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Asymptotic values of strongly normal functions

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Dedicated to the memory of Professor Matts Essén

Abstract. Let f be meromorphic in the open unit disc D and strongly normal; that is,

$$(1-|z|^2)f^{\#}(z) \to 0 \quad \text{as } |z| \to 1,$$

where $f^{\#}$ denotes the spherical derivative of f. We prove results about the existence of asymptotic values of f at points of $C=\partial D$. For example, f has asymptotic values at an uncountably dense subset of C, and the asymptotic values of f form a set of positive linear measure.

1. Introduction

Let D denote the unit disc $\{z:|z|<1\}$, C denote the unit circle $\{z:|z|=1\}$, and $\widehat{\mathbf{C}}$ denote the extended complex plane. Let the function f be meromorphic in D. A curve $\Gamma:z(t)$, $0\le t<1$, in D is a *boundary path* if $|z(t)|\to 1$ as $t\to 1$. The set $\overline{\Gamma}\cap C$ is called the *end* of Γ. We say that f has the *asymptotic value* $a\in\widehat{\mathbf{C}}$ if there is a boundary path $\Gamma:z(t)$, $0\le t<1$, such that

$$f(z(t)) \to a \quad \text{as } t \to 1.$$

Whenever the end of Γ is contained in a subset E of C, we say that f has the asymptotic value a *in* E; if the end of Γ is a singleton $\{\zeta\}$, then we say that f has the (*point*) asymptotic value a *at* ζ.

Recall that f is said to be *normal* if the functions

$$f(\phi(z)), \quad \text{where } \phi(z)=e^{i\theta}\left(\frac{z+a}{1+\bar{a}z}\right), \ |a|<1, \ \theta\in\mathbf{R},$$

form a normal family or, equivalently, if

$$c=\sup_{z\in D}(1-|z|^2)f^{\#}(z)<\infty, \quad \text{where } f^{\#}(z)=\frac{|f'(z)|}{1+|f(z)|^2}. \tag{1.1}$$

The quantity c is the *order of normality* of f. See [14] and [18] for properties of normal functions. For example, the modular function is normal because it omits the three values 0, 1 and ∞. By a theorem of Bagemihl and Seidel [4], all asymptotic values of non-constant normal meromorphic functions are point asymptotic values, and all such point asymptotic values are angular limits, by a theorem of Lehto and Virtanen [16]. Also, non-constant normal analytic functions are in the MacLane class $\mathcal{A}$ since they have point asymptotic values at a dense set of points in C; see [4] and [17, p. 43]. However, there exist normal meromorphic functions in D with no asymptotic values. See [16, p. 58] for an example based on a modification of the modular function.

The class $\mathcal{N}_0$ consists of functions meromorphic in D such that

$$(1.2) \qquad (1-|z|^2)f^{\#}(z)\to 0 \quad \text{as } |z|\to 1.$$

Such *little normal* functions have been characterised in various ways; see [2], and also [10], where they were called *strongly normal*. To our knowledge, no results have been published about the existence of asymptotic values for general functions in $\mathcal{N}_0$. For various subclasses of $\mathcal{N}_0$, however, a great deal is known about the existence of asymptotic values, as we now indicate.

It was noted in [1, p. 31] that the hypothesis (1.2) means that the spherical radius of the largest schlicht disc around $f(z)$ on the Riemann image surface of f tends to 0 as $|z|\to 1$. In particular, every univalent function is in $\mathcal{N}_0$. Such functions have angular limits at all points of C apart from a set of logarithmic capacity zero.

If f is meromorphic in D and

$$(1-|z|^2)f^{\#}(z)=O(1)(1-|z|)^{\varepsilon} \quad \text{as } |z|\to 1,$$

where $\varepsilon>0$, then $f\in\mathcal{N}_0$. It follows from a result of Carleson [9, p. 61] that such functions f have angular limits at all points of C apart from a set of (an appropriate) capacity zero.

The *little Bloch* class $\mathcal{B}_0$ consists of functions analytic in D such that

$$(1-|z|^2)|f'(z)|\to 0 \quad \text{as } |z|\to 1,$$

and these functions evidently lie in $\mathcal{N}_0$. Also, it is easy to see that if $f\in\mathcal{B}_0$ and g has bounded spherical derivative in $\mathbf{C}$ (for example, if g is a rational function), then $g\circ f\in\mathcal{N}_0$. There exist functions in $\mathcal{B}_0$ which have finite angular limits almost nowhere on C, but all such functions must have finite angular limits on a set of Hausdorff dimension 1, by a result of Makarov; see [19, Chapters 8 and 11]. Moreover, Rohde [21] has shown that if f is in $\mathcal{B}_0$ and f has almost no angular limits on C, then for all $\alpha \in \mathbf{C}$ the function f has angular limit α on a set of Hausdorff dimension 1. Also, Gnuschke–Hauschild and Pommerenke [13] have shown that for functions in $\mathcal{B}_0$ the set of point asymptotic values of f has positive linear measure.

In a recent paper [7], the authors showed that a locally univalent meromorphic function in $\mathcal{N}_0$ must have asymptotic values at points of an *uncountably dense* set (that is, the set meets each non-trivial arc of C in an uncountable set) and that the set $\Gamma_P(f,\gamma)$ of point asymptotic values of f in any non-trivial arc γ on C is of positive linear measure. Here we show, by a different method, that the hypothesis of local univalence can be omitted in these results.

