\bigcap_{j\in P}A_j\Big)\cap\Big(\bigcap_{j\in N}B_j\Big) \quad \text{and} \quad t\in\Big(\bigcap_{j\in N}A_j\Big)\cap\Big(\bigcap_{j\in P}B_j\Big).$$

Thus $f_j(s)\in C$ and $f_j(t)\in D$ if $j\in P$, whereas $f_j(t)\in C$ and $f_j(s)\in D$ if $j\in N$. It follows that

$$\begin{aligned}\operatorname{Re}\sum_{j=1}^{n}\lambda_j f_j(s)-\operatorname{Re}\sum_{j=1}^{n}\lambda_j f_j(t)&=\sum_{j=1}^{n}\operatorname{Re}[\lambda_j(f_j(s)-f_j(t))]\\&=\sum_{j=1}^{n}[\alpha_j\operatorname{Re}(f_j(s)-f_j(t))-\beta_j\operatorname{Im}(f_j(s)-f_j(t))]\end{aligned}$$

$$= \sum_{j\in P} \alpha_j \operatorname{Re}(f_j(s)-f_j(t)) + \sum_{j\in N} \alpha_j \operatorname{Re}(f_j(s)-f_j(t))$$
$$- \sum_{j=1}^{n} \beta_j \operatorname{Im}(f_j(s)-f_j(t))$$
$$\geq \sum_{j\in P} |\alpha_j| R + \sum_{j\in N} |\alpha_j| R - \sum_{j=1}^{n} |\beta_j| r$$
$$= R\sum_{j=1}^{n} |\alpha_j| - r\sum_{j=1}^{n} |\beta_j|$$
$$\geq (R-r)\sum_{j=1}^{n} |\alpha_j|$$
$$\geq \frac{R-r}{2}\sum_{j=1}^{n} |\lambda_j|.$$

It follows that either

$$\operatorname{Re}\sum_{j=1}^{n} \lambda_j f_j(s) \geq \frac{R-r}{4}\sum_{j=1}^{n} |\lambda_j| \quad \text{or} \quad -\operatorname{Re}\sum_{j=1}^{n} \lambda_j f_j(t) \geq \frac{R-r}{4}\sum_{j=1}^{n} \lambda_j.$$

Hence

$$\left\|\sum_{j=1}^{n} \lambda_j f_j\right\| \geq \frac{R-r}{4}\sum_{j=1}^{n} |\lambda_j|,$$

as asserted. □

We recall that a sequence $(S_j)_{j=1}^{\infty}$ of nonempty subsets of a set S is said to be a *tree* if S_{2j} and S_{2j+1} are disjoint subsets of S_j for every $j\in\mathbf{N}$. The notion of tree was introduced by Pełczyński [13].

Lemma 2. *Let $(f_n)_{n=1}^{\infty}$ be a sequence in $l_\infty(S;\mathbf{K})$. Let C and D be disjoint nonempty subsets of $\mathbf{K}$, and let*

$$A_n = f_n^{-1}(C) \text{ and } B_n = f_n^{-1}(D) \quad \text{for every } n\in\mathbf{N}.$$

Suppose there exists a tree $(S_j)_{j=1}^{\infty}$ of subsets of S such that

$$S_j \subset A_n \quad \text{for } 2^n \leq j < 2^{n+1},\ n\geq 1,\ j \text{ even},$$

and

$$S_j \subset B_n \quad \text{for } 2^n \leq j < 2^{n+1},\ n\geq 1,\ j \text{ odd}.$$

Then the sequence $(A_n, B_n)_{n=1}^{\infty}$ is independent.

