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A topological rigidity theorem on open manifolds with nonnegative Ricci curvature

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1. Introduction

Let M be an n-dimensional complete Riemannian manifold with nonnegative Ricci curvature. For a base point $p \in M$ we denote by $B(p,r)$ the open geodesic ball with radius r around p and let $\mathrm{vol}[B(p,r)]$ be its volume. Let ω_n be the volume of the unit ball in the Euclidean space $\mathbf{R}^n$ and define α_M by

$$\alpha_M = \lim_{r \to \infty} \frac{\mathrm{vol}[B(p,r)]}{\omega_n r^n}.$$

It follows from the relative volume comparison theorem [BC], [Gr2] that the limit at the right-hand side in the above equality exists and it does not depend on the choice of p. Thus α_M is a global geometric invariant of M. The manifold M has *large volume growth* if $\alpha_M > 0$. Note that, in this case,

$$\mathrm{vol}[B(p,r)] \geq \alpha_M \omega_n r^n \quad \text{for } p \in M \text{ and } r > 0. \tag{1.1}$$

The structure of complete noncompact Riemannian manifolds with nonnegative Ricci curvature and large volume growth has received much attention. Let M be such an n-dimensional manifold. Li [L] and Anderson [A] have each proven that M has finite fundamental group. Li uses the heat equation while Anderson uses volume comparison arguments to prove this theorem. Perelman [P] has shown that there is a small constant $\varepsilon(n) > 0$ depending only on n such that if $\alpha_M > 1 - \varepsilon(n)$, then M is contractible. It has been shown by Cheeger and Colding [CC] that the condition in Perelman's theorem actually implies that M is diffeomorphic to $\mathbf{R}^n$. Shen [S2] has proven that M has finite topological type, provided that

$$\frac{\mathrm{vol}[B(p,r)]}{\omega_n r^n} = \alpha_M + o\left(\frac{1}{r^{n-1}}\right)$$

and, either the conjugate radius $\mathrm{conj}_M \geq c > 0$ or the sectional curvature $\mathrm{K}_M \geq K_0 > -\infty$. Recall that a noncompact manifold is of finite topological type if it is homeomorphic to the interior of a compact manifold with boundary [AG]. Generalizations of Shen's theorem have been made in [OSY], [SS] and [X1].

It should be mentioned that the first important result about topological finiteness of a complete open manifold with nonnegative Ricci curvature is due to Abresch–Gromoll. They have proven that complete open manifolds with small diameter growth, $o(r^{1/n})$, nonnegative Ricci curvature and sectional curvature bounded below have finite topological type. Their theorem is proven by using an inequality referred to as the *excess theorem* (cf. [AG]) which has many interesting applications (cf. [CX], [C], [CC], [G], [OSY], [SS], [S1], [S2], [So], [X1] and [X2]).

In [Pe], Petersen proposed the following conjecture.

Conjecture. ([Pe]) *An n-dimensional complete Riemannian manifold with nonnegative Ricci curvature and $\alpha_M > \frac{1}{2}$ is diffeomorphic to $\mathbf{R}^n$.*

It has been proven in [CX] and [X2] that Petersen's conjecture is true when the sectional curvature of the manifold is bounded below and if the volume of geodesic balls around some point grows properly.

In this paper, we study complete manifolds with nonnegative Ricci curvature and large volume growth. Our purpose is to prove the following topological rigidity theorem which supports Petersen's conjecture and contains no condition on the sectional curvature of the manifolds.

Theorem 1.1. *Given $\alpha \in \left(\frac{1}{2}, 1\right)$, $\varrho_0 > 0$ and an integer $n \geq 2$, there exist positive constants $r_0 = r_0(\alpha, \varrho_0, n)$ and $\varepsilon = \varepsilon(\alpha, \varrho_0, n)$ such that any complete Riemannian n-manifold M with Ricci curvature $\mathrm{Ric}_M \geq 0$, $\alpha_M \geq \alpha$, conjugate radius $\mathrm{conj}_M \geq \varrho_0$ and*

$$\frac{\mathrm{vol}[B(p,r)]}{\omega_n r^n} - \alpha_M \leq \frac{\varepsilon}{r^{n-2+1/n}} \tag{1.2}$$

for some $p \in M$ and all $r \geq r_0$, is diffeomorphic to $\mathbf{R}^n$.

2. Proof of Theorem 1.1

Throughout the paper all geodesics are assumed to have unit speed.