Theorem 1. *Let f be in $\mathcal{N}_0$, let $\alpha \in \widehat{\mathbf{C}}$ and let γ be a non-trivial arc in C. If the set of points of γ at which f has asymptotic value α is at most countable, then f has angular limits at a subset of γ of positive measure.*

As will be clear from the proof of Theorem 1, if we add the hypothesis that 'f takes values arbitrarily close to α near each point of γ', then the conclusion can be strengthened to 'f has angular limits with values in any given neighbourhood of α at a subset of γ of positive measure'.

We have the following corollary of Theorem 1.

Corollary 1. *Any f in $\mathcal{N}_0$ must have angular limits at an uncountably dense subset of C.*

Note that Corollary 1 is false if we assume that f is just normal. For example, the modular function has angular limits at only countably many points of C; see [17, p. 56].

Corollary 1 shows that a non-constant meromorphic function f in $\mathcal{N}_0$ must belong to the meromorphic MacLane class $\mathcal{A}_m$, introduced in [5]. In view of the results of Makarov and Rohde about $\mathcal{B}_0$, mentioned above, it is natural to ask whether 'uncountably dense' can be replaced by 'Hausdorff dimension 1' in Corollary 1.

Our next result also implies Corollary 1. Here $\Gamma_F(f,\gamma)$ denotes the set of angular limits of f in the arc γ.

Theorem 2. *Let f be non-constant and in $\mathcal{N}_0$, and let γ be a non-trivial arc in C. Then $\Gamma_P(f,\gamma)=\Gamma_F(f,\gamma)$ has positive linear measure.*

As in Theorem 1, if we add the hypothesis that 'f takes values arbitrarily close to α near each point of γ', then the conclusion can be strengthened, in this case to '$\Gamma_P(f,\gamma)=\Gamma_F(f,\gamma)$ has positive linear measure in any given neighbourhood of α'.

The plan of the paper is as follows. In Section 2 we prove a topological lemma concerning the existence of asymptotic values of continuous functions and in Section 3 we prove several lemmas about functions in $\mathcal{N}_0$. Section 4 contains the proof of Theorem 1 and Section 5 contains the proof of Theorem 2.

2. A topological lemma

In [15] Hayman proved that certain functions which are meromorphic in $\mathbf{C}$, with relatively few poles, have asymptotic value ∞. A key lemma in his proof states that if f is meromorphic in $\mathbf{C}$, then at least one of the following is true:

(a) there is a path Γ tending to ∞ such that $f(z)\to\infty$ as $z\to\infty$ along Γ;

(b) there is a nested sequence Γ_n of Jordan curves such that $\operatorname{dist}(\Gamma_n,\Gamma_1)\to\infty$ as $n\to\infty$ and f is bounded on $\bigcup_{n=1}^{\infty}\Gamma_n$;

(c) there is a path Γ tending to ∞ on which f is bounded.

This result was extended to continuous functions in $\mathbf{C}$ by Brannan [8], and to continuous functions $u\colon\mathbf{R}^m\to[0,\infty]$, $m\geq 2$, with a strengthened version of case (c), by one of the present authors [20]. Here we need a variant of this last result, which we state in $\mathbf{C}$ though the proof extends readily to $\mathbf{R}^m$. We shall apply this result to real-valued functions on bounded simply connected domains in $\mathbf{C}$, using the fact that such domains are homeomorphic to $\mathbf{C}$.

First recall from [20] that a set E in $\mathbf{C}$ is *solid* if $\widetilde{E}=E$, where $\widetilde{E}$ denotes the union of E and its bounded complementary components; equivalently, E is solid if $\widehat{\mathbf{C}}\setminus E$ is connected. The name *full* is also used for this concept.

Lemma 1. *Let $u\colon\mathbf{C}\to[0,\infty]$ be continuous, with a bounded metric on $[0,\infty]$ giving the usual topology there. Then one of the following holds:*

(a) *there is a path Γ tending to ∞ such that*

$$u(z)\to\infty \quad \text{as } z\to\infty \text{ along } \Gamma; \tag{2.1}$$

(b) *there exist $M<\infty$ and a sequence K_n of solid, compact, connected sets such that $K_1\subset K_2\subset\ldots$, $\operatorname{dist}(\partial K_n,K_1)\to\infty$ as $n\to\infty$ and*

$$u\leq M \quad \text{on } \bigcup_{n=1}^{\infty}\partial K_n;$$

(c) *there exists M_0 such that for all $M\geq M_0$ there are infinitely many unbounded components of $\{z:u(z)\geq M\}$.*

We remark that, since the function u is uniformly continuous on compact sets, we can take the sets K_n, $n=1,2,\ldots$, in case (b), to be bounded by Jordan curves.