Proof. Let $\theta_1,\dots,\theta_p\in\{-1,1\}$ be given. Let T_1 be the unique S_j such that $2\leq j<2^2$ and $S_j\subset\theta_1 A_1$. Let T_2 be the unique S_j such that $2^2\leq j<2^3$ and $S_j\subset$

$T_1 \cap \theta_2 A_2$. In general, if $1<n\leq p$, let T_n be the unique S_j such that $2^n \leq j < 2^{n+1}$ and $S_j \subset T_{n-1} \cap \theta_n A_n$. It follows that

$$\bigcap_{n=1}^{p} \theta_n A_n \supset \bigcap_{n=1}^{p} T_n = T_p \neq \emptyset. \quad \square$$

Lemma 3. *Let $\delta>0$ and let $0<\varepsilon<\frac{1}{6}\delta$. Let*

$$C=\{z\in\mathbf{C}: |z|\leq \delta+\varepsilon \text{ and } \operatorname{Re} z \geq \delta-\varepsilon\}$$

and

$$D=\{w\in\mathbf{C}: |w|\leq \delta+\varepsilon \text{ and } \operatorname{Re} w \leq -\delta+\varepsilon\}.$$

Then

(a) $\operatorname{Re}(z-w)\geq 2(\delta-\varepsilon)$ *and* $|\operatorname{Im}(z-w)|\leq 4\sqrt{\delta\varepsilon}$;
(b) $2(\delta-\varepsilon)>4\sqrt{\delta\varepsilon}$.

Proof. Given $z\in C$ and $w\in D$, it is clear that

$$\operatorname{Re}(z-w)\geq 2(\delta-\varepsilon) \quad \text{and} \quad |\operatorname{Im}(z-w)|\leq 2h,$$

where $(\delta-\varepsilon, h)$ and $(\delta-\varepsilon, -h)$ are the points of intersection of the circle $x^2+y^2=(\delta+\varepsilon)^2$ and the vertical line $x=\delta-\varepsilon$. Hence $h=2\sqrt{\delta\varepsilon}$, and (a) follows. The inequality in (b) is equivalent to the inequality $\delta^2+\varepsilon^2>6\delta\varepsilon$, which is clearly verified if $0<\varepsilon<\frac{1}{6}\delta$. $\square$

Given nonempty subsets J and K of $\mathbf{N}$, we write $J<K$ if $j<k$ for every $j\in J$ and $k\in K$.

Let $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ be sequences in a Banach space E. We recall that $(y_n)_{n=1}^{\infty}$ is said to be a *block sequence* of $(x_n)_{n=1}^{\infty}$ if each y_n can be written as a linear combination

$$y_n = \sum_{j\in J_n} \alpha_j x_j,$$

where $J_1<J_2<J_3<\dots$. The block sequence $(y_n)_{n=1}^{\infty}$ is said to be *l_1-normalized* if $\sum_{j\in J_n}|\alpha_j|=1$ for every n. We write $(y_n)_{n=1}^{\infty}\prec(x_n)_{n=1}^{\infty}$ if $(y_n)_{n=1}^{\infty}$ is an l_1-normalized block sequence of $(x_n)_{n=1}^{\infty}$. The relation $\prec$ is reflexive and transitive.

Let $(\phi_n)_{n=1}^{\infty}$ be a bounded sequence in the dual E' of a Banach space E. Following Hagler and Johnson [7] we define

$$\delta((\phi_n)_{n=1}^{\infty}) = \sup_{x\in S_E} \limsup_{n\to\infty} |\phi_n(x)|.$$

Then one can readily verify that

$$\delta((\psi_n)_{n=1}^{\infty}) \leq \delta((\phi_n)_{n=1}^{\infty})$$

for each sequence $(\psi_n)_{n=1}^{\infty}\prec(\phi_n)_{n=1}^{\infty}$. By using a diagonal procedure, Hagler and Johnson [7] proved the following lemma, which can be found also in [3, p. 220] or [10, Lemma 4.6].

Lemma 4. ([7]) *Given a bounded sequence $(\phi_n)_{n=1}^{\infty} \subset E'$, there always exists a sequence $(\psi_n)_{n=1}^{\infty} \prec (\phi_n)_{n=1}^{\infty}$ such that*

$$\delta((\theta_n)_{n=1}^{\infty}) = \delta((\psi_n)_{n=1}^{\infty})$$

for each sequence $(\theta_n)_{n=1}^{\infty} \prec (\psi_n)_{n=1}^{\infty}$.

We remark that Hagler and Johnson [7] deal with real Banach spaces, and their definition of the function δ is slightly different from the one presented here. Our definition of the function δ coincides with that in [3] and [10], and the proof of Lemma 4 in [3, p. 220] and [10, Lemma 4.6] is valid for real and complex Banach spaces.