Let M be an n-dimensional Riemannian manifold. For a point $p \in M$, we denote the distance from p to x by $d(p,x)$ and set $d_p(x) = d(p,x)$. We denote by crit_p the *criticality radius* of M at p, i.e., crit_p is the smallest critical value for the distance function $d(p,\cdot)\colon M \to \mathbf{R}$. The *criticality radius* of M is defined to be $\mathrm{crit}(M) =$

$\inf_{p\in M} \operatorname{crit}_p$. We refer to [C], [Gr1], [G] and [GS] for the notion of critical points of distance functions and its applications.

In [W], Wei proved an angle version Toponogov comparison theorem for Ricci curvature for which we need some notation.

A *geodesic triangle* $\{\gamma_0,\gamma_1,\gamma_2\}$ consists of three *minimal* geodesics γ_i of length $L[\gamma_i]=l_i$ which satisfy

$$\gamma_i(l_i)=\gamma_{i+1}(0),$$

where the indices i and $i+1$ are taken mod 3.

The *angle* at a corner, say $\gamma_0(0)$, is by definition $\angle(-\gamma_2'(l_2),\gamma_0'(0))$. The angle opposite to γ_i is denoted by α_i.

Lemma 2.1. ([W]) *For any $\varepsilon_0>0$ and $\varrho_0>0$ there exists a constant $\beta_0=\beta_0(n,\varepsilon_0,\varrho_0)>0$ such that if M is an n-dimensional complete Riemannian manifold with $\mathrm{Ric}_M\geq 0$ and conjugate radius $\mathrm{conj}_M\geq\varrho_0$, and $\{\gamma_0,\gamma_1,\gamma_2\}$ is a geodesic triangle contained in $B(p,\beta_0)$ for some $p\in M$, then for the geodesic triangle $\{\tilde\gamma_0,\tilde\gamma_1,\tilde\gamma_2\}$ in the Euclidean plane with $L[\tilde\gamma_i]=L[\gamma_i]=l_i$, we have*

$$\alpha_i>\tilde\alpha_i-\varepsilon_0=\arccos\left(\frac{l_{i-1}^2+l_{i+1}^2-l_i^2}{2l_{i-1}l_{i+1}}\right)-\varepsilon_0. \tag{2.1}$$

Actually, Wei proved a comparison estimate for manifolds with Ricci curvature bounded from below, but the above statement suffices for our purposes.

The lemma below is the key step for proving Theorem 1.1.

Lemma 2.2. *Given $\alpha\in\left(\frac12,1\right)$, $\varrho_0>0$ and an integer $n\geq2$, there exists a positive constant $r_0=r_0(\alpha,\delta,n)$ such that any complete Riemannian n-manifold M with Ricci curvature $\mathrm{Ric}_M\geq0$, $\alpha_M\geq\alpha$ and conjugate radius $\mathrm{conj}_M\geq\varrho_0$ satisfies $\operatorname{crit}_p\geq r_0$, for all $p\in M$.*

Proof. Let $\theta_0=\theta_0(\alpha,n)\in\left(0,\frac12\pi\right)$ be the solution of

$$\int_0^{\theta_0}\sin^{n-2}t\,dt=(1-\alpha)\int_0^{\pi}\sin^{n-2}t\,dt, \tag{2.2}$$

and set $\varepsilon_0=\frac13\left(\frac12\pi-\theta_0\right)$. Let $\beta_0=\beta_0(n,\varepsilon_0,\varrho_0)\equiv\beta_0(n,\alpha,\varrho_0)$ be as in Lemma 2.1 and set

$$T=\frac{\cos\theta_0\cos\varepsilon_0\cos(\theta_0+\varepsilon_0)}{\cos^2(\theta_0+\varepsilon_0)-\sin^2\varepsilon_0}; \tag{2.3}$$

then $T>1$.

We *claim* that

$$\text{crit}_y > r_0 \equiv \frac{\beta_0 \cos(\theta_0+\varepsilon_0)}{4T} \quad \text{for } y\in M, \tag{2.4}$$

which will imply the conclusion of Lemma 2.2.

Suppose on the contrary that the above *claim* is false. Then there is a point $p\in M$ such that $\text{crit}_p \le r_0$ and thus we can find a critical point $q(\neq p)$ of d_p such that $r_1 := d(p,q) \le r_0$.

