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Total curvature and rearrangements

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Abstract. We study to what extent rearrangements preserve the integrability properties of higher order derivatives. It is well known that the second order derivatives of the rearrangement of a smooth function are not necessarily in L^1. We obtain a substitute for this fact. This is done by showing that the total curvature for the graph of the rearrangement of a function is bounded by the total curvature for the graph of the function itself.

1. Introduction

The purpose of this note is to study the regularity properties of the decreasing rearrangement of a function. Let f be a real-valued, bounded and measurable function on an interval $I=[a,b]$. Its decreasing rearrangement f^* is characterised by the following properties:

(a) f^* is bounded and decreasing on I;

(b) f^* is right continuous on $[a,b)$ and left continuous at b;

(c) f^* and f are equimeasurable, i.e.,

$$|\{x\in I:f^*(x)>\lambda\}|=|\{x\in I:f(x)>\lambda\}|$$

for all $\lambda\in\mathbf{R}$.

Here $|E|$ denotes the Lebesgue measure of the measurable set E. We refer to Hardy, Littlewood and Pólya [2] for the classical theory. The monograph by Pólya and Szegő [4] contains a wealth of applications of rearrangements to symmetrization and isoperimetric inequalities.

We recall that

$$\int_I \varphi(f^*)\,dx=\int_I \varphi(f)\,dx \tag{1}$$

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for all continuous functions φ. The basic regularity result for rearrangements is that if $1 \le p \le \infty$ and if the derivative of f belongs to $L^p(I)$, then f^* has the same property. More precisely,

$$\left\| \frac{df^*}{dx} \right\|_p \le \left\| \frac{df}{dx} \right\|_p, \tag{2}$$

where $\|f\|_p = (\int_I |f|^p \, dx)^{1/p}$.

We shall in this paper study how rearrangements preserve the integrability properties of higher order derivatives. We remark that it is easy to give examples of smooth functions f such that $d^2 f^*/dx^2$ does not belong to L^1. For example, letting

$$\begin{aligned} f(x) &= 2x^3 - 9x^2 + 12x, \qquad\qquad 0 \le x \le 3, \\ g(x) &= (8x^3 - 36x^2 + 30x + 153)/32 \end{aligned}$$

then (see Talenti [5])

$$f^*(x) = \begin{cases} f(3-x), & x \in \left[0, \frac{1}{2}\right] \cup \left[\frac{5}{2}, 3\right], \\ g(x), & x \in \left[\frac{1}{2}, \frac{5}{2}\right]. \end{cases}$$

Notice however, that in this case df^*/dx is of bounded variation.

For a bounded function f on $I = [a, b]$ let

$$\|f\|_C = \sup\left\{ \left| \int_I f \varphi'' \, dx \right| : \varphi \in C_0^\infty(a, b) \text{ and } \|\varphi\|_\infty \le 1 \right\}. \tag{3}$$

Here $C_0^\infty(a, b)$ denotes the class of infinitely many times continuously differentiable functions supported in (a, b). We remark that if f is smooth, then

$$\|f\|_C = \int_I |f''| \, dx.$$

We shall establish the following analogue of (2).

Theorem 1.1. *Suppose f is real-valued, bounded and measurable on $[a, b]$. Then*

$$\|f^*\|_C \le \|f\|_C. \tag{4}$$

We shall derive (4) by analysing the total curvature of the graphs of f and f^*, respectively.

Let $\gamma(t)$, $a\le t\le b$, be a simple curve in the plane and let $X=\{\xi_0,\dots,\xi_M\}$ be a partition of $[a,b]$, i.e., $a=\xi_0<\xi_1<\dots<\xi_M=b$ and let

$$e_i=\frac{\gamma(\xi_{i+1})-\gamma(\xi_i)}{|\gamma(\xi_{i+1})-\gamma(\xi_i)|},\quad 0\le i\le M-1.$$

Set

$$\mathcal{B}(\gamma,X)=\sum_{i=1}^{M-1}\delta_i,$$

where δ_i is the length of the shortest arc on $S^1=\{p\in\mathbf{R}^2:|p|=1\}$ joining e_{i-1} and e_i. Finally, the total curvature of γ is

$$\mathcal{B}(\gamma)=\sup_X\mathcal{B}(\gamma,X), \tag{5}$$

where the supremum is taken over all partitions X of $[a,b]$. We refer to Milnor [3] for the basic properties of the total curvature of arcs. We remark that if γ is a smooth curve with curvature k, then it can be shown (Milnor [3]) that

$$\mathcal{B}(\gamma)=\int_\gamma|k|\,ds, \tag{6}$$

where the integration is taken with respect to the arc length of γ. For $f:[a,b]\to\mathbf{R}$ continuous let $T(f)$ denote the total curvature of the graph of f.

Theorem 1.2. *Suppose $f:[a,b]\to\mathbf{R}$ is continuous. Then*

$$T(f^*)\le T(f). \tag{7}$$

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2. Preliminary results

We shall from now on let $I=[a,b]$ be an interval. Let $C(I)$ be the class of continuous and real-valued functions on I. If $f\in C(I)$, then f^* denotes the decreasing rearrangement. Notice that $f^*\in C(I)$ also. For $x\in I$ let $S(x)=a+b-x$. Notice that S maps I onto itself. If $g(x)=f(S(x))$, then

$$g^*=f^*. \tag{8}$$

If $h(x)=-f(x)$, then

$$h^*(x)=-f^*(S(x)). \tag{9}$$

Let $X=\{\xi_0,\dots,\xi_N\}$ be a partition of I and let $\gamma\colon I\to\mathbf{R}^2$ be a simple polygon with nodes at ξ_i, i.e., $\gamma\colon I\to\mathbf{R}^2$ is continuous, one-to-one and its restriction to the intervals $[\xi_i,\xi_{i+1}]$ is linear for $0\le i\le N-1$. Then it is well known (see Milnor [2]) that

$$\mathcal{B}(\gamma)=\mathcal{B}(\gamma,X). \tag{10}$$

In particular, if f is piecewise linear with nodes at ξ_i, $0\le i\le N$, we have

$$T(f)=\sum_{i=1}^{N-1}|\varphi_{i+1}-\varphi_i|, \tag{11}$$

where $\varphi_i\in\left(-\frac{1}{2}\pi,\frac{1}{2}\pi\right)$ is defined by

$$\tan\varphi_i=\frac{f(\xi_i)-f(\xi_{i-1})}{\xi_i-\xi_{i-1}}. \tag{12}$$

For $E\subset\mathbf{R}^d$ we let $\operatorname{Int}(E)$ and ∂E denote the interior and the boundary of the set E. Let

$$\mathcal{D}=\left\{x\in\mathbf{R}:0\le x\le\tfrac{1}{2}\pi\right\}$$

and define $\gamma\colon\mathcal{D}^2\to\mathcal{D}$ by

$$\begin{aligned}\cot\gamma(x,y)&=\cot x+\cot y, &&\text{if } (x,y)\in\operatorname{Int}(\mathcal{D}^2),\\ \gamma(x,y)&=\min\{x,y\}, &&\text{if } (x,y)\in\partial\mathcal{D}^2.\end{aligned}$$

Then γ is continuous on $\mathcal{D}^2$.