Now we can prove the following theorem, which is due to Hagler and Johnson [7] in the case of real Banach spaces (see also [3, Chapter XII]).

Theorem 5. *Let E be a Banach space, whose dual E' contains a bounded sequence $(\phi_n)_{n=1}^{\infty}$ such that $\psi_n \not\to 0$ for the topology $\sigma(E', E)$ for each sequence $(\psi_n)_{n=1}^{\infty} \prec (\phi_n)_{n=1}^{\infty}$. Then E contains a subspace isomorphic to l_1.*

Proof. Without loss of generality we may assume that $(\phi_n)_{n=1}^{\infty} \subset B_{E'}$. By Lemma 4 there exists a sequence $(\psi_n)_{n=1}^{\infty} \prec (\phi_n)_{n=1}^{\infty}$ such that

$$\delta((\theta_n)_{n=1}^{\infty}) = \delta((\psi_n)_{n=1}^{\infty})$$

for each sequence $(\theta_n)_{n=1}^{\infty} \prec (\psi_n)_{n=1}^{\infty}$. It follows from the hypothesis that $\psi_n(x) \not\to 0$ for some $x \in S_E$, and therefore $\delta((\psi_n)_{n=1}^{\infty}) > 0$. Let $\delta = \delta((\psi_n)_{n=1}^{\infty})$ and let $0 < \varepsilon < \frac{1}{6}\delta$.

Let $0 < \varepsilon' < \frac{1}{5}\varepsilon$, let $N_1 = \mathbf{N}$, and let $A = (m_j)_{j=1}^{\infty}$ and $B = (n_j)_{j=1}^{\infty}$ be infinite subsets of N_1 such that $m_j < n_j < m_{j+1}$ for every j. Since $\frac{1}{2}(\psi_{m_j} - \psi_{n_j})_{j=1}^{\infty} \prec (\psi_n)_{n=1}^{\infty}$, it follows that

$$\delta\left(\tfrac{1}{2}(\psi_{m_j} - \psi_{n_j})_{j=1}^{\infty}\right) = \delta((\psi_n)_{n=1}^{\infty}) = \delta.$$

Hence, there exists $x_1 \in S_E$ and an infinite subset $J \subset \mathbf{N}$ such that

$$\left|\tfrac{1}{2}(\psi_{m_j} - \psi_{n_j})(x_1)\right| \geq \delta - \varepsilon' \quad \text{for every } j \in J.$$

Hence the sequence $\left(\frac{1}{2}(\psi_{m_j} - \psi_{n_j})(x_1)\right)_{j=1}^{\infty}$ admits a cluster point $c_1 \in \mathbf{C}$, with $|c_1| \geq \delta - \varepsilon'$. After multiplying x_1 by a suitable $\gamma \in \mathbf{C}$, with $|\gamma| = 1$, we may assume that c_1 is real, and $c_1 \geq \delta - \varepsilon'$. After passing to a subsequence we may assume that

$$\tfrac{1}{2}(\psi_{m_j} - \psi_{n_j})(x_1) \in \Delta(c_1; \varepsilon'),$$

and therefore

$$\tfrac{1}{2}\operatorname{Re}(\psi_{m_j} - \psi_{n_j})(x_1) \geq \delta - 2\varepsilon' \quad \text{for every } j \in J.$$

Since $(\psi_{m_j})_{j=1}^{\infty} \prec (\psi_n)_{n=1}^{\infty}$ and $(\psi_{n_j})_{j=1}^{\infty} \prec (\psi_n)_{n=1}^{\infty}$, it follows that

$$\delta((\psi_{m_j})_{j=1}^{\infty}) = \delta((\psi_{n_j})_{j=1}^{\infty}) = \delta((\psi_n)_{n=1}^{\infty}) = \delta,$$

and therefore

$$\limsup_{j\to\infty} |\psi_{m_j}(x_1)| \le \delta \quad \text{and} \quad \limsup_{j\to\infty} |\psi_{n_j}(x_1)| \le \delta.$$

