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Superposition operators between the Bloch space and Bergman spaces

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Abstract. Using a geometric method, we characterize all entire functions that transform the Bloch space into a Bergman space by superposition in terms of their order and type. We also prove that all superposition operators induced by such entire functions act boundedly. Similar results hold for superpositions from BMOA into Bergman spaces and from the Bloch space into certain weighted Hardy spaces.

Introduction

If X and Y are metric spaces of analytic functions in the disk and φ is a complex-valued function in the plane such that $\varphi\circ f\in Y$ whenever $f\in X$, we say that φ *acts by superposition* from X into Y. If X and Y contain the linear functions, then φ must be an entire function. The *superposition operator* $S_\varphi\colon X\to Y$ with *symbol* φ is then defined by $S_\varphi(f)=\varphi\circ f$. Note that if X and Y are also linear spaces, the operator S_φ is linear if and only if φ is a linear function that fixes the origin. The central questions are:

When does a superposition operator map one space into another? When is it bounded?

Even though analogous concepts also make sense in the context of real-valued functions and their theory has a long history (see [AZ]), the study of such natural questions on spaces of analytic functions has only begun fairly recently. The operators S_φ that map one Bergman space into another, or into the area Nevanlinna class, were characterized in terms of their symbols in [CG]; cf. [C] for Hardy spaces. Results of similar nature were obtained in [BFV] for superpositions between various spaces of Dirichlet type.

In this paper we give a complete description of the superpositions between a Bergman space A^p and the Bloch space $\mathcal{B}$ (in both directions) in terms of the order and type of φ. That is, we measure in a precise way "how much slower" the Bloch functions grow than those in A^p. We recall the basic definitions and facts which motivate our results.

Let dA denote the Lebesgue area measure in the unit disk $\mathbf{D}$, normalized so that $A(\mathbf{D})=1$. If $0<p<\infty$, the *Bergman space* A^p is the set of all analytic functions f in the unit disk $\mathbf{D}$ with finite $L^p(\mathbf{D}, dA)$ norm:

$$\|f\|_p^p = \int_{\mathbf{D}} |f(z)|^p \, dA(z) = \frac{1}{\pi} \int_0^1 \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta \, r \, dr < \infty.$$

Note that $\|f\|_p$ is a true norm if and only if $1 \le p < \infty$. When $0<p<1$, A^p is a complete space with respect to the translation-invariant metric defined by $d_p(f,g) = \|f-g\|_p^p$. It is clear that $A^q \subset A^p$ if $0<p<q<\infty$. The monograph [HKZ] contains plenty of information about Bergman spaces and so does [DS]; see also [Z].

The maximum growth of functions in Bergman spaces will be essential. If $f \in A^p$, by the subharmonicity of $|f|^p$ and the area version of the sub-mean value property applied to the disk of radius $1-|z|$ centered at z, we readily obtain the standard estimate

$$|f(z)| \le \frac{\|f\|_p}{(1-|z|)^{2/p}} \quad \text{for all } z \text{ in } \mathbf{D}. \tag{1}$$

An analytic function f in $\mathbf{D}$ is said to belong to the *Bloch space* $\mathcal{B}$ if

$$\|f\|_{\mathcal{B}} = |f(0)| + \sup_{z \in \mathbf{D}} (1-|z|^2)|f'(z)| < \infty.$$

Any such function satisfies the growth condition

$$|f(z)| \le \left(\log \frac{1}{1-|z|} + 1 \right) \|f\|_{\mathcal{B}} \quad \text{for all } z \text{ in } \mathbf{D}. \tag{2}$$

This follows by integrating the inequality $|f'(\zeta)| \le (\|f\|_{\mathcal{B}} - |f(0)|)(1-|\zeta|^2)^{-1}$ along the line segment from 0 to z. For further basic facts about the Bloch space the reader may consult [ACP] or Chapter 5 of [Z].