Proof. To prove Lemma 1, we need some further notation and results from [20]. For each M in $(0,\infty)$ we let $\mathcal{U}_M$ denote the set of components of $\{z:u(z)<M\}$. Then, for U in $\mathcal{U}_M$, we put $\Omega_U=\bigcup\{\widetilde{V}:V\in\mathcal{U}_M \text{ and } \widetilde{U}\subset\widetilde{V}\}$. The set Ω_U is solid, and $u(z)=M$, for $z\in\partial\Omega_U$. It is shown in [20, p. 313] that if $\Omega_U=\mathbf{C}$ for some M and some U in $\mathcal{U}_M$, then case (b) holds. Thus we can assume that

$$\Omega_U\neq\mathbf{C} \quad \text{for each } M\in(0,\infty) \text{ and each } U\in\mathcal{U}_M. \tag{2.2}$$

It follows from (2.2) that, for each $M\in(0,\infty)$, the set $\{z:u(z)\geq M\}$ has at least one unbounded component. We outline the argument; see [20, proof of Lemma 3] for more details. If for some M, $0<M<\infty$, there exists an unbounded Ω_U, $U\in\mathcal{U}_M$, such that $\Omega_U\neq\mathbf{C}$, then $\widehat{\mathbf{C}}\setminus\Omega_U$ is a compact, connected subset of $\widehat{\mathbf{C}}$, so $\partial\Omega_U\cup\{\infty\}$ is a compact, connected subset of $\widehat{\mathbf{C}}$, from which it follows that each component of $\partial\Omega_U$ is unbounded and so lies in an unbounded component of $\{z:u(z)\geq M\}$. On the other hand, if for some M, $0<M<\infty$, all Ω_U, $U\in\mathcal{U}_M$, are bounded, then the complement of the union of these Ω_U is an unbounded, connected subset of $\{z:u(z)\geq M\}$.

We now suppose that case (c) is false and deduce that case (a) holds. Then there is an increasing sequence M_j, $j=1,2,\ldots$, tending to ∞ with the property that there are only finitely many unbounded components of $\{z:u(z)\geq M_j\}$, for $j=1,2,\ldots$. For each j this finite number is non-zero, as noted above.

Evidently there is at least one component E_1 of $\{z:u(z)\geq M_1\}$ which contains an unbounded component of $\{z:u(z)\geq M_j\}$ for each $j=1,2,\ldots$. Then there is at least one component E_2 of $\{z:u(z)\geq M_2\}$ in E_1 which contains an unbounded component of $\{z:u(z)\geq M_j\}$ for each $j=2,3,\ldots$. Continuing in this way, we obtain unbounded components E_j of $\{z:u(z)\geq M_j\}$, $j=1,2,\ldots$, such that $E_1\supset E_2\supset\ldots$.

For $j=1,2,\ldots$, let G_j denote the (unbounded) component of $\{z:u(z)>M_j\}$ such that $G_j\supset E_{j+1}$. Then $G_j\supset G_{j+1}$, for $j=1,2,\ldots$. Thus if $z_j\in G_j$ and the path Γ is of the form $\Gamma_1\cup\Gamma_2\cup\ldots$, where Γ_j joins z_j to z_{j+1} in G_j, then we deduce that $u(z)$ tends to ∞ as z proceeds along Γ. For a general continuous function u we cannot conclude that the path Γ tends to ∞, since Γ may accumulate at an unbounded, closed, connected subset of $\mathbf{C}$ on which $u=\infty$. To overcome this problem, we consider the set $E=\overline{\Gamma}$. Then E is an unbounded, closed, connected set with the property that

$$u(z)\to\infty \quad \text{as } z\to\infty,\ z\in E.$$

Since u is uniformly continuous on compact sets, we can choose a decreasing continuous function $\delta\colon[0,\infty)\to(0,\frac{1}{2}]$ such that if

$$E_\delta=\bigcup_{\zeta\in E}\{z:|z-\zeta|<\delta(|\zeta|)\},$$

then

$$u(z)\to\infty \quad \text{as } z\to\infty,\ z\in E_\delta.$$

To complete the proof, we use the fact that the set E_δ must contain a path tending to ∞; see [20, Theorem 2]. □

3. Properties of $\mathcal{N}_0$

We recall the following results of Dragosh [12, Theorem 1 and Theorem 2], which were proved using the Lehto–Virtanen maximum principle; see Section 4. Here and in what follows we put

$$\delta(x)=\frac{1+(1+x^2)^{1/2}}{x}\exp(-(1+x^2)^{1/2}),$$

which is a decreasing function on $(0,\infty)$.

Theorem A. *Let f be meromorphic in D with order of normality c, $0<c<\infty$. Let γ be an open subarc of C and let γ_n be a sequence of arcs in D which converges to γ in the Hausdorff metric. Put $M_n=\sup_{z\in\gamma_n}|f(z)|$. If f is unbounded near any point of γ, then*

$$\liminf_{n\to\infty} M_n \geq \delta(c). \tag{3.1}$$

Dragosh used Theorem A to give a sufficient condition for membership of the class $\mathcal{L}_m$ of functions f non-constant and meromorphic in D such that the level sets of f 'end at points'. To be precise, let $d(r,\lambda)$ denote the supremum of the diameters of the components of the set

$$\{z:|f(z)|=\lambda,\ r<|z|<1\},\quad \text{where } \lambda>0 \text{ and } 0<r<1.$$

Then $f\in\mathcal{L}_m$ if, for each $\lambda>0$, we have

$$d(r,\lambda)\to 0 \quad \text{as } r\to 1;$$

see [17] and [5] for more details of this notion.