Proposition 2.1. *The function γ has the following properties:*
(i) $\gamma(x,y)=\gamma(y,x)$ *for* $(x,y)\in\mathcal{D}^2$;
(ii) $\gamma(x,\frac{1}{2}\pi)=x$ *for* $x\in\mathcal{D}$;
(iii) $0\le\gamma(x,y)\le\min\{x,y\}\le x$ *for* $(x,y)\in\mathcal{D}^2$;
(iv) $0<\partial\gamma(x,y)/\partial x<1$ *for* $(x,y)\in\mathrm{Int}(\mathcal{D}^2)$;
(v) $\partial\gamma(x,y)/\partial x<\partial\gamma(x,z)/\partial x$ *if* $x\in\mathrm{Int}(\mathcal{D})$ *and* $0<y<z<\frac{1}{2}\pi$.

Proof. The first three properties are obvious from the definition of γ. The last two follow from the identity

$$\frac{\partial\gamma(x,y)}{\partial x}=\frac{\cot^2 x+1}{\cot^2\gamma+1},\quad (x,y)\in\mathrm{Int}(\mathcal{D}^2),$$

which completes the proof of the proposition. □

The function γ will be used for computing the rearrangements of piecewise linear functions. The following lemma gives its basic role.

Lemma 2.1. *Let $I_1=(a_1,b_1)$ and $I_2=(a_2,b_2)$ be disjoint, open and bounded intervals of positive length. Let I be an interval of length $|I_1|+|I_2|$. Set $E=I_1\cup I_2$ and assume $f\colon E\to\mathbf{R}$ has a linear restriction to the subintervals I_1 and I_2 with $f(I_1)=f(I_2)$. Let $(\alpha,\beta)\in\mathrm{Int}(D^2)$, assume $|f'|=\tan\alpha$ in I_1 and $|f'|=\tan\beta$ in I_2 and set $\gamma=\gamma(\alpha,\beta)$. Then there is a decreasing linear function $g\colon I\to\mathbf{R}$ such that*

$$g'=-\tan\gamma$$

and

$$|\{x\in I:g(x)>\lambda\}|=|\{x\in E:f(x)>\lambda\}| \tag{13}$$

for all $\lambda\in\mathbf{R}$.

Proof. Let $J=(A,B)$, $A<B$, the range of f, i.e.,

$$J=f(E)=f(I_1)=f(I_2).$$

We may assume $f(b_1)=B$, otherwise we replace f by $f(a_1+b_1-x)$ on I_1. Similarly, we may assume $f(a_2)=B$ so $f(a_1)=f(b_2)=A$.

There is also no loss in generality in assuming $a_1<b_1=a_2<b_2$ so that f is continuous in $E=(a_1,b_2)$. Elementary geometry shows that if g is the linear function on E with $g(a_1)=B$ and $g(b_2)=A$, then g satisfies (13) and $g'=-\tan\gamma$. The lemma is proved. □

We shall next show some inequalities involving the function γ. We first define $a_n, b_n: \mathcal{D}^n\times\mathcal{D}^n\to\mathbf{R}$ by $a_1(x,y)=\gamma(x,y)$ and $b_1(x,y)=x+y$ if $x,y\in\mathcal{D}$.

If $n\geq 2$ and $x,y\in\mathcal{D}^n$, we set

$$a_n(x,y)=\gamma(x_1,y_1)+\sum_{i=1}^{n-1}|\gamma(x_i,y_i)-\gamma(x_{i+1},y_{i+1})|,$$

$$b_n(x,y)=x_1+y_1+\sum_{i=1}^{n-1}(|x_i-x_{i+1}|+|y_i-y_{i+1}|).$$

We next define $\alpha_n,\beta_n:\mathcal{D}^n\times\mathcal{D}^n\times\mathcal{D}\to\mathbf{R}$ by

$$\alpha_1(x,y,t)=\gamma(x,y)+|t-\gamma(x,y)| \quad\text{and}\quad \beta_1(x,y,t)=x+y+|t-x|$$

for $x,y,t\in\mathcal{D}$. If $n\geq 2$ and if $x,y\in\mathcal{D}^n$, $t\in\mathcal{D}$, we set

$$\alpha_n(x,y,t)=a_n(x,y)+|t-\gamma(x_n,y_n)|,$$
$$\beta_n(x,y,t)=b_n(x,y)+|t-x_n|.$$

We can now give some basic inequalities.

Proposition 2.2. *Let $n\geq 1$ and let $x,y\in\mathcal{D}^n$ and $t\in\mathcal{D}$. Then*

$$a_n(x,y)\leq b_n(x,y), \tag{14}$$
$$\alpha_n(x,y,t)\leq \beta_n(x,y,t). \tag{15}$$

We shall base the proof of Proposition 2.2 on the following lemma.

Lemma 2.2. *Suppose $f:\mathcal{D}\to\mathbf{R}$ satisfies $0\leq f'\leq 1$. Let $\theta\in\mathcal{D}$ and $A\in\mathbf{R}$ and set*

$$g(x)=x+|x-\theta|-f(x)-|f(x)-A|.$$

Then $g(x)\geq g(\theta)$ for all $x\in\mathcal{D}$.

Proof. Let $h(x)=f(x)+|f(x)-A|$. Clearly

$$0\leq h'\leq 2 \quad\text{in } \mathcal{D}.$$

If $0\leq\theta<\frac{1}{2}\pi$, we have that $g'=2-h'\geq 0$ in the interval $\left(\theta,\frac{1}{2}\pi\right)$. If $0<\theta\leq\frac{1}{2}\pi$, we see that $g'=-h'\leq 0$ in $(0,\theta)$ so in all cases $g(x)\geq g(\theta)$. □

Proof of Proposition 2.2. We begin by verifying the case $n=1$. If $x,y,t\in\mathcal{D}$, we have that

$$a_1(x,y)\leq x\leq x+y=b_1(x,y)$$

which establishes (14) in this case. If $t \geq \gamma(x,y)$, then $\alpha_1(x,y,t)=t \leq x+|t-x| \leq \beta_1(x,y,t)$. If $0 \leq t \leq \gamma(x,y)$, we have

$$\alpha_1(x,y,t) = 2\gamma(x,y) - t \leq 2\gamma(x,y) \leq x+y \leq \beta_1(x,y,t)$$

which establishes (15) when $n=1$. Let now $n \geq 2$ and assume that (14) and (15) hold in the range $1,2,\dots,n-1$. For $x \in \mathbf{R}^n$ let $\hat{x} \in \mathbf{R}^{n-1}$ be the vector $(x_2, x_3, \dots, x_n)$ and set $x^* = (x_2, \hat{x})$. Let $e_n = b_n - a_n$ and $\varepsilon_n = \beta_n - \alpha_n$. If $x, y \in \mathcal{D}^n$ and $t \in \mathbf{R}$, it follows from Lemma 2.2 that

$$e_n(x,y) \geq e_n(x^*, y^*) = e_{n-1}(\hat{x}, \hat{y}) \geq 0.$$

Similarly $\varepsilon_n(x,y,t) \geq \varepsilon_n(x^*, y^*, t) = \varepsilon_{n-1}(\hat{x}, \hat{y}, t) \geq 0$. Hence the proposition follows by induction. □

3. The main inequality

We shall in this section develop the main step in the proof of Theorem 1.2. We begin by defining $\Gamma\colon \mathcal{D}^3 \to \mathcal{D}$ by setting

$$\Gamma(x,y,z) = \gamma(x, \gamma(y,z)) \quad \text{for } x,y,z \in \mathcal{D}.$$

Notice that if $(x,y,t) \in \mathrm{Int}(\mathcal{D}^3)$, then

$$\cot \Gamma(x,y,z) = \cot x + \cot y + \cot z, \tag{16}$$

so Γ is a symmetric function. We shall now define $A_n, B_n\colon \mathcal{D}^n \times \mathcal{D}^n \times \mathcal{D}^n \to \mathbf{R}$ by setting $A_1(x,y,z) = x+z+2\Gamma(x,y,z)$ and $B_1(x,y,z) = x+2y+z$. It is easily seen that

$$A_1 \leq B_1. \tag{17}$$

For $n \geq 2$ and $x,y,z \in \mathcal{D}^n$ we now set

$$A_n(x,y,z) = x_1 + \Gamma(\omega_1) + \sum_{i=1}^{n-1} |\Gamma(\omega_{i+1}) - \Gamma(\omega_i)| + \Gamma(\omega_n) + z_n,$$

$$B_n(x,y,z) = \sum_{i=1}^{n-1} (|x_{i+1}-x_i| + |y_{i+1}-y_i| + |z_{i+1}-z_i|) + x_n + y_1 + y_n + z_1.$$

Here $\omega_j = (x_j, y_j, z_j)$, $1 \leq j \leq n$.