Let Γ_{qp} (resp. Γ_{pq}) be the set of unit vectors in T_qM (resp. T_pM) corresponding to the set of normal minimal geodesics of M from q to p (resp. p to q). For any $\theta\in[0,\frac{1}{2}\pi]$, let $\Gamma_{pq}(\theta)=\{u\in S_pM \mid \angle(u,\Gamma_{pq})\le\theta\}$. Since q is a critical point of d_p, we have $\Gamma_{qp}(\frac{1}{2}\pi)=S_qM$ which implies that Γ_{pq} contains at least two distinct vectors. Thus there exists a constant $t_0>0$ such that $\text{vol}(\Gamma_{pq}(\theta_0))\ge v(\theta_0)+t_0$, where $v(\theta_0)$ is the volume of a geodesic ball of radius θ_0 in an $(n-1)$-unit sphere. It then follows from (2.2) that $\text{vol}(\Gamma_{pq}(\theta_0))\ge(1-\alpha)\alpha_{n-1}+t_0$, where α_m is the volume of $S^m(1)$, the unit m-sphere.

For each $u\in S_pM$, we denote by $\tau(u)$ the distance to the cut point of p along the geodesic $\exp_p(tu)$, $t\in[0,\infty)$.

Sublemma. *For any $u\in\Gamma_{pq}(\theta_0)$, we have $\tau(u)<\frac{1}{4}\beta_0$.*

Proof. We proceed by contradiction. Suppose that there is a $v\in\Gamma_{pq}(\theta_0)$ with $\tau(v)\ge\frac{1}{4}\beta_0$. Then the geodesic $\gamma(t):=\exp_p(tv)$, $t\in[0,\frac{1}{4}\beta_0]$, is a minimizing one. Take a vector $w\in\Gamma_{pq}$ with $\angle(w,v)\le\theta_0$. Let $\gamma_1\colon[0,r_1]\to M$ be a minimal geodesic of M from p to q with $\gamma_1'(0)=w$. Let $z=\gamma(\frac{1}{4}\beta_0)$, set $r_2=d(q,z)$ and take a minimal geodesic $\gamma_2(t)$, $t\in[0,r_2]$, from q to z. Since $\max\{d(p,q),d(p,z)\}=\frac{1}{4}\beta_0$, one can see from the triangle inequality that the geodesic triangle $\{\gamma,\gamma_1,\gamma_2\}$ is contained in $B(p,\beta_0)$. Thus we can apply Lemma 2.1 to $\{\gamma,\gamma_1,\gamma_2\}$ to get

$$\theta_0 \ge \angle(\gamma'(0),\gamma_1'(0)) > \arccos\left(\frac{\frac{1}{16}\beta_0^2+r_1^2-r_2^2}{2\cdot\frac{1}{4}\beta_0 r_1}\right)-\varepsilon_0,$$

that is,

$$r_2^2 < r_1^2-\tfrac{1}{2}\beta_0 r_1\cos(\theta_0+\varepsilon_0)+\tfrac{1}{16}\beta_0^2. \tag{2.5}$$

Now we use the fact that q is a critical point of d_p to get a minimal geodesic γ_3 from q to p such that $\angle(\gamma_2'(0),\gamma_3'(0))\le\frac{1}{2}\pi$. Applying Lemma 2.1 to the geodesic triangle $\{\gamma_2,\gamma_3,\gamma\}$, we obtain

$$\frac{\pi}{2} > \arccos\left(\frac{r_1^2+r_2^2-\frac{1}{16}\beta_0^2}{2r_1r_2}\right)-\varepsilon_0,$$

which is equivalent to

$$\tfrac{1}{16}\beta_0^2 < r_1^2 + r_2^2 + 2r_1r_2 \sin\varepsilon_0. \tag{2.6}$$

Substituting (2.5) into (2.6), one gets

$$\beta_0 \cos(\theta_0+\varepsilon_0) < 4r_1 + 4r_2 \sin\varepsilon_0 < 4r_1 + 4\sqrt{r_1^2 - \tfrac{1}{2}\beta_0 r_1 \cos(\theta_0+\varepsilon_0)\tfrac{1}{16}\beta_0^2}\,\sin\varepsilon_0. \tag{2.7}$$

Observe that for any $t\in R$, we have

$$t^2 - \tfrac{1}{2}\beta_0 \cos(\theta_0+\varepsilon_0)t + \tfrac{1}{16}\beta_0^2 > 0,$$

and

$$\left|t - \tfrac{1}{4}\beta_0 \cos(\theta_0+\varepsilon_0)\right| < \sqrt{t^2 - \tfrac{1}{2}\beta_0 \cos(\theta_0+\varepsilon_0)t + \tfrac{1}{16}\beta_0^2}\,. \tag{2.8}$$

Now consider the function $f\colon \mathbf{R}\to\mathbf{R}$ defined by

$$f(t) = t + \sqrt{t^2 - \tfrac{1}{2}\beta_0 \cos(\theta_0+\varepsilon_0)t + \tfrac{1}{16}\beta_0^2}\,\sin\varepsilon_0.$$