Theorem B. *Let $c^*\simeq 0.663$ be the unique solution of the equation $\delta(c)=1$. If f is meromorphic in D with order of normality $c<c^*$, then $f\in\mathcal{L}_m$.*

Next, we state a result about functions in the class $\mathcal{L}_m$, given in [5, Theorem 2]. Here we need the notion of a *tract* of f for ∞, which is a family of components D_λ of $\{z:|f(z)|>\lambda\}$, $\lambda>0$, such that $D_{\lambda_2}\subset D_{\lambda_1}$, for $\lambda_2>\lambda_1$, and $\bigcap_{\lambda>0} D_\lambda=\emptyset$. The set $E=\bigcap_{\lambda>0}\overline{D_\lambda}$ is called the *end* of the tract, and the function f has asymptotic value ∞ at each point of E.

Theorem C. *Let f be in $\mathcal{L}_m$ and suppose that γ is a non-trivial arc of C such that no level curve of f ends at any point of γ. Then exactly one of the following statements holds*:

(a) *for each interior point ζ of γ there exists a path Γ_ζ in D ending at ζ, such that f is bounded on $\bigcup\{\Gamma_\zeta:\zeta\in\gamma\}$;*

(b) *there exists a tract of f for ∞ with end containing γ.*

We use Theorems A, B and C to prove the following result about $\mathcal{N}_0$.

Lemma 2. *Let f be in $\mathcal{N}_0$. Then*

(a) *if f is bounded on a sequence γ_n of arcs in D which converges in the Hausdorff metric to an open arc γ in C, then f is bounded near each point of γ;*

(b) *there is a dense set of points in C, each of which is the end of a path in D on which f is bounded.*

Proof. To prove part (a) suppose that f is unbounded near some point ζ_0 of γ. For some $M>1$, we have

$$M_n = \sup_{z\in\gamma_n} |f(z)| \le M < \infty. \tag{3.2}$$

Since $f\in\mathcal{N}_0$, we can choose a boundary neighbourhood D_0 of ζ_0 in D such that $\gamma_0 = C\cap\partial D_0 \subset \gamma$ and

$$(1-|z|^2)f^{\#}(z) < c, \quad z\in D_0, \tag{3.3}$$

where c is so small that $\delta(c)>M$.

Let $\phi\colon D\to D_0$ be conformal and put $g(t)=f(\phi(t))$. Then, by the Schwarz–Pick lemma,

$$(1-|t|^2)|\phi'(t)| \le 1-|\phi(t)|^2,$$

so

$$(1-|t|^2)g^{\#}(t) \le (1-|\phi(t)|^2)f^{\#}(\phi(t)) < c, \quad t\in D.$$

Thus the order of normality of g is at most c. Now, for $n=1,2,\dots$, choose a component γ'_n of $\gamma_n\cap D_0$ in such a way that γ'_n tends to γ_0 in the Hausdorff metric. Then $\phi^{-1}(\gamma'_n)$ is a sequence of arcs in D tending to the open arc $\phi^{-1}(\gamma_0)$, and g is unbounded near $\phi^{-1}(\zeta_0)$, which is in $\phi^{-1}(\gamma_0)$. Since $|g(t)|\le M_n$, for $t\in\phi^{-1}(\gamma'_n)$, we deduce by Theorem A that

$$\liminf_{n\to\infty} M_n \ge \delta(c) > M,$$

which contradicts (3.2).

To prove part (b) suppose that γ_0 is a non-trivial arc of C. We can choose a boundary neighbourhood D_0 in D such that $\gamma_0=C\cap\partial D_0$ and (3.3) holds with $c<c^*$. As in the proof of part (a), we take $\phi\colon D\to D_0$ to be conformal, so $g(t)=f(\phi(t))$ is normal of order at most c. Thus $g\in\mathcal{L}_m$ by Theorem B. Also, since g is normal, it cannot have a tract for ∞ with end containing an arc, by the theorem of Bagemihl and Seidel; see [4]. Hence, by applying Theorem C to g on the arc $\phi^{-1}(\gamma_0)$, we deduce either that a level curve of f ends at a point of γ_0 or that f is uniformly bounded on a family of boundary paths Γ_ζ with endpoints at interior points ζ in γ_0. This proves part (b). □

Next we need a result about the level sets of functions in $\mathcal{N}_0$.

Lemma 3. *Let f be in $\mathcal{N}_0$, let Ω be a simply connected Jordan domain in D such that each component of $\partial\Omega\cap D$ is part of a level set of the form $\{z:|f(z)|=\lambda\}$, where $0<\lambda\le\lambda_0$, and let ϕ be a conformal map of D onto Ω. Then*

(a) *the function $g=f\circ\phi\in\mathcal{N}_0$;*

(b) *if there is some $\mu>0$ such that the components of $\{z:|f(z)|\ge\mu\}$ in Ω are all compact, then f is bounded in Ω near $\partial\Omega$; that is, there is a compact subset K of Ω such that f is bounded in $\Omega\setminus K$.*

Proof. To prove part (a) suppose, for a contradiction, that for some sequence t_n in D, we have $|t_n|\to 1$ and

$$(1-|t_n|^2)g^{\#}(t_n)\ge\varepsilon>0, \quad n=1,2,\ldots . \tag{3.4}$$

Without loss of generality, we have $t_n\to t_0\in C$ and $\phi(t_n)\to z_0\in\overline{D}$. If $z_0\in C$, then (3.4) together with the inequality

$$(1-|t|^2)g^{\#}(t)\le(1-|\phi(t)|^2)f^{\#}(\phi(t)), \quad t\in D,$$

contradict the fact that $f\in\mathcal{N}_0$. If $z_0\notin C$, then $z_0\in\partial\Omega\cap D$. Hence $|f(z)|=\lambda$ for z near z_0 on $\partial\Omega$, so $|g(t)|=\lambda$ for t near t_0 on C. Since f is analytic near z_0, we deduce that g has an analytic continuation to a neighbourhood of t_0, which contradicts (3.4). Hence $g\in\mathcal{N}_0$.