We can now formulate the main result of this section.

Theorem 3.1. *Let $n \geq 1$ and suppose $\omega \in \mathcal{D}^n \times \mathcal{D}^n \times \mathcal{D}^n$. Then*

$$A_n(\omega) \leq B_n(\omega). \tag{18}$$

We will next introduce some notation. Let $U_n = \mathcal{D}^n \times \mathcal{D}^n \times \mathcal{D}^n$ and let

$$\Delta_n = B_n - A_n. \tag{19}$$

Put

$$\delta_n = \min_{U_n} \Delta_n \tag{20}$$

and let

$$D_n = \min\{\delta_1, \dots, \delta_n\}.$$

From (17) follows

$$\delta_1 = D_1 = 0. \tag{21}$$

Also set

$$\Omega_n = \{\omega \in U_n : \Delta_n(\omega) = \delta_n\}$$

and notice that $\Omega_n \neq \phi$ since Δ_n is continuous on U_n. For $\omega = (x, y, z) \in U_n$ and $1 \leq j \leq n$ let $\Gamma_j(\omega) = \Gamma(x_j, y_j, z_j)$.

Lemma 3.1. *Suppose $n \geq 2$, $\delta_n < 0$ and $D_{n-1} = 0$. Then n is odd and for all $\omega \in \Omega_n$,*

$$\Gamma_{2j}(\omega) < \min\{\Gamma_{2j-1}(\omega), \Gamma_{2j+1}(\omega)\}, \quad 2 \leq 2j < n, \tag{22}$$

$$\Gamma_1(\omega) > \Gamma_2(\omega), \qquad \Gamma_n(\omega) > \Gamma_{n-1}(\omega), \tag{23}$$

$$\Gamma_{2j+1}(\omega) > \max\{\Gamma_{2j}(\omega), \Gamma_{2j+2}(\omega)\}, \quad 2 \leq 2j < n-2. \tag{23}$$

Proof. Let $\omega = (x, y, z) \in U_n$, $x, y, z \in \mathcal{D}^n$.

For $p = (p_1, \dots, p_n) \in \mathbf{R}^n$ let $\hat{p} = (p_2, \dots, p_n)$. Let $\widehat{\omega} = (\hat{x}, \hat{y}, \hat{z}) \in U_{n-1}$. If $\Gamma_1(\omega) \leq \Gamma_2(\omega)$, then using that $\Delta_{n-1}(\widehat{\omega}) \geq D_{n-1} = 0$ we get

$$\Delta_n(\omega) = \Delta_{n-1}(\widehat{\omega}) + x_2 + |x_1 - x_2| - x_1 + y_1 + |y_1 - y_2| - y_2 + z_1 - z_2 + |z_1 - z_2| \geq 0.$$

Similarly, if $\Gamma_n(\omega) \leq \Gamma_{n-1}(\omega)$, then $\Delta_n(\omega) \geq 0$, which shows (23). Let now $1 < i < n$ and let $W = (X, Y, Z)$, where $X, Y, Z \in \mathcal{D}^{n-1}$,

$$\begin{cases} X_j = x_j, \\ Y_j = y_j, \\ Z_j = z_j \end{cases}$$

for $1\leq j<i$ and

$$\begin{cases} X_j = x_{j+1}, \\ Y_j = y_{j+1}, \\ Z_j = z_{j+1} \end{cases}$$

for $i\leq j\leq n-1$. If $\Gamma_i(\omega)$ is between $\Gamma_{i-1}(\omega)$ and $\Gamma_{i+1}(\omega)$, then

$$\begin{aligned}\Delta_n(\omega) = \Delta_{n-1}(W) &+ |x_{i-1}-x_i|+|x_i-x_{i+1}|-|x_{i-1}-x_{i+1}| \\ &+|y_{i-1}-y_i|+|y_i-y_{i+1}|-|y_{i-1}-y_{i+1}| \\ &+|z_{i-1}-z_i|+|z_i-z_{i+1}|-|z_{i-1}-z_{i+1}|\geq 0.\end{aligned}$$

Using (23), we now see that (22) holds. Again using (23), we see that n must be odd. Finally (23) yields (24), which completes the proof of the lemma. □

For $f\in C(\mathcal{D})$ we let $m(f)$ denote the minimum of f on $\mathcal{D}$, i.e.,

$$m(f) = \min\{f(x) : x\in\mathcal{D}\}.$$

We shall now consider functions $g\in C(\mathcal{D})$ of the form

$$g(x) = |x-\alpha|+|x-\beta|-|f(x)-a|-|f(x)-b|+c, \tag{25}$$

where $\alpha,\beta\in\mathcal{D}$ and $a,b,c\in\mathbf{R}$. If (25) holds, we will say that g has the function f as its base. We say that $g\in\mathcal{M}_0$ if $g\in C(\mathcal{D})$ has the form (25) and

$$f(\xi) < \min\{a,b\}$$

whenever $g(\xi)=m(g)$. If

$$f(\xi) > \max\{a,b\}$$

whenever $g(\xi)=m(g)$ we will say that $g\in\mathcal{M}_1$.

For $\varrho\in\mathbf{R}$ set $f_\varrho(x)=\varrho x$. Let Λ be the class of all $f\in C(\mathcal{D})$ such that f is continuously differentiable on $\mathrm{Int}(\mathcal{D})$ with

$$0<f'<1 \quad \text{on } \mathrm{Int}(\mathcal{D}).$$

Lemma 3.2. *Suppose $g\in\mathcal{M}_1$ has f as its base function. Let $\xi\in\mathcal{D}$. If $f=f_0$, then $g(\xi)=m(g)$ if and only if $\xi\in[\alpha,\beta]$. If $f=f_1$, then $g(\xi)=m(g)$ if and only if $\max\{\alpha,\beta\}\leq\xi\leq\frac{1}{2}\pi$. If $f\in\Lambda$, then $g(\xi)=m(g)$ if and only if $\xi=\max\{\alpha,\beta\}$. Here the parameters α and β are defined by the relation* (25).

Proof. We may without loss of generality assume $\alpha\leq\beta$ and set $h(x)=|x-\alpha|+|x-\beta|$.