The inclusion $\mathcal{B} \subset A^p$ $(0<p<\infty)$ follows easily from (2). Moreover, since the exponents in both (1) and (2) are sharp, it becomes intuitively clear that the Bloch space should be contained in any Bergman space "exponentially", thinking in terms of superpositions. However, our main result, Theorem 3, will show that functions such as $\varphi(z) = \exp cz$, $c \ne 0$, fail to map $\mathcal{B}$ into A^p. In fact, the superposition operator S_φ maps $\mathcal{B}$ into A^p if and only if the entire function φ is either of order less than one, or of order one and type zero. Thus, the characterization of the superpositions does not depend on p. Our proof resembles the ideas employed in the proofs of some of the main results of [BFV] (Theorem 12 and Theorem 19). Although here we no longer need the inequalities of Trudinger type, our argument still requires several delicate technical details. The key point is a geometric construction of a special univalent function in the Bloch space (Lemma 2) which may have some independent interest.

1. Superposition from a Bergman space into the Bloch space

Given that any Bergman space A^p is larger than the Bloch space $\mathcal{B}$, it should be intuitively obvious that a superposition from the former to the latter is possible via constant functions only. However, some work is still required. This is due to the fact that no uniform bounds are available (because we do not have any *a priori* information on the boundedness of the superposition operator).

Proposition 1. *Let $0<p<\infty$ and let φ be entire. Then the superposition operator S_φ maps A^p into $\mathcal{B}$ if and only if φ is a constant function.*

Proof. Suppose that S_φ maps A^p into $\mathcal{B}$. To show that φ is constant, it suffices to prove that φ' is constant. Indeed, if $\varphi'\equiv a$ then φ is a linear function: $\varphi(z)=az+b$. Taking $f\in A^p\setminus\mathcal{B}$, we have that $S_\varphi(f)=af+b\in\mathcal{B}$, which implies $a=0$ and therefore φ is constant.

Now there are two possibilities for the function φ': either

$$\frac{\varphi'(w)}{w}\to 0, \quad \text{as } |w|\to\infty, \tag{3}$$

or

$$|\varphi'(w_n)|\geq\delta|w_n| \tag{4}$$

for some fixed $\delta>0$ and some sequence $\{w_n\}_{n=1}^{\infty}$ such that $|w_n|\to\infty$.

Note that (3), in particular, yields a standard Cauchy estimate $|\varphi'(w)|\leq A|w|$ for some $A>0$ and all sufficiently large w, which in turn implies that $\varphi'(w)$ is a linear function. By letting $w\to\infty$ and applying (3), it follows that φ' is constant and we are done.

It is, thus, only left to show that (4) is impossible. To this end, let $\alpha\in(0,2/p)$ and consider the function

$$f(z)=\frac{1}{(1-z)^\alpha}, \quad z\in\mathbf{D}.$$

By integrating in polar coordinates centered at $z=1$, it is easily checked that $f\in A^p$. Hence $\varphi\circ f\in\mathcal{B}$, and therefore

$$M_f=\sup_{z\in\mathbf{D}}(1-|z|^2)|f'(z)|\,|\varphi'(f(z))|<\infty.$$

From here we deduce that

$$|\varphi'(f(z))|\le\frac{M_f|1-z|^{\alpha+1}}{\alpha(1-|z|^2)}\quad\text{for all } z \text{ in } \mathbf{D}. \tag{5}$$

Partition the complex plane into a finite number of equal sectors with common vertex at the origin and of aperture at most $\frac{1}{2}\pi\alpha$ each. At least one of them will contain infinitely many points w_n. From now on we consider only this infinite subsequence and use for it the same labelling $\{w_n\}_{n=1}^{\infty}$ in order not to burden the notation. We may rotate the sequence if necessary since the function φ_t given by $\varphi_t(z)=\varphi(e^{it}z)$ has the same properties as φ. Thus, after eliminating the terms of modulus at most one, we may assume that

$$|\arg w_n|<\tfrac{1}{4}\pi\alpha \text{ and } |w_n|>1\quad\text{for all } n.$$

Consider the preimages of w_n under the function f:

$$z_n=1-w_n^{-1/\alpha}. \tag{6}$$

By our assumptions on $\{w_n\}_{n=1}^{\infty}$, each z_n belongs to the Stolz domain

$$S=\bigl\{z\in\mathbf{D}:|1-z|<1 \text{ and } |\arg(1-z)|<\tfrac{1}{4}\pi\bigr\}.$$