To prove part (b), note that the function $u=|g|$ cannot satisfy case (a) or case (c) of Lemma 1 in D, since each of these cases implies the existence of non-compact components of $\{z:|g(z)|\ge\mu\}$ in D for arbitrarily large μ and hence non-compact components of $\{z:|f(z)|\ge\mu\}$ in Ω for arbitrarily large μ. Thus $u=|g|$ satisfies case (b) of Lemma 1, so g is bounded in D near C by Lemma 2 and the remark following the statement of Lemma 1, because $g\in\mathcal{N}_0$. Hence f is bounded in Ω near $\partial\Omega$, as required. □

4. Proof of Theorem 1

Without loss of generality, we may assume in the proof that $\alpha=\infty$, since we obtain a function in $\mathcal{N}_0$ by composing f with a rotation of the Riemann sphere taking α to ∞.

We shall assume that f is in $\mathcal{N}_0$ and has angular limits almost nowhere in γ, and then deduce that f has asymptotic value ∞ at points of an uncountable subset of γ. The first step is to show that there is at least one point in γ where f has asymptotic value ∞. By Lemma 2, part (b), we can choose a cross-cut γ' of D with distinct endpoints in γ on which f is bounded, say $|f|\leq M'$. Then let $D(\gamma)$ denote the component of $D\backslash\gamma'$ such that $\partial D(\gamma)\cap C\subset\gamma$.

Since $D(\gamma)$ is homeomorphic to $\mathbf{C}$, we can apply Lemma 1 to $u=|f|$ in $D(\gamma)$. If case (a) occurs, then f has asymptotic value ∞ in γ, as required. Case (b) does not occur since, by Lemma 2, part (a), and the remark following Lemma 1, this would imply that f is bounded near interior points of $\partial D(\gamma)\cap C$ and so f would have angular limits at almost every point of this arc by Fatou's theorem, contrary to our assumption. If case (c) of Lemma 1 holds, then there exists $M_0>M'$ such that, for all $M>M_0$, there are infinitely many non-compact components of $\{z:|f(z)|\geq M\}$ in $D(\gamma)$.

We now consider components D_λ of sets of the form $\{z:|f(z)|>\lambda\}$, where $\lambda>0$. Following the usage in [11, p. 123], we say that such a component D_λ is *unbounded* if ∂D_λ meets C, and D_λ is *bounded* otherwise. From the above argument, it follows that if Lemma 1, case (c) holds, then we can choose $\mu>\lambda>M_0$ and unbounded components D_μ and D_λ of $\{z:|f(z)|>\mu\}$ and $\{z:|f(z)|>\lambda\}$, respectively, such that

$$D(\gamma)\supset D_\lambda\supset D_\mu.$$

We call such a pair of unbounded components (D_λ, D_μ) an *unbounded component pair* for f in $D(\gamma)$.

Lemma 4. *Let f and $D(\gamma)$ be as above, and suppose that (D_λ, D_μ) is an unbounded component pair for f in $D(\gamma)$. Then*

(a) *D_λ contains an unbounded component pair $(D_{\lambda'}, D_{\mu'})$, with $\lambda'>\lambda+1$;*

(b) *D_λ contains a tract of f for ∞, so f has asymptotic value ∞ at some point of γ.*

First note that if D_λ contains an unbounded component $D_{\mu'}$, where $\mu'>\lambda+1$, then we can choose λ' with $\mu'>\lambda'>\lambda+1$ and take $D_{\lambda'}$ to be the component of $\{z:|f(z)|>\lambda'\}$ that contains $D_{\mu'}$.

Otherwise, for $\mu'>\lambda+1$, the components of $\{z:|f(z)|>\mu'\}$ in D_λ are all bounded. Now let $\widetilde{D}_\lambda$ denote the union of D_λ and its compact complementary components, and let ϕ be a conformal map from D onto $\widetilde{D}_\lambda$. Note that $\widetilde{D}_\lambda$ is a Jordan domain, because $f\in\mathcal{L}_m$; see the proof of Lemma 2 (b). Thus ϕ can be extended to a homeomorphism from ∂D onto $\partial\widetilde{D}_\lambda$. Also, each component of $\partial\widetilde{D}_\lambda\cap D$ is part of the level set $\{z:|f(z)|=\lambda\}$. Thus, by Lemma 3, the function $g(t)=f(\phi(t))$ is in $\mathcal{N}_0$ and $|g|$ is bounded near C, by μ'' say. Hence g has finitely many poles in D and finite angular limits a.e. on C, by Fatou's theorem.