If $f=f_0$, then $g=h+C$ for some constant C, which concludes the lemma in this case. Suppose now that $g(\xi)=m(g)$ and $f\in\Lambda\cup\{f_1\}$. Since f is increasing, we have for $x\geq\xi$ that

$$f(x)\geq f(\xi)>\max\{a,b\}$$

so from (25) follows that

$$g(x)=h(x)-2f(x)+C, \quad x\geq\xi.$$

Since f is strictly increasing and h is non-increasing on $(-\infty,\beta)$, we see that if ξ were less than β, then

$$g(\beta)<g(\xi),$$

which contradicts the definition of ξ. Hence $\xi\geq\beta$ if $f\in\Lambda\cup\{f_1\}$. If $x>\beta=\max\{\alpha,\beta\}$, then $h(x)=2x-\alpha-\beta$. If now $f\in\Lambda$, then g is strictly increasing on $\left(\beta,\frac{1}{2}\pi\right)$, so $g(\xi)=m(g)$ if and only if $\xi=\beta$ in this case. If $f=f_1$, then it is easily seen that $g(x)=g(\beta)$ for $x\geq\beta$ which completes the proof of the lemma. □

A straightforward modification of the proof of Lemma 3.2 yields the following result.

Lemma 3.3. *Suppose $g\in\mathcal{M}_0$ has f as its base function. Let $\xi\in\mathcal{D}$. If $f=f_0$, then $g(\xi)=m(g)$ if and only if $\xi\in[\alpha,\beta]$. If $f=f_1$, then $g(\xi)=m(g)$ if and only if $0\leq\xi\leq\min\{\alpha,\beta\}$. If $f\in\Lambda$, then $g(\xi)=m(g)$ if and only if $\xi=\min\{\alpha,\beta\}$. Here the parameters α and β are defined by the relation* (25).

Let $V\subset\{1,2,\dots,n\}$, $x\in\mathbf{R}^n$ and $t\in\mathbf{R}$. We define $q_V(x,t)$ as the point $y\in\mathbf{R}^n$ with $y_i=x_i$ for $i\notin V$ and $y_i=t$ when $i\in V$. If $\omega=(x,y,z)\in U_n$, we put $Q_V(\omega,t)=(q_V(x,t),y,z)$ and

$$E_n^{\omega,V}(t)=\Delta_n(Q_V(\omega,t)). \tag{26}$$

In the special case when $V=\{k\}$, $1\leq k\leq n$, we will write $E_n^{\omega,k}=E_n^{\omega,V}$. For $\omega=(x,y,z)\in U_n$ we set

$$\theta_i(\omega)=\gamma(y_i,z_i) \quad \text{and} \quad \lambda_{i,\omega}(t)=\gamma(t,\theta_i(\omega)).$$

We observe that $E_n^{\omega,k}$ has $\lambda_{k,\omega}$ as its base function. We remark that if $\omega\in\Omega_n$, then under the conditions of Lemma 3.1 we have

$$E_n^{\omega,k}\in\mathcal{M}_1 \tag{27}$$

for k odd and

$$E_n^{\omega,k}\in\mathcal{M}_0 \tag{28}$$

for k even.

The following result is an immediate consequence of the previous two lemmas. The verification is left to the reader.

Lemma 3.4. *Suppose $n\geq 3$, $\delta_n<0$ and $D_{n-1}=0$. Assume $\omega=(x,y,z)\in\Omega_n$ and $1<k<n$. If k is odd, then*

$$\delta_n = E_n^{\omega,k}(\max\{x_{k-1},x_{k+1}\})$$

and if k is even, then

$$\delta_n = E_n^{\omega,k}(\min\{x_{k-1},x_{k+1}\}).$$

If $\theta_k(\omega)=0$ and $1<k<n$, then

$$\delta_n = E_n^{\omega,k}(t) \quad \textit{for all } t\in[x_{k-1},x_{k+1}].$$

If $\theta_k(\omega)>0$, then

$$x_k\geq \max\{x_{k-1},x_{k+1}\} \quad \textit{for } k \textit{ odd}$$

and

$$x_k\leq \min\{x_{k-1},x_{k+1}\} \quad \textit{for } k \textit{ even.}$$

We shall next analyse the function $E_n^{\omega,V}$.

Lemma 3.5. *Suppose $n\geq 3$, $\delta_n<0$ and $D_{n-1}=0$. Assume $j\geq 1$ satisfies $2j<n$ and set $V=\{1,2,\dots,2j\}$. Let $\xi\in\mathcal{D}$ and assume $\omega=(x,y,z)\in\Omega_n$ satisfies*

$$x_1=x_2=\dots=x_{2j}=\xi.$$

If $\xi\leq x_{2j+1}$, then

$$\delta_n = E_n^{\omega,V}(x_{2j+1})$$

so $Q_V(\omega,x_{2j+1})\in\Omega_n$.

Proof. We need only treat the case when $\xi<x_{2j+1}$. Setting $\theta_i=\theta_i(\omega)$ we see from Lemma 3.1 that

$$\gamma(\xi,\theta_{2k-1})>\gamma(\xi,\theta_{2k}),\quad 1\leq k\leq j.$$

From Proposition 2.1 follows that for all $t\in\mathcal{D}$

$$\gamma(t,\theta_{2k-1})\geq\gamma(t,\theta_{2k}) \quad\text{and}\quad \frac{\partial\gamma(t,\theta_{2k-1})}{\partial t}\geq\frac{\partial\gamma(t,\theta_{2k})}{\partial t},$$

whenever $1\leq k\leq j$.

Also $\gamma(\xi,\theta_{2j})<\Gamma_{2j+1}(\omega)$. Letting

$$a=\sup\{u\in[\xi,x_{2j+1}]:\gamma(t,\theta_{2j})\leq\Gamma_{2j+1}(\omega) \text{ for } \xi\leq t\leq u\}$$

we have that $\xi<a\leq x_{2j+1}$. If $t\in[\xi,a]$, then

$$E_n^{\omega,V}(t)=-2t+2\sum_{k=1}^{j}(\gamma(t,\theta_{2k})-\gamma(t,\theta_{2k-1}))+\Phi,$$

where Φ is independent of t. Hence $E_n^{\omega,V}$ is decreasing on $[\xi,a]$ so $\delta_n=E_n^{\omega,V}(a)$ and $Q_V(\omega,a)\in\Omega_n$. In particular, $\gamma(a,\theta_{2j})<\Gamma_{2j+1}(\omega)$ so we cannot have $a\in(\xi,x_{2j+1})$, i.e., $a=x_{2j+1}$, which yields the lemma. □

Lemma 3.6. *Suppose* $n \geq 3$, $\delta_n < 0$ *and* $D_{n-1} = 0$. *Assume that* $j \geq 1$ *satisfies* $2j < n$ *and put* $V = \{2j, 2j+1\}$. *Assume* $\omega = (x, y, z) \in \Omega_n$ *satisfies*

$$x_{2j} = x_{2j+1} \leq x_{2j-1}.$$

Then

$$\delta_n = E_n^{\omega,V}(x_{2j-1})$$

so $Q_V(\omega, x_{2j-1}) \in \Omega_n$.