Therefore there exists a constant c such that

$$|1-z_n|\le c(1-|z_n|)\quad\text{for all } n \text{ in } \mathbf{N}. \tag{7}$$

From (4), (5), (7), and (6), respectively, we get

$$\delta|w_n|\le|\varphi'(f(z_n))|\le\frac{M_f|1-z_n|^{\alpha+1}}{\alpha(1-|z_n|^2)}\le\frac{cM_f}{\alpha}|1-z_n|^{\alpha}=\frac{cM_f}{\alpha|w_n|}.$$

It follows that

$$\delta|w_n|^2\le c\frac{M_f}{\alpha},$$

which contradicts the assumption that $|w_n|\to\infty$. This completes the proof. $\square$

2. Superposition from the Bloch space into a Bergman space

Proposition 1 tells us that $\mathcal{B}$ is "very small" inside the Bergman space A^p. The following questions are, thus, much more challenging than the one answered in the previous section.

For which entire functions φ does S_φ map $\mathcal{B}$ into A^p? Which ones among them induce a bounded operator S_φ?

We remind the reader that a (possibly nonlinear) operator acting between two metric spaces is said to be *bounded* if it maps bounded sets into bounded sets.

2.1. The key lemma

A *univalent function* in $\mathbf{D}$ is an analytic function which is one-to-one in the disk. By the Riemann mapping theorem, for any given simply connected domain Ω (other than the plane itself) there is such a function f (called a *Riemann map*) that takes $\mathbf{D}$ onto Ω and the origin to a prescribed point. Denoting by $\operatorname{dist}(w, \partial\Omega)$ the Euclidean distance of the point w to the boundary of the domain Ω, the Riemann map f has the following property (see Corollary 1.4 of [P2], for example):

$$\tag{8} \tfrac{1}{4}(1-|z|^2)|f'(z)| \le \operatorname{dist}(f(z), \partial\Omega) \le (1-|z|^2)|f'(z)| \quad \text{for all } z \text{ in } \mathbf{D}.$$

This estimate plays an important role in the geometric theory of functions. In particular, (8) tells us that a function f univalent in $\mathbf{D}$ belongs to $\mathcal{B}$ if and only if the image domain $f(\mathbf{D})$ does not contain arbitrarily large disks.

The auxiliary construction of a conformal map onto a specific Bloch domain with the maximal (logarithmic) growth along a certain polygonal line displayed below might be of some independent interest. Thus, we state it separately as a lemma. Loosely speaking, such a domain can be imagined as a "highway from the origin to infinity" of width 2δ. Somewhat similar constructions of simply connected domains as the images of functions in various function spaces can be found in the recent papers [BFV] and [DGV].

Lemma 2. *For each positive number δ and for every sequence $\{w_n\}_{n=0}^\infty$ of complex numbers such that $w_0=0$, $|w_1|\ge 5\delta$, $\arg w_1 < \frac{1}{2}\pi$, $\arg w_n \searrow 0$, and*

$$\tag{9} |w_n| \ge \max\left\{3|w_{n-1}|, \sum_{k=1}^{n-1} |w_k - w_{k-1}|\right\} \quad \text{for all } n \ge 2,$$

there exists a domain Ω with the following properties:

(i) *Ω is simply connected*;

(ii) Ω *contains the infinite polygonal line* $L=\bigcup_{n=1}^{\infty}[w_{n-1},w_n]$, *where* $[w_{n-1},w_n]$ *denotes the line segment from* w_{n-1} *to* w_n;

(iii) *any Riemann map* f *of* $\mathbf{D}$ *onto* Ω *belongs to* $\mathcal{B}$;

(iv) $\operatorname{dist}(w,\partial\Omega)=\delta$ *for each point* w *on* L.