Thus we can choose a finite Blaschke product B such that gB is analytic in D and hence $|gB|$ is bounded there by μ''. But $|gB|$ is not bounded in D by λ, since there exist points of D in $\phi^{-1}(D_\mu)$ where $|g|>\mu>\lambda$, and these points are arbitrarily close to C because D_μ is unbounded. Hence, by the extended maximum principle, the angular limits of gB exceed λ in modulus on a set $E\subset C$ of positive length, and this must also hold for g. Since $|f|=\lambda$ on $\partial\widetilde{D}_\lambda\cap D$, it follows that the set $\phi(E)$ is contained in γ and has positive harmonic measure with respect to $\widetilde{D}_\lambda$ and therefore positive length, by the domain extension principle. But f has an asymptotic value, and hence an angular limit, at each point of $\phi(E)$, which contradicts our initial assumption about f. This proves part (a).

We deduce from part (a) that $D(\gamma)$ contains a sequence of unbounded component pairs $(D_{\lambda_n}, D_{\mu_n})$, $n=0,1,2,\dots$, such that $D_{\lambda_0}=D_\lambda$ and

$$D_{\lambda_n}\supset D_{\lambda_{n+1}},\quad \lambda_{n+1}>\lambda_n+1,\ n=0,1,2,\dots .$$

Therefore the sequence D_{λ_n}, $n=0,1,2,\dots$, determines a tract for ∞ of f. If we now choose $z_n\in D_{\lambda_n}$, $n=0,1,2,\dots$, such that $|z_n|\to 1$ as $n\to\infty$, and take the path Γ to be of the form $\Gamma_1\cup\Gamma_2\cup\dots$, where Γ_n joins z_n to z_{n+1} in D_{λ_n}, then Γ tends to C in $D(\gamma)$, since Γ cannot accumulate at any point of $\overline{D(\gamma)}\setminus C$. Thus f has asymptotic value ∞ along Γ, so f has asymptotic value ∞ at a point of γ. This proves part (b) of Lemma 4.

From Lemma 4 and the discussion before it, we deduce that if f is in $\mathcal{N}_0$ and has angular limits almost nowhere in γ, then f has asymptotic value ∞ at some point of γ; in particular, $D(\gamma)$ must contain an unbounded component pair for f.

The next lemma will enable us to deduce that there are uncountably many points in γ at which f has asymptotic value ∞. The argument is a modification of the proof of [6, Theorem 1].

Lemma 5. *Let f and $D(\gamma)$ be as above, and suppose that (D_λ, D_μ) is an unbounded component pair for f in $D(\gamma)$. Then*

(a) *there exists $\lambda'>\lambda+1$ such that D_λ contains distinct unbounded component pairs $(D^i_{\lambda'}, D^i_{\mu'})$, $i=1,2$;*

(b) *D_λ contains uncountably many tracts of f for ∞.*

By Lemma 4, part (a), there is a sequence of unbounded component pairs $(D_{\lambda_n}, D_{\mu_n}), n=0,1,2,\dots$, such that $D_{\lambda_0}=D_\lambda$ and

$$D_{\lambda_n}\supset D_{\lambda_{n+1}},\quad \lambda_{n+1}>\lambda_n+1,\ n=0,1,2,\dots .$$

The D_{λ_n}, $n=0,1,2,\dots$, determine a tract for ∞ of f, the end of which must be a point of γ, say ζ_0, since f is normal. Hence f has asymptotic value ∞ at ζ_0 and

(4.1) $$\operatorname{diam} D_{\lambda_n}\to 0\quad \text{as } n\to\infty.$$

If the assertion in part (a) is false, then (because $\lambda_n > \lambda_0 + 1 = \lambda + 1$, for $n \geq 1$) each of the unbounded component pairs $(D_{\lambda_n}, D_{\mu_n})$ is unique in D_{λ_0}. It follows that, for $n \geq 1$, the set

$$G_n = D_{\lambda_0} \setminus \overline{\widetilde{D}_{\lambda_n}} \tag{4.2}$$

contains no unbounded component pair $(D'_{\lambda_n}, D'_{\mu_n})$. Hence the components, if any, of $\{z : |f(z)| > \mu_n\}$ in G_n are bounded; for otherwise we could take an unbounded component $D'_{\mu_n} \subset G_n$ and the corresponding component D'_{λ_n} of $\{z : |f(z)| > \lambda_n\}$, with $D'_{\mu_n} \subset D'_{\lambda_n} \subset G_n$, to give a different unbounded component pair in D_{λ_0}. Now $\widetilde{G}_n$ is a Jordan domain because $f \in \mathcal{L}_m$, so $\widetilde{G}_n$ satisfies the hypotheses for Ω in Lemma 3. Thus, by Lemma 3, part (b), we deduce that

$$f \text{ is bounded in } \widetilde{G}_n \text{ near } \partial\widetilde{G}_n \text{ for } n \geq 1. \tag{4.3}$$

In the remainder of the proof, we consider two cases. First suppose that the set $\partial\widetilde{D}_{\lambda_0} \cap C$ has positive harmonic measure with respect to $\widetilde{D}_{\lambda_0}$. In this case we can apply the following result [6, Lemma 1].