Proof. Put $\xi = x_{2j} = x_{2j+1}$. We need only treat the case when $\xi < x_{2j-1}$. Setting $\theta_i = \theta_i(\omega)$ we find from Lemma 3.1 that

$$\gamma(\xi, \theta_{2j}) < \gamma(\xi, \theta_{2j+1}) \quad \text{and} \quad \gamma(\xi, \theta_{2j}) < \Gamma_{2j-1}(\omega),$$

so from Proposition 2.1 it follows that for all $t \in \mathcal{D}$

$$\gamma(t, \theta_{2j}) \leq \gamma(t, \theta_{2j+1}) \quad \text{and} \quad \frac{\partial \gamma(t, \theta_{2j})}{\partial t} \leq \frac{\partial \gamma(t, \theta_{2j+1})}{\partial t}.$$

Suppose now that $2j+1 = n$. Let

$$a = \sup\{u \in [\xi, x_{2j-1}] : \gamma(t, \theta_{2j}) \leq \Gamma_{2j-1}(\omega) \quad \text{for all } t \in [\xi, u]\}.$$

If $t \in [\xi, a]$, then

$$E_n^{\omega,V}(t) = 2(\gamma(t, \theta_{2j}) - \gamma(t, \theta_{2j+1})) + \Phi,$$

where Φ is independent of t. Hence $E_n^{\omega,V}$ is decreasing on $[\xi, a]$ so $\delta_n = E_n^{\omega,V}(a)$ and $Q_V(\omega, a) \in \Omega_n$.

In particular, $\gamma(a, \theta_{2j}) < \Gamma_{2j-1}(\omega)$, so we cannot have $a \in (\xi, x_{2j-1})$, i.e., $a = x_{2j-1}$ which establishes the lemma in this case.

We shall now treat the remaining case, so we assume now that $2j+1 < n$. In this case $\gamma(\xi, \theta_{2j+1}) > \Gamma_{2j+2}(\omega)$ so we now set $b = \sup\{u \in [\xi, x_{2j-1}] : \gamma(t, \theta_{2j}) \leq \Gamma_{2j-1}(\omega)$ and $\gamma(t, \theta_{2j+1}) \geq \Gamma_{2j+2}(\omega)$ for all $t \in [\xi, u]\}$. If $t \in [\xi, b]$, then

$$E_n^{\omega,V}(t) = -t + |t - x_{2j+2}| + 2(\gamma(t, \theta_{2j}) - \gamma(t, \theta_{2j+1})) + \psi,$$

where ψ is independent of t. Hence $E_n^{\omega,V}$ is decreasing on $[\xi, b]$ so $\delta_n = E_n^{\omega,V}(b)$ and $Q_V(\omega, b) \in \Omega_n$. In particular, $\gamma(b, \theta_{2j}) < \Gamma_{2j-1}(\omega)$ and $\gamma(b, \theta_{2j+1}) > \Gamma_{2j+2}(\omega)$, so we cannot have $b \in (\xi, x_{2j-1})$, i.e., $b = x_{2j-1}$. This concludes the proof of the lemma. □

The next lemma will provide the crucial part of the proof of Theorem 3.1. For $\xi \in \mathbf{R}$ we let $Q_n(\xi)$ denote the point in $\mathbf{R}^n$ with all components equal to ξ.

Lemma 3.7. *Suppose $n \geq 2$, $\delta_n < 0$ and $D_{n-1} = 0$. Assume $W = (X, Y, Z) \in \Omega_n$. Then there exists a $\xi \in \mathcal{D}$ such that $(Q_n(\xi), Y, Z) \in \Omega_n$.*

Proof. Let $\Omega_n(W) = \{\omega = (x, y, z) \in \Omega_n : y = Y \text{ and } z = Z\}$ and notice $W \in \Omega_n(W)$. For $\omega = (x, y, z) \in U_n$ let $N(\omega)$ be the largest integer $p \in \{1, \dots, n\}$ such that $x_i = x_1$ for $1 \leq i \leq p$. Set

$$N = \max\{N(\omega) : \omega \in \Omega_n(W)\}$$

and pick $\omega = (x, y, z) \in \Omega_n(W)$ such that $N = N(\omega)$. Assume that $N < n$. We shall show that this assumption leads to a contradiction. Note that n is odd by Lemma 3.1, so that $n \geq 3$.

Suppose first that $N = n-1$. From Lemma 3.2 follows that $\delta_n = E_n^{\omega,n}(x_N)$ so $\zeta = Q_n(\omega, x_N) \in \Omega_n(\omega)$ with $N(\zeta) = n$. This contradicts the definition of N.

Suppose next that $N < n-1$. Put $\theta_i = \gamma(y_i, z_i)$. From Lemma 3.4 it follows that $\delta_n = E_n^{\omega,N+1}(x_N)$, if $\theta_{N+1} = 0$. Hence, if $\theta_{N+1} = 0$ we have $\zeta = Q_{N+1}(\omega, x_N) \in \Omega_n(W)$ with $N(\zeta) \geq N+1$. Again this contradicts the definition of N, so we must have $\theta_{N+1} > 0$.

We can therefore from now on assume that $\theta_{N+1} > 0$ and $1 \leq N \leq n-2$. Also recall that n must be an odd integer.

We first treat the case when N is even, say $N = 2j$. Since $N+1$ must be odd with $\theta_{N+1} > 0$ it follows from Lemma 3.4 that $x_{N+1} \geq x_N$. Setting $V = \{1, \dots, N\}$ it follows from Lemma 3.5 that $\zeta = Q_V(\omega, x_{N+1}) \in \Omega_n(W)$. But $N(\zeta) \geq N+1$, which again leads to a contradiction.

It remains only to treat the case when N is odd and $\theta_{N+1} > 0$. Setting $\varrho_N = \min\{x_N, x_{N+2}\}$ it follows from Lemma 3.4 that $x_{N+1} \leq \varrho_N \leq x_N$. Putting now $\eta = Q_{N+1}(\omega, \varrho_N)$, we also see from Lemma 3.4 that $\eta \in \Omega_n(W)$. If $\varrho_N = x_N$ then $N(\eta) \geq N+1$, which is a contradiction. If $\varrho_N < x_N$, then $\varrho_N = x_{N+2}$ so if $\eta = (\xi, Y, Z)$, then $\xi_{N+1} = \xi_{N+2} = \varrho_N < x_N$. Hence η fulfils the assumptions of Lemma 3.6. Setting $S = \{N+1, N+2\}$, we therefore have $q = Q_S(p, x_N) \in \Omega_n(W)$. But $N(q) \geq N+2$ which again contradicts the definition of N.

So in all cases the assumption $N < n$ is impossible, which yields the lemma. □

We can now prove the main result of this section.

Proof of Theorem 3.1. Since $\Delta_n(0) = 0$, we see that $\delta_n \leq 0$ for all $n \geq 1$. Hence it is enough to show that $D_n = 0$ for all $n \geq 1$. From (17) it follows that $\delta_1 = D_1 = 0$. We shall now proceed by induction.

Suppose $n \geq 2$ and

$$D_{n-1} = 0. \tag{29}$$

We shall prove that $D_n=0$. It is enough to show that $\delta_n=0$. We shall argue by contradiction, so assume

$$\delta_n < 0. \tag{30}$$

Define the mapping $\varrho\colon \mathbf{R}^n \to \mathbf{R}^n$ by

$$\varrho(x) = (x_n, \dots, x_1) \quad \text{for } x = (x_1, \dots, x_n).$$

For $\omega=(x,y,z)\in U_n$ we set

$$R(\omega) = (\varrho(z), \varrho(y), \varrho(x)) \in U_n.$$

Since $\Delta_n(R(\omega))=\Delta_n(\omega)$, we have that

$$R\colon \Omega_n \longrightarrow \Omega_n.$$

From Lemma 3.7 follows the existence of $\xi\in\mathcal{D}$ and $y,z\in\mathbf{R}^n$ such that if $x=Q_n(\xi)$, then $\omega=(x,y,z)\in\Omega_n$.