Hence there exists $j_0 \in J$ such that

$$|\psi_{m_j}(x_1)| \le \delta + \varepsilon' \text{ and } |\psi_{n_j}(x_1)| \le \delta + \varepsilon' \quad \text{for every } j \in J,\ j \ge j_0.$$

We claim that

$$\operatorname{Re} \psi_{m_j}(x_1) \ge \delta - 5\varepsilon' \text{ and } \operatorname{Re} \psi_{n_j}(x_1) \le -\delta + 5\varepsilon' \quad \text{for every } j \in J,\ j \ge j_0.$$

Indeed

$$\begin{aligned}\operatorname{Re} \psi_{m_j}(x_1) &= \operatorname{Re}(\psi_{m_j} - \psi_{n_j})(x_1) + \operatorname{Re} \psi_{n_j}(x_1) \\ &\ge \operatorname{Re}(\psi_{m_j} - \psi_{n_j})(x_1) - |\psi_{n_j}(x_1)| \ge 2(\delta - 2\varepsilon') - (\delta + \varepsilon') = \delta - 5\varepsilon',\end{aligned}$$

and the other inequality is proved similarly. Thus we have found two disjoint infinite subsets N_2 and N_3 of N_1 such that

$$\begin{aligned} |\psi_n(x_1)| &\le \delta + \varepsilon && \text{for every } n \in N_2 \cup N_3, \\ \operatorname{Re} \psi_n(x_1) &\ge \delta - \varepsilon && \text{for every } n \in N_2, \\ \operatorname{Re} \psi_n(x_1) &\le -\delta + \varepsilon && \text{for every } n \in N_3. \end{aligned}$$

Let $0 < \varepsilon'' < \frac{1}{11}\varepsilon$. Let $P = (p_k)_{k=1}^{\infty}$, $Q = (q_k)_{k=1}^{\infty}$, $R = (r_k)_{k=1}^{\infty}$ and $S = (s_k)_{k=1}^{\infty}$ be infinite subsets of $\mathbf{N}$ such that $P \cup Q \subset N_2$, $R \cup S \subset N_3$ and $p_k < q_k < r_k < s_k < p_{k+1}$ for every k. Since $\frac{1}{4}(\psi_{p_k} - \psi_{q_k} + \psi_{r_k} - \psi_{s_k})_{k=1}^{\infty} \prec (\psi_n)_{n=1}^{\infty}$, it follows that

$$\delta\big(\tfrac{1}{4}(\psi_{p_k} - \psi_{q_k} + \psi_{r_k} - \psi_{s_k})_{k=1}^{\infty}\big) = \delta((\psi_n)_{n=1}^{\infty}) = \delta.$$

Hence the preceding argument yields a point $x_2 \in S_E$ and an infinite set $K \subset \mathbf{N}$ such that

$$\tfrac{1}{4} \operatorname{Re}(\psi_{p_k} - \psi_{q_k} + \psi_{r_k} - \psi_{s_k})(x_2) \ge \delta - 2\varepsilon'' \quad \text{for every } k \in K.$$

The preceding argument yields also $k_0 \in K$ such that

$$\begin{aligned} |\psi_{p_k}(x_2)| &\le \delta + \varepsilon'', & |\psi_{q_k}(x_2)| &\le \delta + \varepsilon'', \\ |\psi_{r_k}(x_2)| &\le \delta + \varepsilon'' \quad \text{and} & |\psi_{s_k}(x_2)| &\le \delta + \varepsilon'' \end{aligned}$$

for every $k\in K$, $k\geq k_0$. As before we can prove that

$$\begin{aligned}\operatorname{Re}\psi_{p_k}(x_2)&\geq\delta-11\varepsilon'', &\operatorname{Re}\psi_{q_k}(x_2)&\leq-\delta+\varepsilon'',\\ \operatorname{Re}\psi_{r_k}(x_2)&\geq\delta-11\varepsilon'' \quad\text{and} &\operatorname{Re}\psi_{s_k}(x_2)&\leq-\delta+11\varepsilon''\end{aligned}$$

for every $k\in K$, $k\geq k_0$. Thus we have found two disjoint infinite subsets N_4 and N_5 of N_2 and two disjoint infinite subsets N_6 and N_7 of N_3 such that

$$\begin{aligned}|\psi_n(x_2)|&\leq\delta+\varepsilon &&\text{for every } n\in N_4\cup N_5\cup N_6\cup N_7,\\ \operatorname{Re}\psi_n(x_2)&\geq\delta-\varepsilon &&\text{for every } n\in N_4\cup N_6,\\ \operatorname{Re}\psi_n(x_2)&\leq-\delta+\varepsilon &&\text{for every } n\in N_5\cup N_7.\end{aligned}$$