Proof. It is clear from (9) that $|w_n|\nearrow\infty$, as $n\to\infty$. We construct the domain Ω as follows. First connect the points w_n by a polygonal line L as indicated in the statement. Let $D(z,\delta)=\{w:|z-w|<\delta\}$ and define

$$\Omega=\bigcup\{D(z,\delta):z\in L\},$$

i.e. let Ω be a δ-thickening of L. In other words, Ω is the union of simply connected cigar-shaped domains

$$C_n=\bigcup\{D(z,\delta):z\in[w_{n-1},w_n]\}.$$

By our choice of w_n, it is easy to check inductively that $|w_n-w_k|\geq 5\delta$ whenever $n>k$. Since our construction implies that

$$C_n\subset\{w:|w_{n-1}|-\delta<|w|<|w_n|+\delta\},$$

wee see immediately that

(a) for all m and n, $C_m\cap C_n\neq\emptyset$ if and only if $|m-n|\leq 1$;

(b) for all n, $C_n\cap C_{n+1}$ is either $D(w_n,\delta)$ or the interior of the convex hull of $\overline{D(w_n,\delta)\cup\{a_n\}}$ for some point a_n outside of $\overline{D(w_n,\delta)}$.

Thus, each $\Omega_N=\bigcup_{n=1}^{N}C_n$ is also simply connected. Since

$$\Omega=\bigcup_{N=1}^{\infty}\Omega_N \quad\text{and}\quad \Omega_N\subset\Omega_{N+1}\text{ for all }N,$$

we conclude that Ω is also simply connected (like in [DGV], Section 4.2, p. 56). By construction, $\operatorname{dist}(w,\partial\Omega)\leq\delta$ for all w in Ω, hence any Riemann map onto Ω will belong to $\mathcal{B}$. It is also clear that (iv) holds. $\square$

2.2. Characterizations of superposition operators

In what follows we will need a few basic properties of the hyperbolic metric. Recall that the *hyperbolic distance* between two points z and w in the disk is defined as

$$\varrho(z,w)=\inf_{\gamma}\int_{\gamma}\frac{|d\zeta|}{1-|\zeta|^2}=\frac{1}{2}\log\frac{1+\left|\dfrac{z-w}{1-\bar{z}w}\right|}{1-\left|\dfrac{z-w}{1-\bar{z}w}\right|},$$

where the infimum is taken over all rectifiable curves γ in $\mathbf{D}$ that join z with w.

The hyperbolic metric ϱ_Ω on an arbitrary simply connected domain Ω (not the entire plane) is defined via the corresponding pullback to the disk: if f is a Riemann map of $\mathbf{D}$ onto Ω then

$$\varrho_\Omega(f(z), f(w)) = \varrho(z, w) = \inf_\Gamma \int_{f^{-1}(\Gamma)} \frac{|d\zeta|}{1-|\zeta|^2},$$

where the infimum is taken over all rectifiable curves Γ in Ω from $f(z)$ to $f(w)$. The metric ϱ_Ω does not depend on the choice of the Riemann map f. For more details we refer the reader to Sections 1.2 and 4.6 of [P2].

From the definition of the hyperbolic metric we notice the following important feature of Riemann maps: if $f(0)=0$ then

$$\varrho_\Omega(0, f(z)) = \varrho(0, z) \ge \frac{1}{2} \log \frac{1}{1-|z|}, \quad z \in \mathbf{D}. \tag{10}$$

Another fundamental property, which is easily deduced from (8), is that

$$\varrho_\Omega(w_1, w_2) \le \inf_\Gamma \int_\Gamma \frac{|dw|}{\operatorname{dist}(w, \partial\Omega)}, \tag{11}$$

where the infimum is taken over all rectifiable curves Γ in Ω from w_1 to w_2.

Denote by D^p_α $(\alpha > -1)$ the weighted Dirichlet space of all analytic functions f in $\mathbf{D}$ for which

$$\int_{\mathbf{D}} |f'(z)|^p (1-|z|^2)^\alpha \, dA(z) < \infty$$

and by B^p the standard analytic Besov space D^p_{p-2}, $1<p<\infty$; see [DGV] for some of its properties and [BFV] for superposition operators between such spaces. Since $\mathcal{B}$ can be viewed as a limit space of all B^p as $p \to \infty$, the results about superpositions from $\mathcal{B}$ into A^q might be inspired by those for the superpositions from B^p into A^q by letting $p \to \infty$. Also, it is well known that

$$\int_{\mathbf{D}} |f(z)|^q \, dA(z) \asymp |f(0)|^q + \int_{\mathbf{D}} |f'(z)|^q (1-|z|^2)^q \, dA(z),$$