Since $\varrho(x)=x$ in this case, we have that $R(\omega)=(\varrho(z),\varrho(y),x)\in\Omega_n$. Using Lemma 3.7 one more time, we see that there is an $\eta\in\mathcal{D}$ such that if $p=Q_n(\eta)$, then $V=(p,\varrho(y),x)\in\Omega_n$. Hence $W=R(V)\in\Omega_n$. Since $W=(x,y,p)$, we see by setting $\theta=\gamma(\xi,\eta)$ that

$$\begin{aligned}\delta_n &= \Delta_n(W)\\ &= y_1+y_n-(\gamma(y_1,\theta)+\gamma(y_n,\theta))+\sum_{i=1}^{n-1}(|y_i-y_{i+1}|-|\gamma(y_i,\theta)-\gamma(y_{i+1},\theta)|)\geq 0,\end{aligned}$$

by Proposition 2.1. This contradicts the assumption (30) which completes the proof by induction. □

4. Total curvature of piecewise linear functions

Let $I=[a,b]$ be an interval and let $f\in C(I)$. We will say that f is unimodular if there exists a $c\in[a,b]$ such that the restrictions $f|_{[a,c]}$ and $f|_{[c,b]}$ are both monotone. We shall begin by showing that if f is unimodular and piecewise linear, then $T(f^*)\leq T(f)$.

Lemma 4.1. *Let $n \geq 1$ and assume*

$$x_n < x_{n-1} < \dots < x_1 < x_0 \leq \xi_0 < \xi_1 < \dots < \xi_{n-1} < \xi_n.$$

Put $a = x_n$ and $b = \xi_n$. Suppose $y_0 > y_1 > \dots > y_n$ and assume that f is piecewise linear on $[a, b]$ with nodes $\{x_n, x_{n-1}, \dots, x_0, \xi_0, \dots, \xi_n\}$. Assume that $f(x_i) = f(\xi_i) = y_i$, $0 \leq i \leq n$. Then

$$T(f^*) \leq T(f).$$

Proof. We define for $1 \leq i \leq n$ the angles $\alpha_i, \beta_i \in (0, \frac{1}{2}\pi)$ by

$$\tan \alpha_i = \frac{y_i - y_{i-1}}{x_i - x_{i-1}} \quad \text{and} \quad \tan \beta_i = \frac{y_{i-1} - y_i}{\xi_i - \xi_{i-1}}.$$

Notice that $f'(x) = \tan \alpha_i$ for $x \in (x_i, x_{i-1})$, and $f'(x) = -\tan \beta_i$ for $x \in (\xi_{i-1}, \xi_i)$. It is easily seen that

$$T(f) = \alpha_1 + \beta_1 + \sum_{i=1}^{n-1} (|\alpha_{i+1} - \alpha_i| + |\beta_{i+1} - \beta_i|).$$

Let $\varepsilon = 1$ if $\xi_0 > x_0$ and $\varepsilon = 0$ otherwise. From Lemma 2.1 it follows that

$$T(f^*) = \varepsilon \gamma(\alpha_1, \beta_1) + \sum_{i=1}^{n-1} |\gamma(\alpha_{i+1}, \beta_{i+1}) - \gamma(\alpha_i, \beta_i)|.$$

Hence the lemma follows from Proposition 2.2. □

We will need the following variant of Lemma 4.1.

Lemma 4.2. *Let $m > n \geq 1$ and assume*

$$x_n < x_{n-1} < \dots < x_1 < x_0 \leq \xi_0 < \xi_1 < \dots < \xi_{m-1} < \xi_m.$$

Put $a = x_n$ and $b = \xi_m$. Suppose $y_0 > y_1 > \dots > y_m$ and assume that f is piecewise linear on $[a, b]$ with nodes $\{x_n, x_{n-1}, \dots, x_0, \xi_0, \dots, \xi_m\}$. Assume that $f(x_i) = y_i$ for $0 \leq i \leq n$ and $f(\xi_i) = y_i$ for $0 \leq i \leq m$. Then

$$T(f^*) \leq T(f).$$

Proof. We define for $1 \leq i \leq n$ the angle $\alpha_i \in (0, \frac{1}{2}\pi)$ by

$$\tan \alpha_i = \frac{y_i - y_{i-1}}{x_i - x_{i-1}}.$$

For $1 \le i \le m$ we define $\beta_i \in (0, \frac{1}{2}\pi)$ by

$$\tan \beta_i = \frac{y_{i-1} - y_i}{\xi_i - \xi_{i-1}}.$$

It is easily seen that

$$T(f) = \alpha_1 + \beta_1 + \sum_{i=1}^{n-1} (|\alpha_{i+1} - \alpha_i| + |\beta_{i+1} - \beta_i|) + |\beta_n - \beta_{n+1}| + T(g),$$

where $g = f|_{[\xi_n, b]}$. Let $\varepsilon = 1$ if $\xi_0 > x_0$ and $\varepsilon = 0$ otherwise. From Lemma 2.1 it follows that

$$T(f^*) = \varepsilon\gamma(\alpha_1, \beta_1) + \sum_{i=1}^{n-1} (|\gamma(\alpha_{i+1}, \beta_{i+1}) - \gamma(\alpha_i, \beta_i)|) + |\beta_{n+1} - \gamma(\alpha_n, \beta_n)| + T(g).$$

Hence the lemma follows from Proposition 2.2. □

We can now analyse the total curvature of the rearrangement of a unimodular piecewise linear function.

Lemma 4.3. *Let $I = [a, b]$ be an interval. If $f \in C(I)$ is unimodular and piecewise linear, then*

$$T(f^*) \le T(f).$$

Proof. Let $c \in [a, b]$ be such that $f|_{[a,c]}$ and $f|_{[c,b]}$ are monotone. We may without loss of generality assume that f is non-decreasing on $[a, c]$; otherwise we consider $-f$ instead. The result is trivial if f is also non-decreasing on $[c, b]$ so we may assume that f is non-increasing on $[c, b]$. The result is also trivial if $f(c) \in \{f(a), f(b)\}$, so we will assume that $f(c) > \max\{f(a), f(b)\}$.

Put $x_0 = \inf\{x \in I : f(x) = f(c)\}$ and $\xi_0 = \sup\{x \in I : f(x) = f(c)\}$. Clearly $f(x) = f(c)$ for all $x \in [x_0, \xi_0]$. By approximation, it is enough to treat the case when f is strictly increasing on $[a, x_0]$ and f is strictly decreasing on $[\xi_0, b]$. Also, we may assume that $f(b) \le f(a)$; otherwise we consider $g(x) = f(a+b-x)$. Set $M = \{x \in I : x \text{ is a node for } f\}$ and set $V = \{f(x) : x \in M\}$. Let $y_0 > ... > y_m$ be listing of the distinct numbers in V. For $1 \le i \le n$ let $x_i = \inf\{x \in I : f(x) = y_i\}$ and $\xi_i = \sup\{x \in I : f(x) = y_i\}$. For $n < i \le m$ let ξ_i be the unique solution of the equation $f(x) = y_i$, $x \in I$.

Clearly, the function f can be viewed as a piecewise linear function with nodes $\{x_n, ..., x_0, \xi_0, ..., \xi_m\}$. If $m = n$ the lemma follows from Lemma 4.1. If $m > n$, then the lemma follows from Lemma 4.2. □

Let $I=[a,b]$ be an interval. We let $\mathcal{N}(I)$ denote the class of functions $f\in C(I)$ that satisfy the following two properties:

(i) There are two points $c_1,c_2\in I$ such that $a<c_1<c_2<b$ and the restrictions $f|_{[a,c_1]}$, $f|_{[c_1,c_2]}$ and $f|_{[c_2,b]}$ are all monotone.