Let

$$C=\{z\in\mathbf{C}:|z|\leq\delta+\varepsilon \text{ and } \operatorname{Re}z\geq\delta-\varepsilon\}$$

and

$$D=\{w\in\mathbf{C}:|w|\leq\delta+\varepsilon \text{ and } \operatorname{Re}w\leq-\delta+\varepsilon\}.$$

Proceeding by induction we can find a sequence $(x_n)_{n=1}^{\infty}\subset S_E$, and a tree $(N_j)_{j=1}^{\infty}$ of subsets of $\mathbf{N}$ such that

$$\psi_k(x_n)\in C \quad \text{for } k\in N_j,\ 2^n\leq j<2^{n+1},\ n\geq1,\ j \text{ even},$$

and

$$\psi_k(x_n)\in D \quad \text{for } k\in N_j,\ 2^n\leq j<2^{n+1},\ n\geq2,\ j \text{ odd}.$$

Let

$$A_n=\{\psi\in B_{E'}:\psi(x_n)\in C\} \text{ and } B_n=\{\psi\in B_{E'}:\psi(x_n)\in D\} \quad \text{for every } n\in\mathbf{N},$$

and let

$$\Psi_j=\{\psi_k:k\in N_j\} \quad \text{for every } j\in\mathbf{N}.$$

Then $(\Psi_j)_{j=1}^{\infty}$ is a tree of subsets of $B_{E'}$ such that

$$\Psi_j\subset A_n \quad \text{for } 2^n\leq j<2^{n+1},\ n\geq1,\ j \text{ even},$$

and

$$\Psi_j\subset B_n \quad \text{for } 2^n\leq j<2^{n+1},\ n\geq1,\ j \text{ odd}.$$

By Lemma 2 the sequence $(A_n,B_n)_{n=1}^{\infty}$ is independent. By Lemma 3 the sets C and D verify the hypothesis of Lemma 1. Thus it follows from Lemma 1 that $(x_n)_{n=1}^{\infty}$ is a basic sequence in E which is equivalent to the canonical Schauder basis of l_1. □

Theorem 5, together with the Rosenthal–Dor theorem (see [16], [4] or [3, Chapter XI]), yields the following corollary.

Corollary 6. *Let E be a Banach space which contains no subspace isomorphic to l_1. Then*

(a) *each infinite dimensional subspace M of E contains a normalized sequence which converges to zero for $\sigma(E, E')$;*

(b) *each infinite dimensional subspace N of E' contains a normalized sequence which converges to zero for $\sigma(E', E)$.*

Proof. (a) By the Riesz lemma there exists a sequence $(x_n)_{n=1}^{\infty} \subset S_M$ such that $\|x_n - x_m\| \geq \frac{1}{2}$ whenever $n \neq m$. By the Rosenthal–Dor theorem we may assume that $(x_n)_{n=1}^{\infty}$ is $\sigma(E, E')$-Cauchy. Hence $x_{n+1} - x_n \to 0$ for $\sigma(E, E')$ and $\|x_{n+1} - x_n\| \geq \frac{1}{2}$ for every n. Thus it suffices to normalize the sequence $(x_{n+1} - x_n)_{n=1}^{\infty}$.

(b) Let N be an infinite dimensional subspace of E', and let $(\phi_n)_{n=1}^{\infty}$ be a sequence in S_N such that $\|\phi_n - \phi_m\| \geq \frac{1}{2}$ whenever $n \neq m$. By the Rosenthal–Dor theorem we may assume that either $(\phi_n)_{n=1}^{\infty}$ is $\sigma(E', E'')$-Cauchy, or else $(\phi_n)_{n=1}^{\infty}$ is a basic sequence which is equivalent to the canonical Schauder basis of l_1.