Theorem D. *Let G be a simply connected Jordan domain with $G \subset D$, and suppose that $E = \partial G \cap C$ has positive harmonic measure with respect to G. Then there is a subset E_1 of E of positive length such that each ζ in E_1 is the vertex of an open Stolz angle S_ζ contained in G.*

Applying Theorem D with $G = \widetilde{D}_{\lambda_0}$, we deduce that if $\zeta \in E_1 \setminus \{\zeta_0\}$, then $S_\zeta \subset \widetilde{G}_m$, for some Stolz angle S_ζ and some $m \geq 1$, by (4.1) and (4.2). Thus, by (4.3), we can assume that f is bounded in each such S_ζ. Plessner's theorem [11, p. 147] then gives a contradiction to our initial assumption that f has angular limits almost nowhere in γ.

Otherwise, the set $\partial\widetilde{D}_{\lambda_0} \cap C$ has harmonic measure zero with respect to $\widetilde{D}_{\lambda_0}$. In this case we claim that

$$\limsup_{\substack{z \to \zeta \\ z \in \widetilde{D}_{\lambda_0}}} |f(z)| \leq \lambda_0 \quad \text{for } \zeta \in (\partial\widetilde{D}_{\lambda_0} \cap C) \setminus \{\zeta_0\}. \tag{4.4}$$

To prove (4.4) suppose that $\zeta_1 \in (\partial\widetilde{D}_{\lambda_0} \cap C) \setminus \{\zeta_0\}$. Then by (4.1) there exists $m \geq 1$ and a boundary neighbourhood

$$N_1 = \{z \in D : |z - \zeta_1| < \varrho_1\}, \quad \varrho_1 > 0,$$

such that $N_1 \cap \widetilde{D}_{\lambda_0} \subset \widetilde{G}_m$, and f is bounded in $N_1 \cap \widetilde{D}_{\lambda_0}$ by (4.3). If H is a component of $N_1 \cap \widetilde{D}_{\lambda_0}$ which contains ζ_1 in its closure, then H is regular for the Dirichlet problem, since $\widetilde{D}_{\lambda_0}$ is a Jordan domain. Also $\partial H \cap C$ has harmonic measure zero with respect to H, by the domain extension principle. The function

$$h(\zeta) = \begin{cases} |f(\zeta)| & \text{for } \zeta \in \partial H \cap D, \\ \lambda_0 & \text{for } \zeta \in \partial H \cap C, \end{cases}$$

is bounded on ∂H and continuous there, except possibly at the points (at most two) of $\partial H \cap C \cap \{z : |z-\zeta_1| = \varrho_1\}$. Hence the Dirichlet problem for h in H has a unique solution, h say, which is bounded in H and continuous on $\overline{H}$, except possibly at the points just mentioned. In particular,

$$\lim_{\substack{z \to \zeta_1 \\ z \in H}} h(z) = \lambda_0.$$

Now the function

$$u(z) = |f(z)| - h(z), \quad z \in H,$$

is subharmonic and bounded above in H, with boundary value zero at each point of ∂H, except for a subset of harmonic measure zero. Thus, by the extended maximum principle, $u \leq 0$ in H. Hence (4.4) holds.

It follows from (4.4) that $\partial \widetilde{D}_{\lambda_1}$ cannot include any point of $(\partial \widetilde{D}_{\lambda_0} \cap C) \setminus \{\zeta_0\}$, so $\partial \widetilde{D}_{\lambda_1} \cap D$ is a simple path Γ_1 approaching ζ_0 at both ends, on which $|f| = \lambda_1$. Since f has asymptotic value ∞ at ζ_0, and hence angular limit ∞ at ζ_0, we have obtained a contradiction to the following theorem of Anderson, Clunie and Pommerenke; see [1, p. 31]. Here $C_E(f,\zeta)$, $\zeta \in C$, denotes the *cluster set* of f along a set $E \subset D$ such that $\zeta \in \overline{E}$; that is, the set of all limits of sequences of the form $f(z_n)$, where $z_n \to \zeta$, $z_n \in E$.

Theorem E. *Let f be in $\mathcal{N}_0$, let Γ be a path in D ending at ζ in C, and let S be any Stolz angle with vertex at ζ. Then*

$$C_S(f,\zeta) \subset C_\Gamma(f,\zeta).$$

This proves Lemma 5, part (a), and part (b) follows immediately.

To complete the proof of Theorem 1 we use the fact that f can have at most one tract for ∞ ending at each point ζ of C. Indeed, if a normal function f has asymptotic value α along two paths approaching ζ in C, then $f(z)$ must tend to α as z tends to ζ between the paths. It is sufficient to prove this result for $\alpha = 0$, and we can do this by combining results from [18] and [19]. First we state a version of the maximum principle of Lehto and Virtanen. This involves the real function $\delta(x)$ defined in Section 3.

Theorem F. *Let f be meromorphic in D and normal of order c, let A be an open arc of C and let B_λ, $0<\lambda<\pi$, be the lens-shaped domain in D bounded by A and the circular arc in D making angle λ with A. Let G be a domain in B_λ with $\partial G\subset\overline{B}_\lambda\setminus A$. If*

$$|f(z)|\le\delta<\delta(\varkappa)\quad \textit{for } z\in\partial G\setminus\partial B_\lambda,$$

where $\varkappa=c\lambda/\sin\lambda$, then

$$|f(z)|\le\eta\quad \textit{for } z\in G,$$

where $\eta=\eta(\delta,c,\lambda)$ is the smallest positive solution of

$$\delta=\eta\exp\left(-\frac{\varkappa}{2}\left(\eta+\frac{1}{\eta}\right)\right).$$

Note that, for fixed c and λ, we have $\eta(\delta,c,\lambda)\to0$ as $\delta\to0$. Theorem F is given in [18, Theorem 9.1], with the extra assumption made there that $\overline{G}\subset D$. To deduce the above version, we can apply this special case to a sequence of lens-shaped regions approximating B_λ from within, as described in [19, part (b) of the proof of Theorem 4.2].