(ii) Set $m=\min\{f(a),f(b)\}$ and $M=\max\{f(a),f(b)\}$. Then $m<f(x)<M$ for all $x\in(a,b)$.

We shall next establish the inequality $T(f^*)\le T(f)$ for the case when $f\in\mathcal{N}(I)$ and f is piecewise linear.

Lemma 4.4. *Let $n\ge 1$ and assume that $x_0<\ldots<x_n$, $\xi_n<\ldots<\xi_0$, $\eta_0<\ldots<\eta_n$ and $y_0>\ldots>y_n$. Assume also that $x_n\le\xi_n$, $\xi_0\le\eta_0$, $a<x_0$ and $\eta_n<b$. Suppose $f\in C([a,b])$ is piecewise linear with nodes $\{a,x_0,\ldots,x_n,\xi_n,\ldots,\xi_0,\eta_0,\ldots,\eta_n,b\}$. Suppose furthermore that $f(a)>y_0$, $f(b)<y_n$ and $y_i=f(x_i)=f(\xi_i)=f(\eta_i)$ for $0\le i\le n$. Then*

$$T(f^*)\le T(f).$$

Proof. Let $y_{-1}=f(a)$, $x_{-1}=a$, $y_{n+1}=f(b)$, $\eta_{n+1}=b$ and define $a_i,b_i,c_i\in\left(0,\frac{1}{2}\pi\right)$ by

$$\tan a_i=\frac{y_i-y_{i-1}}{x_{i-1}-x_i},\quad \tan b_i=\frac{y_i-y_{i-1}}{\xi_i-\xi_{i-1}},\quad \tan c_i=\frac{y_i-y_{i-1}}{\eta_{i-1}-\eta_i}.$$

It is easily seen that

$$\begin{aligned}T(f)=|a_1-a_0|&+\sum_{i=1}^{n-1}(|a_{i+1}-a_i|+|b_{i+1}-b_i|+|c_{i+1}-c_i|)\\&+b_1+c_1+a_n+b_n+|c_{n+1}-c_n|.\end{aligned}$$

Let $\theta=|a_0-\Gamma(a_1,b_1,c_1)|$ if $\xi_0=\eta_0$ and $\theta=a_0+\Gamma(a_1,b_1,c_1)$ otherwise. Let $\varphi=|c_{n+1}-\Gamma(a_n,b_n,c_n)|$ if $x_n=\xi_n$ and $\varphi=c_{n+1}+\Gamma(a_n,b_n,c_n)$ otherwise. From the definition of Γ and Lemma 2.1 it follows that

$$T(f^*)=\theta+\sum_{i=1}^{n-1}|\Gamma_{i+1}-\Gamma_i|+\varphi,$$

where $\Gamma_i=\Gamma(a_i,b_i,c_i)$, $1\le i\le n$. We now set $\omega=(a_1,\ldots,a_n,b_1,\ldots,b_n,c_1,\ldots,c_n)\in U_n$. We find by Theorem 3.1 that

$$T(f)-T(f^*)\ge\Delta_n(\omega)+|a_0-a_1|+|c_{n+1}-c_n|-a_0+a_1-c_{n+1}+c_n\ge\Delta_n(\omega)\ge 0$$

which establishes the lemma. $\square$

We can now study rearrangements of piecewise linear functions of the class $\mathcal{N}(I)$.

Lemma 4.5. *Let $I=[a,b]$ and suppose $f\in\mathcal{N}(I)$ is piecewise linear. Then*

$$T(f^*)\le T(f).$$

Proof. Let $a<c_1<c_2<b$ be such that f has a monotone restriction to each of the intervals $[a,c_1]$, $[c_1,c_2]$ and $[c_2,b]$. We may assume that f is non-increasing on $[a,c_1]$ since otherwise we consider $-f$.

We may also assume that the restriction of f is non-monotone on any of the intervals $[a,c_2]$ of $[c_1,b]$ since otherwise f is unimodular and the result follows from Lemma 4.3. Hence f must be non-increasing on the intervals $[a,c_1]$ and $[c_2,b]$ and non-decreasing on $[c_1,c_2]$. Consequently,

$$f(b)<f(c_1)<f(c_2)<f(a).$$

Let $I_1=[a,c_2]$ and $I_2=[c_1,b]$. Put $A_k=\inf\{x\in I_k:f(x)=f(c_k)\}$ and $B_k=\sup\{x\in I_k: f(x)=f(c_k)\}$. Then

$$a<A_1\le B_1<A_2\le B_2<b.$$

By approximation it is enough to treat the case when f is strictly monotone on the intervals $[a,A_1],[B_1,A_2]$ and $[B_2,b]$. Let A_0 solve the equation $f(x)=f(c_2)$, $x\in[a,A_1]$, and let B_3 solve the equation $f(x)=f(c_1)$, $x\in[B_2,b]$. Let $R=\{\xi_0,\dots,\xi_m\}$ be the set of nodes of f and let $\hat{a}=\sup\{\xi\in R:\xi<A_0\}$ and $\hat{b}=\inf\{\xi\in R:\xi>B_3\}$.

It is easy to see that possibly after introducing additional nodes, we have that $g=f|_{[\hat{a},\hat{b}]}$ satisfies the assumptions of Lemma 4.4. Let $f_1=f|_{[a,A_0]}$, $f_2=f|_{[B_3,b]}$. Then

$$T(f)=T(g)+T(f_1)+T(f_2)$$

and

$$T(f^*)=T(g^*)+T(f_1)+T(f_2)$$

which yields the lemma. □

5. Proof of the main results

We shall in this section finish the proofs of our main results. We begin with the following lemma.

Lemma 5.1. *Let $I=[a,b]$ be an interval. If $f\in C(I)$ is piecewise linear, then*

$$T(f^*)\le T(f). \tag{31}$$

Proof. Let $n\ge 2$ be the number of nodes of f. The result is trivial if $n=2$. If $n=3$, the result follows from Lemma 4.3. We shall prove (31) by induction over the number of nodes of f.

We shall therefore assume that $n\ge 4$ and that (31) holds for all piecewise linear functions with less than n nodes.

Let $V=\{1,\dots,n\}$, $V^*=\{2,\dots,n-1\}$, and let $\xi_1=a<\xi_2<\dots<\xi_n=b$ be the nodes of f. Set $\eta_i=f(\xi_i)$, $m=\min\{\eta_i:i\in V\}$, $M=\max\{\eta_i:i\in V\}$, $m^*=\min\{\eta_i:i\in V^*\}$ and $M^*=\max\{\eta_i:i\in V^*\}$. We will first treat the case when $M^*=M$. Pick $j\in V^*$ such that $\eta_j=M=M^*$. Set $g_1=f|_{[a,\xi_j]}$ and $g_2=f|_{[\xi_j,b]}$. Let G_1 be the increasing rearrangement of g_1, and put $G_2=g_2^*$. Define $\theta,\varphi\in\left[0,\frac{1}{2}\pi\right)$ by

$$\tan\theta=f'(\xi_j-)\quad\text{and}\quad\tan\varphi=-f'(\xi_j+). \tag{32}$$

Then

$$T(f)=T(g_1)+T(g_2)+\theta+\varphi.$$

Define $\theta^*,\varphi^*\in\left[0,\frac{1}{2}\pi\right)$ by

$$\tan\theta^*=G_1'(\xi_j-)\quad\text{and}\quad\tan\varphi^*=-G_2'(\xi_j+). \tag{33}$$

Set $G(x)=G_1(x)$ if $a\le x\le\xi_j$ and $G(x)=G_2(x)$ if $\xi_j\le x\le b$. Now

$$T(G)=T(G_1)+T(G_2)+\theta^*+\varphi^*.$$

By the induction assumption $T(G_1)\le T(g_1)$ and $T(G_2)\le T(g_2)$. Since $0\le\theta^*\le\theta$ and $0\le\varphi^*\le\varphi$, we find that $T(G)\le T(f)$. Because G and f are equimeasurable, $f^*=G^*$. Since G is unimodular, we have $T(f^*)=T(G^*)\le T(G)\le T(f)$, which establishes the induction step in this case.