In the first case $\phi_{n+1} - \phi_n \to 0$ for $\sigma(E', E'')$ and $\|\phi_{n+1} - \phi_n\| \geq \frac{1}{2}$ for every n. Thus it suffices to normalize the sequence $(\phi_{n+1} - \phi_n)_{n=1}^{\infty}$.

In the second case, by Theorem 5 there exists a sequence $(\psi_n)_{n=1}^{\infty} \prec (\phi_n)_{n=1}^{\infty}$ which converges to zero for $\sigma(E', E)$. Since $(\phi_n)_{n=1}^{\infty}$ is equivalent to the canonical Schauder basis of l_1, so is $(\psi_n)_{n=1}^{\infty}$. Hence $\|\psi_n\| > a > 0$ for every n, and it suffices to normalize the sequence $(\psi_n)_{n=1}^{\infty}$. □

Corollary 7. *Let E be a Banach space whose dual E' contains no subspace isomorphic to l_1. Then*

(a) *each infinite dimensional subspace M of E contains a normalized sequence which converges to zero for $\sigma(E, E')$;*

(b) *each infinite dimensional subspace N of E' contains a normalized sequence which converges to zero for $\sigma(E', E'')$.*

Proof. By applying Corollary 6(b) to E', we obtain (a). By applying Corollary 6(a) to E' we obtain (b). □

Corollary 6(a) and Corollary 7(b) rely only on the Rosenthal–Dor theorem, and are probably known, but we have included them here for the sake of completeness.

Corollary 6(b) yields a very short proof of the Josefson–Nissenzweig theorem (see [9], [11] or [3, Chapter XII]).

Theorem 8. ([9], [11]) *If E is an infinite dimensional Banach space, then E' contains a normalized sequence which converges to zero for $\sigma(E', E)$.*

Proof. If E contains no subspace isomorphic to l_1, then the conclusion is a direct consequence of Corollary 6(b). If E contains a subspace isomorphic to l_1, then

E admits a quotient isomorphic to l_2, by a result of Aron, Diestel and Rajappa [1] (when E is separable, this follows also from a result of Pelczyński [13]). Since l_2 is separable, the desired conclusion follows at once. □

We recall that a Banach space E is said to have the *Schur property* if weakly convergent sequences in E are norm convergent. From Corollary 6(a) and Corollary 7(a) we immediately get the following result.

Corollary 9. *Let E be a Banach space which contains a closed, infinite dimensional subspace with the Schur property. Then both E and E' contain subspaces isomorphic to l_1.*

Since l_1 has the Schur property, we immediately obtain the following corollary.

Corollary 10. *If a Banach space E contains a subspace isomorphic to l_1, then E' also contains a subspace isomorphic to l_1.*

Pethe and Thakare [14] have shown that *if E contains a subspace isomorphic to l_1, then E' does not have the Schur property* (the argument of the proof of Theorem 8 can be used also to prove this). Using this we obtain the following result.

Corollary 11. *If E is an infinite dimensional Banach space with the Schur property, then none of the upper duals $E^{(n)}$ $(n\geq 1)$ have the Schur property.*

Proof. By Corollaries 9 and 10, $E^{(n)}$ has a subspace isomorphic to l_1 for each $n\geq 0$. By the aforementioned result of Pethe and Thakare, $E^{(n)}$ does not have the Schur property for each $n\geq 1$. □

We recall that a Banach space E is said to have the *Dunford–Pettis property* if given weakly null sequences $(x_n)_{n=1}^\infty \subset E$ and $(\phi_n)_{n=1}^\infty \subset E'$, one has that $\phi_n(x_n)\to 0$. It is known that if E has the Dunford–Pettis property, $(x_n)_{n=1}^\infty \subset E$ and $(\phi_n)_{n=1}^\infty \subset E'$, then $\phi_n(x_n)\to 0$ whenever one of the two sequences is weakly null, and the other is weakly Cauchy.

Theorem 12. *The dual E' of a Banach space E contains a subspace isomorphic to l_1 whenever E or E' contains a closed, infinite dimensional subspace with the Dunford–Pettis property.*

Proof. Suppose E' contains no subspace isomorphic to l_1, and let M and N be closed, infinite dimensional subspaces of E and E', respectively. We will show that neither M nor N has the Dunford–Pettis property.