hence $A^q = D^q_q$ as sets. One of the main results of [BFV], Theorem 24 there, tells us that $S_\varphi(B^p) \subset D^q_q$ if and only if either φ has order less than $p/(p-1)$ or it has order $p/(p-1)$ and finite type. This seems to suggest that the characterization of all superpositions from $\mathcal{B}$ into A^q should be related to the entire functions of order one (letting $p \to \infty$), but in which way exactly? In the earlier papers [CG] (in the results for the Nevanlinna class) and [BFV] the cut usually occurred at the level of functions of infinite type. The appearance of the functions of given order and type zero in the result below seems to be a novelty in this context. The intuitive reason behind it lies in the fact that $\mathcal{B}$ is not the smallest space that contains all B^p spaces.

Theorem 3. *Let* $0<p<\infty$ *and let* φ *be entire. Then the following statements are equivalent:*

(a) *the superposition operator* S_φ *maps* $\mathcal{B}$ *into* A^p;

(b) S_φ *is a bounded operator from* $\mathcal{B}$ *into* A^p;

(c) φ *is an entire function either of order less than one, or of order one and type zero.*

Proof. We need only show that (a) $\Rightarrow$ (c) $\Rightarrow$ (b).

In order to prove that (c) implies (b), let us suppose that φ is an entire function of order one and type zero. Writing $M(r)=\max\{|\varphi(z)|:|z|=r\}$, this means that

$$\limsup_{r\to\infty}\frac{\log\log M(r)}{\log r}=1$$

(see [Bo], for example) and the quantity

$$E(r)=\frac{\log M(r)}{r} \tag{12}$$

tends to zero as $r\to\infty$.

Let f be an arbitrary function in $\mathcal{B}$ of norm at most K. By (2), we have

$$|f(z)|\le\left(\log\frac{1}{1-|z|}+1\right)K. \tag{13}$$

By our assumption on $E(r)$, for some sufficiently large R_0 it follows that

$$E(|w|)<\frac{1}{2pK},\quad\text{whenever } |w|>R_0. \tag{14}$$

Thus, whenever $|f(z)|>R_0$ we get

$$\begin{aligned}|\varphi(f(z))|&\le M(|f(z)|)=e^{|f(z)|E(|f(z)|)}\\&\le\exp\left(K\left(\log\frac{1}{1-|z|}+1\right)E(|f(z)|)\right)\le\frac{e^{1/2p}}{(1-|z|)^{1/2p}}\end{aligned}$$

by the definition of $M(r)$ and by (12), (13), and (14), respectively.

If, on the contrary, $|f(z)|\le R_0$ then $|\varphi(f(z))|\le M(R_0)$ by the maximum modulus principle. Combining the two possible cases, we obtain

$$\|\varphi\circ f\|_p^p\le e^{1/2}\int_{\mathbf{D}}\frac{dA(z)}{(1-|z|)^{1/2}}+M(R_0)^p=C,$$

where C depends only on φ, p, and K, but not upon f. This shows that S_φ is a bounded operator from $\mathcal{B}$ into A^p.

The reasoning is similar, but simpler, when φ has order $\varrho<1$: use the estimate $M(r)\le\exp r^{\varrho+\varepsilon}$ for a small enough ε and large enough r.

For the proof of (a) $\Rightarrow$ (c), let us assume that $S_\varphi\colon\mathcal{B}\to A^p$ but (c) is false. Thus, either φ has order greater than one, or it has order one and type $\sigma\neq0$. The former case is easier to handle, so we consider only the latter.