It follows from (2.8) that f is strictly increasing. Thus from (2.7) and $r_1 \le r_0$, we have

$$\beta_0 \cos(\theta_0+\varepsilon_0) < 4r_0 + 4\sqrt{r_0^2 - \tfrac{1}{2}\beta_0 r_0 \cos(\theta_0+\varepsilon_0) + \tfrac{1}{16}\beta_0^2}\,\sin\varepsilon_0.$$

By (2.4), $4r_0 < \beta_0 \cos(\theta_0+\varepsilon_0)$ and so we get from the above inequality that

$$(\beta_0 \cos(\theta_0+\varepsilon_0) - 4r_0)^2 < (16r_0^2 - 8\beta_0 r_0 \cos(\theta_0+\varepsilon_0) + \beta_0^2)\sin^2\varepsilon_0. \tag{2.9}$$

Substituting (2.4) into (2.9) and simplifying, we obtain

$$(T-1)^2\cos^2(\theta_0+\varepsilon_0) < (T^2 + (1-2T)\cos^2(\theta_0+\varepsilon_0))\sin^2\varepsilon_0,$$

that is

$$(\cos^2(\theta_0+\varepsilon_0) - \sin^2\varepsilon_0)T^2 - 2\cos^2(\theta_0+\varepsilon_0)\cos^2\varepsilon_0 T + \cos^2(\theta_0+\varepsilon_0)\cos^2\varepsilon_0 < 0.$$

Solving this inequality, one finds

$$\frac{\cos(\theta_0+2\varepsilon_0)\cos\varepsilon_0\cos(\theta_0+\varepsilon_0)}{\cos^2(\theta_0+\varepsilon_0)-\sin^2\varepsilon_0} < T < \frac{\cos\theta_0\cos\varepsilon_0\cos(\theta_0+\varepsilon_0)}{\cos^2(\theta_0+\varepsilon_0)-\sin^2\varepsilon_0},$$

which clearly contradicts the choice of T. This completes the proof of the sublemma. □

Now we continue on the proof of the *claim*. Let

$$dV(\exp_p(t\xi)) = \sqrt{g(t;\xi)}\,dt\,d\mu_p(\xi)$$

be the volume form in the geodesic spherical coordinates around p, where $d\mu_p(\xi)$ is the Riemannian measure on S_pM induced by the Euclidean Lebesgue measure on T_pM (cf. [Ch, p. 112]). Since $\mathrm{Ric}_M \geq 0$, the Bishop–Gromov comparison theorem [BC], [Gr2] gives $\sqrt{g(t;v)} \leq t^{n-1}$ for all $t > 0$. Thus, for any $r \geq \frac{1}{4}\beta_0$, we have from the sublemma that

$$\begin{aligned}
\mathrm{vol}[B(p,r)] &= \int_{S_pM} d\mu_p(\xi) \int_0^{\min\{\tau(\xi),r\}} \sqrt{g(t;\xi)}\,dt \\
&= \int_{\Gamma_{pq}(\theta_0)} d\mu_p(\xi) \int_0^{\min\{\tau(\xi),r\}} \sqrt{g(t;\xi)}\,dt \\
&\quad + \int_{S_pM\setminus\Gamma_{pq}(\theta_0)} d\mu_p(\xi) \int_0^{\min\{\tau(\xi),r\}} \sqrt{g(t;\xi)}\,dt \qquad (2.10)\\
&\leq \int_{\Gamma_{pq}(\theta_0)} d\mu_p(\xi) \int_0^{\beta_0/4} t^{n-1}\,dt + \int_{S_pM-\Gamma_{pq}(\theta_0)} d\mu_p(\xi) \int_0^{r} t^{n-1}\,dt \\
&\leq \mathrm{vol}\big[B\big(\tfrac{1}{4}\beta_0\big)\big] + (\alpha\alpha_{n-1} - t_0) \int_0^r t^{n-1}\,dt \\
&= \mathrm{vol}\big[B\big(\tfrac{1}{4}\beta_0\big)\big] + \left(\alpha\omega_n - \frac{t_0}{n}\right) r^n,
\end{aligned}$$

where $B\big(\frac{1}{4}\beta_0\big)$ denotes the $\frac{1}{4}\beta_0$-ball in $\mathbf{R}^n$.