By Corollary 7(a) there exists a sequence $(x_n)_{n=1}^\infty \subset S_M$ which converges to zero for $\sigma(E,E')$, and therefore for $\sigma(M,M')$. Let $(\phi_n)_{n=1}^\infty \subset S_{E'}$ be such that $\phi_n(x_n)=1$ for every n. By the Rosenthal–Dor theorem we may assume that $(\phi_n)_{n=1}^\infty$

is $\sigma(E', E'')$-Cauchy, and therefore $\sigma(M', M'')$-Cauchy. Hence M does not have the Dunford–Pettis property.

By Corollary 7(b) there exists a sequence $(\psi_n)_{n=1}^{\infty} \subset S_N$ which converges to zero for $\sigma(E', E'')$, and therefore for $\sigma(N, N')$. Let $(y_n)_{n=1}^{\infty} \subset S_E$ be such that $\psi_n(y_n) \geq \frac{1}{2}$ for every n. By Corollary 10, E contains no subspace isomorphic to l_1. By the Rosenthal–Dor theorem we may assume that $(y_n)_{n=1}^{\infty}$ is $\sigma(E, E')$-Cauchy. If $J: E \hookrightarrow E''$ denotes the canonical embedding, then $(Jy_n)_{n=1}^{\infty}$ is $\sigma(E'', E''')$-Cauchy, and therefore $\sigma(N', N'')$-Cauchy. Hence N does not have the Dunford–Pettis property. □

Theorem 12 improves a result of Fakhoury [5, Corollary 12], which asserts that *if E is an infinite dimensional Banach space with the Dunford–Pettis property, then E' contains a subspace isomorphic to l_1*. A closely related result of Rosenthal [17, Corollary 2] asserts that *a Banach space E contains a subspace isomorphic to l_1 whenever E' contains a closed subspace N which has the Dunford–Pettis property, but does not have the Schur property.*

To end this paper we show the connection between Theorem 5 and the separable quotient problem. The question of whether every infinite dimensional Banach space has an infinite dimensional quotient with a Schauder basis was raised by Pełczyński [12] in 1964. The question of whether every infinite dimensional Banach space has a separable, infinite dimensional quotient was raised by Rosenthal [15] in 1969. Since Johnson and Rosenthal [8, Theorem IV.1(i)] proved that every separable, infinite dimensional Banach space has an infinite dimensional quotient with a Schauder basis, it follows that Pełczyński's question and Rosenthal's question are equivalent. The following theorem completes results of Johnson and Rosenthal [8] and Hagler and Johnson [7].

Theorem 13. *Let E be a Banach space.*

(a) *If E' contains an infinite dimensional subspace with a separable dual, then E has an infinite dimensional quotient with a boundedly complete Schauder basis.*

(b) *If E' contains a subspace isomorphic to l_1, then E has a quotient isomorphic to c_0 or l_2. In particular, E has an infinite dimensional quotient with a shrinking Schauder basis.*

Proof. (a) is due to Johnson and Rosenthal [8, Theorem IV.1(iii)].

(b) We distinguish two cases.

(i) Suppose that E' contains a basic sequence $(\phi_n)_{n=1}^{\infty}$, which is equivalent to the canonical Schauder basis of l_1, and which converges to zero for $\sigma(E', E)$. Then E has a quotient isomorphic to c_0, by a result of Johnson and Rosenthal [8, Remark III.1].

(ii) Suppose that E' contains a subspace isomorphic to l_1, but $\psi_n \not\to 0$ for $\sigma(E', E)$ for each sequence $(\psi_n)_{n=1}^{\infty}$ which is equivalent to the canonical Schauder basis of l_1. Hence there is a basic sequence $(\phi_n)_{n=1}^{\infty}$ in E' which is equivalent to the canonical Schauder basis of l_1, but $\psi_n \not\to 0$ for $\sigma(E', E)$ for each sequence $(\psi_n)_{n=1}^{\infty} \prec (\phi_n)_{n=1}^{\infty}$. Then E has a subspace isomorphic to l_1, by Theorem 5. Thus E has a quotient isomorphic to l_2, by the aforementioned result of Aron, Diestel and Rajappa [1]. □

Case (ii) is due to Hagler and Johnson [7] in the case of real Banach spaces.