Suppose now that f has asymptotic value 0 at ζ along a simple path Γ. It is sufficient to show that $f(z)$ tends to 0 as z tends to ζ between Γ and the radius R_ζ of D with endpoint at ζ. To do this we show that, for each $\varepsilon>0$, there is a Jordan domain in D, in which $|f|\le\varepsilon$ and in the closure of which Γ and R_ζ both eventually lie.

Let Γ_δ be a subpath of Γ such that $|f|\le\delta$ on Γ_δ, where δ is so small that $\eta\left(\delta,c,\frac{3}{4}\pi\right)\le\varepsilon$. Following the proof of [19, Theorem 4.3], we construct a pair of open discs $D^\pm$ such that the circles $C^\pm=\partial D^\pm$ each pass through ζ, making the angle $\frac{3}{4}\pi$ with C, and also $\Gamma_\delta\cap(C^+\cup C^-)\neq\emptyset$. Then the radius R_ζ lies eventually in $D^+\cap D^-$. We may also assume that $\Gamma_\delta\cap(D^+\cup D^-)$ is connected. Then let $V^\pm$ be the component of $D^\pm\setminus\Gamma_\delta$ that contains points of the unit circle C, and put $G^\mp=D^\pm\setminus(V^\pm\cup\Gamma_\zeta)$. Since $V^\pm$ are disjoint, we have $D^+\cap D^-\subset G^+\cup G^-\cup\Gamma_\delta$. It follows that

$$\operatorname{int}(G^+\cup G^-\cup\Gamma_\delta)$$

is a Jordan domain in D, in the closure of which Γ and R_ζ both eventually lie and in which $|f|\le\varepsilon$, by Theorem F. This completes the proof of Theorem 1.

5. Proof of Theorem 2

To prove Theorem 2, we establish the following lemma.

Lemma 6. *Let f be in $\mathcal{N}_0$, and let γ be a non-trivial arc of C. If $\Gamma_P(f,\gamma)$ has linear measure zero, then f has asymptotic value ∞ at a point of γ.*

As noted earlier, we can replace ∞ here by any α in $\widehat{\mathbf{C}}$, so Theorem 2 follows immediately from Lemma 6.

The proof of Lemma 6 is similar to the proof of the first part of Theorem 1, that is, up to Lemma 4. There we assumed that f has angular limits at almost no points of γ and deduced that f has asymptotic value ∞ at some point of γ. Here our initial assumption is that $\Gamma_P(f,\gamma)$ has linear measure zero, and we again wish to deduce that f has asymptotic value ∞ at some point of γ.

We need the following slight generalisation of a theorem of Collingwood and Cartwright.

Theorem G. *Let f be meromorphic in D and bounded in a simply connected Jordan domain Ω in D such that $\partial\Omega\cap C$ is an arc γ of C. Let f have angular limits $w_1\neq w_2$ at interior points $\zeta_1\neq\zeta_2$ of γ, and let L be a polygonal path joining w_1 and w_2, with the property that for any line M normal to L the points w_1 and w_2 lie in different components of $\mathbf{C}\setminus M$. Then, for each line segment L' of L, we have*

(a) *the orthogonal projection of $\Gamma_P(f,\gamma)$ on the interior of L' includes all of the interior of L';*

(b) *the set of points of $\Gamma_P(f,\gamma)$ that can be projected orthogonally onto L' has positive linear measure.*

In the original result [11, p. 120], the path L consisted of a single line segment. A similar proof works in this more general case, since the polygonal path L has the property that any smooth path from w_1 to w_2 meets any normal line to L.

In the part of the proof of Theorem 1 before Lemma 4, we can replace the use of Fatou's theorem by that of part (b) of Theorem G in order to deduce that there is an unbounded component pair for f in $D(\gamma)$. In the proof of Lemma 4, part (a), a little more work is required. There we have a function $g=f\circ\phi$ that is meromorphic in D and bounded near C. Moreover, the angular limits of g on C include a point w_1 with $|w_1|=\lambda$, and a point w_2 with $|w_2|>\lambda$. Thus we can apply part (b) of Theorem G to the function g on an annulus $\Omega=\{z:r_0<|z|<1\}$, taking the path L from w_1 to w_2 to consist of (at most) two line segments, one of which is the shortest line segment from w_2 to the circle $\{w:|w|=\lambda\}$. We deduce that the set $\Gamma_P(f,C)\cap\{w:|w|>\lambda\}$ has positive linear measure. Since each angular limit of g with modulus greater than λ is also a point of $\Gamma_P(f,\gamma)$, we deduce that $\Gamma_P(f,\gamma)$ has positive linear measure, which contradicts our initial assumption. Thus the proof of Lemma 4, part (a), goes through with this new assumption. This completes the proof of Lemma 6 and hence that of Theorem 2.

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