Since φ has order one and type different from zero, there exists an ε and a sequence $\{r_n\}_{n=1}^\infty$ such that $r_n\to\infty$, as $n\to\infty$, and

$$\frac{\log M(r_n)}{r_n}\ge\varepsilon>0 \quad \text{for all } n. \tag{15}$$

In other words, there exists a sequence $\{w_n\}_{n=1}^\infty$ such that $|w_n|=r_n$ and

$$|\varphi(w_n)|=M(r_n)\ge e^{\varepsilon|w_n|} \quad \text{for all } n. \tag{16}$$

Let us fix a constant δ so that $\delta>12/\varepsilon p$. We can now choose an infinite subsequence, denoted again $\{w_n\}_{n=1}^\infty$, so that the sequence $\{\arg w_n\}_{n=1}^\infty$ in $[0,2\pi]$ is convergent and all points w_n lie in an angular sector of opening $\frac12\pi$. We may further assume that they are all located in the first quadrant and the arguments $\arg w_n$ decrease to 0, by applying symmetries or rotations if necessary. There is no loss of generality in doing this because the entire functions ψ and φ_t, defined by $\psi(z)=\overline{\varphi(\bar z)}$ and $\varphi_t(z)=\varphi(e^{it}z)$, respectively, have the same order and type as φ.

Select inductively a further subsequence, labelled again $\{w_n\}_{n=1}^\infty$, in such a way that $w_0=0$, $|w_1|\ge5\delta$, and (9) holds. Next, use Lemma 2 to construct a domain Ω with the properties (i)–(iv) indicated there. Let f be a Riemann map of $\mathbf{D}$ onto Ω that fixes the origin.

Now let z_n be the points in $\mathbf{D}$ for which $w_n=f(z_n)$. Since $|w_n|\to\infty$, as $n\to\infty$, it follows that $|z_n|\to1$. By applying estimate (10) for the hyperbolic metric, the triangle inequality, inequality (11) and property (iv) from Lemma 2, as well as the properties (9) of the points w_n, respectively, we obtain the following chain of inequalities:

$$\begin{aligned}\frac12\log\frac{1}{1-|z_n|}&\le\varrho_\Omega(0,w_n)\le\sum_{k=1}^n\varrho_\Omega(w_{k-1},w_k)\le\sum_{k=1}^n\int_{[w_{k-1},w_k]}\frac{|dw|}{\operatorname{dist}(w,\partial\Omega)}\\&=\sum_{k=1}^n\int_{[w_{k-1},w_k]}\frac{|dw|}{\delta}=\frac1\delta\sum_{k=1}^n|w_k-w_{k-1}|\le\frac3\delta|w_n|.\end{aligned}$$

This shows that

$$|w_n|\ge\frac\delta6\log\frac{1}{1-|z_n|}. \tag{17}$$

It follows from (16) and (17) that

$$|\varphi(w_n)| \geq \exp\left(\frac{\varepsilon\delta}{6}\log\frac{1}{1-|z_n|}\right) = \frac{1}{(1-|z_n|)^{\varepsilon\delta/6}}. \tag{18}$$

On the other hand, $f \in \mathcal{B}$, hence by assumption (a): $\varphi \circ f \in A^p$. By (1), we have

$$|\varphi(w_n)| = |(\varphi\circ f)(z_n)| \leq \frac{\|\varphi\circ f\|_p}{(1-|z_n|)^{2/p}} \tag{19}$$

for all n. However, (18) and (19) contradict each other since $\frac{1}{6}\varepsilon\delta > 2/p$ by our initial choice of δ. The proof is now complete. □

2.3. Superpositions between some related spaces

We would like to emphasize that Theorem 3 used very few facts typical only of Bergman spaces. One could consider instead the weighted Hardy spaces H_α^∞ (sometimes also denoted $A^{-\alpha}$ and called *Korenblum spaces*). A function f analytic in $\mathbf{D}$ is said to belong to H_α^∞, $0<\alpha<\infty$, if and only if

$$\sup_{z\in\mathbf{D}} (1-|z|^2)^\alpha |f(z)| < \infty.$$

The point here is that, by (1), $A^p \subset H^\infty_{2/p}$, while it can be easily seen that $\alpha < 1/p$ implies $H_\alpha^\infty \subset A^p$. Thus, it follows that

(a) S_φ maps H_α^∞ into $\mathcal{B}$ if and only if $\varphi \equiv \text{const}$;

(b) S_φ maps $\mathcal{B}$ into H_α^∞ if and only if the entire function φ is either of order one and type zero, or of order less than one; also, whenever this happens, S_φ acts boundedly from $\mathcal{B}$ into H_α^∞.

A similar statement can easily be formulated for weighted Bergman spaces.