Not every Banach space verifies the hypotheses in Theorem 13. Indeed Gowers [6] has constructed an infinite dimensional Banach space E whose dual contains no infinite dimensional subspace with a separable dual, and no subspace isomorphic to l_1. In particular, E' contains no infinite dimensional subspace which is either reflexive or isomorphic to c_0 or l_1. This answered negatively a question raised by Rosenthal [17].

From Theorem 12 and Theorem 13(b) we immediately obtain the following corollary.

Corollary 14. *If E or E' contains a closed, infinite dimensional subspace with the Dunford–Pettis property, then E has a quotient isomorphic to c_0 or l_2. In particular, E has an infinite dimensional quotient with a shrinking Schauder basis.*

References

1. Aron, R., Diestel, J. and Rajappa, A., Weakly continuous functions on Banach spaces containing l_1, in *Banach Spaces (Columbia, Mo., 1984)* (Kalton, N. and Saab, E., eds.), Lecture Notes in Math. **1166**, pp. 1–3, Springer-Verlag, Berlin–Heidelberg, 1985.
2. Diestel, J., A survey of results related to the Dunford–Pettis property, in *Integration, Topology and Geometry in Linear Spaces (Chapel Hill, N. C., 1979)* (Graves, W., ed.), Contemporary Math. **2**, pp. 15–60, Amer. Math. Soc., Providence, R. I., 1980.
3. Diestel, J., *Sequences and Series in Banach Spaces*, Springer-Verlag, New York, 1984.
4. Dor, L., On sequences spanning a complex l_1 space, *Proc. Amer. Math. Soc.* **47** (1975), 461–465.
5. Fakhoury, H., Sur les espaces de Banach ne contenant pas $l^1(\mathbf{N})$, *Math. Scand.* **41** (1977), 277–289.
6. Gowers, W., A Banach space not containing c_0, l_1 or a reflexive subspace, *Trans. Amer. Math. Soc.* **344** (1994), 407–420.
7. Hagler, J. and Johnson, W., On Banach spaces whose dual balls are not weak* sequentially compact, *Israel J. Math.* **28** (1977), 325-330.

8. JOHNSON, W. and ROSENTHAL, H., On w^*-basic sequences and their applications to the study of Banach spaces, *Studia Math.* **43** (1972), 77–92.
9. JOSEFSON, B., Weak sequential convergence in the dual of a Banach space does not imply norm convergence, *Ark. Mat.* **13** (1975), 79–89.
10. MUJICA, J., Separable quotients of Banach spaces, *Rev. Mat. Univ. Complut. Madrid* **10** (1997), 299–330.
11. NISSENZWEIG, A., w* sequential convergence, *Israel J. Math.* **22** (1975), 266–272.
12. PELCZYŃSKI, A., Some problems on bases in Banach and Fréchet spaces, *Israel J. Math.* **2** (1964), 132–138.
13. PELCZYŃSKI, A., On Banach spaces containing $L_1(\mu)$, *Studia Math.* **30** (1968), 231–246.
14. PETHE, P. and THAKARE, N., Note on Dunford–Pettis property and Schur property, *Indiana Univ. Math. J.* **27** (1978), 91–92.
15. ROSENTHAL, H., On quasi-complemented subspaces of Banach spaces, with an appendix on compactness of operators from $L^p(\mu)$ to $L^r(\nu)$, *J. Funct. Anal.* **4** (1969), 176–214.
16. ROSENTHAL, H., A characterization of Banach spaces containing l_1, *Proc. Nat. Acad. Sci. U.S.A.* **71** (1974), 2411–2413.
17. ROSENTHAL, H., Some recent discoveries in the isomorphic theory of Banach spaces, *Bull. Amer. Math. Soc.* **84** (1978), 803–831.

Received June 18, 2002

Jorge Mujica
Universidade Estadual de Campinas
Instituto de Matemática, Estatística
e Computação Científica
Caixa Postal 6065
13083-970 Campinas, SP
Brazil
email: mujica@ime.unicamp.br