It follows from (2.10) that

$$\lim_{r\to\infty} \frac{\mathrm{vol}[B(p,r)]}{\omega_n r^n} \leq \alpha - \frac{t_0}{n\omega_n},$$

which is a contradiction and completes the proof of our *claim*. □

Before stating our next lemma, we fix some notation. Let M be a complete open Riemannian n-manifold. Let R_p denote the (point set) union of rays issuing from p; then R_p is a closed subset of M. Define a function h_p by

$$h_p(x) = d(x, R_p). \qquad (2.11)$$

We set for $r>0$

$$\mathcal{H}(p,r)=\max_{x\in S(p,r)} h_p(x), \tag{2.12}$$

where $S(p,r)$ is the geodesic sphere of radius r with center p.

Let M be an n-dimensional Riemannian manifold. If, for any point $x\in M$ and any $(k+1)$-mutually orthogonal unit tangent vectors $e, e_1, \dots, e_k \in T_xM$, we have $\sum_{i=1}^k K(e\wedge e_i)\geq 0$, we say that the kth Ricci curvature of M is nonnegative and denote this fact by $\mathrm{Ric}_M^{(k)}\geq 0$. Here, $K(e\wedge e_i)$ denotes the sectional curvature of the plane spanned by e and e_i, $1\leq i\leq k$.

Lemma 2.3. *Given positive numbers ϱ_0, r_0 and integers $n\geq 2$ and k, $1\leq k\leq n-1$, there is a $\delta=\delta(\varrho_0, r_0, n, k)>0$ such that any complete Riemannian n-manifold M with $\mathrm{Ric}_M^{(k)}\geq 0$, $\mathrm{conj}_M\geq\varrho_0$, $\mathrm{crit}_p\geq r_0$ and*

$$\mathcal{H}(p,r)\leq \delta r^{1/(k+1)} \tag{2.13}$$

for some $p\in M$ and all $r\geq r_0$, has infinite criticality radius at p and so is diffeomorphic to $\mathbf{R}^n$.

Lemma 2.3 is similar to Lemma 3.1 in [X2] where there is a lower bound on the sectional curvature instead of a lower bound on the conjugate radius of M. In the proof of Lemma 3.1 in [X2], one can use the Toponogov comparison theorem owing to the lower bound condition on the sectional curvature. We will not include the proof of Lemma 2.3 since it can be carried out easily by using the Toponogov type estimate of Dai–Wei [DW] and modifying the arguments in [S2] and [X2].

Theorem 1.1 is a special case of the following more general result.

Theorem 2.4. *Given $\alpha\in\left(\frac{1}{2},1\right)$, $\varrho_0>0$ and integers $n\geq 2$ and k, $1\leq k\leq n$, there are positive constants $r_0=r_0(\alpha,\varrho_0,n,k)$ and $\varepsilon=\varepsilon(\alpha,\varrho_0,n)$ such that any complete Riemannian n-manifold M with $\mathrm{Ric}_M^{(k)}\geq 0$, $\alpha_M\geq\alpha$, conjugate radius $\mathrm{conj}_M\geq\varrho_0$ and*

$$\frac{\mathrm{vol}[B(p,r)]}{\omega_n r^n}-\alpha_M\leq\frac{\varepsilon}{r^{n-(n+k)/(k+1)}} \tag{2.14}$$

for some $p\in M$ and all $r\geq r_0$, has infinite criticality radius at p and thus is diffeomorphic to $\mathbf{R}^n$.

Proof. Once we have Lemma 2.2 and Lemma 2.3, the proof of Theorem 2.4 becomes routine. We only give an outline of it. Let $r_0=r_0(\alpha,\varrho_0,n)$ be as in Lemma 2.2;

then $\text{crit}_p \geq r_0$. Let $\delta=\delta(\varrho_0, r_0, n, k)=\delta(\alpha, \varrho_0, n, k)$ be as in Lemma 2.3. We define the number $\varepsilon=\varepsilon(\alpha, \varrho_0, n, k)$ in Theorem 2.4 as

$$\varepsilon = \min\left\{ \frac{\alpha r_0^{n-(n+k)/(k+1)}}{4^n}, \frac{\alpha \delta^{n-1}}{4(3^n-1)} \right\}. \tag{2.15}$$

With the above choice of ε, by Lemma 2.3, in order to prove Theorem 2.4, we need only to show that

$$\mathcal{H}(p, r) \leq \delta r^{1/(k+1)} \quad \text{for } r \geq r_0. \tag{2.16}$$

The condition (2.14) and the nonnegativity of the Ricci curvature of M enable us to use the relative volume comparison theorem and the same discussions as in the proof of Theorem 3.2 in [X2] to show that

$$h_p(x) \leq \delta r^{1/(k+1)} \quad \text{for } x \in S(p, r) \text{ and } r \geq r_0. \tag{2.17}$$

From the definition of $\mathcal{H}(p, r)$, we know that (2.16) holds. □

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