It should especially be worth pointing out that there is a variant of Theorem 3 with another well-known space instead of $\mathcal{B}$. Recall that BMOA is the space of all Hardy H^1 functions whose boundary values have bounded mean oscillation. For the precise definition and the basic properties of BMOA we refer the reader to the survey papers [Ba] and [G].

Since BMOA$\subset\mathcal{B}$, it makes sense to ask whether $\mathcal{B}$ can be replaced by the smaller space BMOA in Theorem 3. This is indeed the case. Namely, a well-known result of Pommerenke [P1] states that every univalent function in the Bloch space also belongs to BMOA, so the function used in the proof above works for this space as well. All standard norms on BMOA are equivalent and the injection operator from BMOA into $\mathcal{B}$ is bounded, so an inequality for BMOA totally analogous to

(13) also holds. Thus, we can either give a proof that (c) implies (b) as above or, alternatively, factor the superposition operator from BMOA into A^p through $\mathcal{B}$ and use the boundedness of $S_\varphi: \mathcal{B} \to A^p$. The rest of the proof is obtained by following the same steps as in the proof of Theorem 3.

In fact, the above reasoning shows that the Bloch space in Theorem 3 can be replaced by the class of all univalent functions in $\mathcal{B}$ (that is, in BMOA).

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References

[ACP] Anderson, J. M., Clunie, J. and Pommerenke, C., On Bloch functions and normal functions, *J. Reine Angew. Math.* **270** (1974), 12–37.

[AZ] Appell, J. and Zabrejko, P. P., *Nonlinear Superposition Operators,* Cambridge Tracts in Mathematics **95**, Cambridge Univ. Press, Cambridge, 1990.

[Ba] Baernstein, A. II, Analytic functions of bounded mean oscillation, in *Aspects of Contemporary Complex Analysis (Durham, 1979)*, pp. 3–36, Academic Press, London–New York, 1980.

[Bo] Boas, R. P., Jr., *Entire Functions*, Academic Press, New York, 1954.

[BFV] Buckley, S. M., Fernández, J. L. and Vukotić, D., Superposition operators on Dirichlet type spaces, in *Papers on Analysis: A Volume Dedicated to Olli Martio on the Occasion of his 60th Birthday*, pp. 41–61, Rep. Univ. Jyväskylä Dept. Math. Stat. **83**, University of Jyväskylä, Jyväskylä, 2001.

[C] Cámera, G. A., Nonlinear superposition on spaces of analytic functions, in *Harmonic Analysis and Operator Theory (Caracas, 1994),* Contemp. Math. **189**, pp. 103–116, Amer. Math. Soc., Providence, RI, 1995.

[CG] Cámera, G. A. and Giménez, J., The nonlinear superposition operators acting on Bergman spaces, *Compositio Math.* **93** (1994), 23–35.

[DGV] Donaire, J. J., Girela, D. and Vukotić, D., On univalent functions in some Möbius invariant spaces, *J. Reine Angew. Math.* **553** (2002), 43–72.

[DS] Duren, D. L. and Schuster, A. P., *Bergman Spaces,* Math. Surveys and Monographs **100**, Amer. Math. Soc., Providence, RI, 2004.

[G] Girela, D., Analytic functions of bounded mean oscillation, in *Complex Function Spaces (Mekrijärvi, 1999),* Univ. Joensuu Dept. Math. Rep. Ser. **4**, pp. 61–170, Univ. Joensuu, Joensuu, 2001.

[HKZ] HEDENMALM, H., KORENBLUM, B. and ZHU, K. H., *Theory of Bergman Spaces,* Grad. Texts in Math. **199**, Springer-Verlag, New York, 2000.

[P1] POMMERENKE, C., Schlichte Funktionen und analytische Funktionen von beschränkter mittlerer Oszillation, *Comment. Math. Helv.* **52** (1977), 591–602.

[P2] POMMERENKE, C., *Boundary Behaviour of Conformal Maps,* Grundlehren Math. Wiss. **299**, Springer-Verlag, Berlin, 1992.

[Z] ZHU, K. H., *Operator Theory in Function Spaces,* Monographs Textbooks Pure Appl. Math. **139**, Marcel Dekker, New York–Basel, 1990.

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