If $m^*=m$, the previous reasoning applied to $-f$ shows again that $T(f^*)\le T(f)$. We are now left with the case $m<m^*\le M^*<M$. We may assume $f(\xi_n)<M^*$, since otherwise we consider $-f$. Pick $j\in V^*$ such that $\eta_j=M^*<M$. Set $g_1=f|_{[a,\xi_j]}$, $g_2=f|_{[\xi_j,b]}$, and let $\theta,\varphi\in\left[0,\frac{1}{2}\pi\right)$ be defined by (32). Then

$$T(f)=T(g_1)+T(g_2)+\theta+\varphi.$$

Let $g(x)=f(x)$ for $a\le x\le\xi_j$, $g(x)=g_2^*(x)$ for $\xi_j\le x\le b$. Then g and f are equimeasurable so $f^*=g^*$. Furthermore, $g\in C(I)$ is piecewise linear. Let $\varphi^*\in\left[0,\frac{1}{2}\pi\right)$ be defined by $\tan\varphi^*=-g'(\xi_j+)$. Since $0\le\varphi^*\le\varphi$, we have from the induction assumption that

$$T(g)=T(g_1)+T(g_2^*)+\theta+\varphi^*\le T(f). \tag{34}$$

Set $\mu=\min\{f(x):x\in[a,\xi_j]\}$. Then $\mu\le M^*$ and if $\mu=M^*$, we must have $f(x)=M^*$ for $\xi_2\le x\le\xi_j$ and consequently g is decreasing on $[a,b]$. Hence, if $\mu=M^*$, we have $f^*=g$, so (31) follows from (34) in this case.

We suppose now that $\mu<M^*$ and pick k, $1\le k<j$, such that $\eta_k=\mu$. Put $h_1=f|_{[a,\xi_k]}$ and $h_2=f|_{[\xi_k,\xi_j]}$. Let H_1 be the decreasing rearrangement of h_1 and H_2 the increasing rearrangement of h_2. Define H by

$$H(x)=\begin{cases} H_1(x) & \text{for } a\le x\le\xi_k,\\ H_2(x) & \text{for } \xi_k\le x\le\xi_j,\\ g(x) & \text{for } \xi_j\le x\le b.\end{cases}$$

Then H and f are equimeasurable, $H\in C(I)$ is piecewise linear and arguing as in the derivation of (34) one finds

$$T(H)\le T(g)\le T(f).$$

By the construction the function $H\in\mathcal{N}(I)$ so $T(f^*)=T(H^*)\le T(H)\le T(f)$. The proof of the induction step is complete, which establishes the lemma. □

Proof of Theorem 1.2. Let $X=\{\xi_0,\dots,\xi_n\}$, $n\ge1$, be a partition of I. For $f\in C(I)$ let

$$T(f,X)=\mathcal{B}(\gamma,X),$$

where γ is the graph of f. Let $\theta_i\in\left(-\frac{1}{2}\pi,\frac{1}{2}\pi\right)$,

$$\tan\theta_i=\frac{f(\xi_i)-f(\xi_{i-1})}{\xi_i-\xi_{i-1}},\quad 1\le i\le n.$$

Then

$$T(f,X)=\sum_{i=1}^{n-1}|\theta_{i+1}-\theta_i|.$$

Notice that if $f_n\in C(I)$, $f_n\to f$ uniformly, then $T(f_n,X)\to T(f,X)$. Also $f_n^*\to f^*$ uniformly.

Pick $f_n\in C(I)$ such that $f_n\to f$ uniformly and f_n is a piecewise linear function for all n, such that f_n has its nodes on the graph of f. Then $T(f_n)\le T(f)$ so

$$T(f^*,X)=\lim_{n\to\infty}T(f_n^*,X)\le\limsup_{n\to\infty}T(f_n^*)\le T(f)$$

by Lemma 5.1. Since

$$T(f^*)=\sup T(f^*,X),$$

where X ranges over all partitions of I, we have proved the theorem. □

Lemma 5.2. *Suppose $f\in C(I)$, $I=[a,b]$, is piecewise linear. Then*

$$\|f\|_C=\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}T(\varepsilon f).$$

Also

$$\|f^*\|_C\le\|f\|_C.$$

Proof. Let $a=\xi_0<\xi_1<\ldots<\xi_n=b$ be the nodes of f. Set

$$Q_i=\frac{f(\xi_i)-f(\xi_{i-1})}{\xi_i-\xi_{i-1}},\quad 1\le i\le n.$$

Now

$$\frac{1}{\varepsilon}T(\varepsilon f)=\frac{1}{\varepsilon}\sum_{i=1}^{n-1}|\arctan(\varepsilon Q_{i+1})-\arctan(\varepsilon Q_i)|\to\sum_{i=1}^{n-1}|Q_{i+1}-Q_i|=\|f\|_C$$

as $\varepsilon\downarrow 0$. Since f^* is piecewise linear, the lemma follows from Theorem 1.2. □

We shall next prove Theorem 1.1 in the case of smooth functions. We will use the Green function

$$G(x,\xi)=\begin{cases}(1-\xi)x, & x\le\xi,\\ (1-x)\xi, & x\ge\xi.\end{cases}\tag{35}$$

For a measure μ on $(0,1)$ set

$$G\mu(x)=\int_0^1 G(x,\xi)\,d\mu(\xi).$$

Lemma 5.3. *Let $I=[a,b]$. Suppose f is twice continuously differentiable on I. Then*

$$\|f^*\|_C\le\|f\|_C.$$

Proof. By rescaling there is no loss in generality in assuming that $I=[0,1]$. Let $h=f''$. Then

$$f(x)=(1-x)f(0)+xf(1)-Gh(x)$$

and $\|f\|_C=\int_0^1|h(x)|\,dx$.

Let $X=\{\xi_0,\ldots,\xi_n\}$, $0=\xi_0<\ldots<\xi_n=1$ be a partition of I. Let $F=A_X(f)$ denote the piecewise linear function in I whose set of nodes equals X and $F(\xi_i)=f(\xi_i)$. We claim that

$$\|A_X(f)\|_C\le\|f\|_C.\tag{36}$$

If g is twice continuously differentiable with $g''\geq 0$, then $G=A_X(g)$ is convex. Hence

$$\|A_X(g)\|_C=G'(1-)-G'(0+)=g'(p)-g'(q)$$

for some $p,q\in(0,1)$. As $g''\geq 0$, we have that $g'(p)-g'(q)\leq g'(1)-g'(0)=\int_I g''\,dx=\|g\|_C$. Hence $\|A_X(g)\|_C\leq\|g\|_C$. Notice that we can write $f=f_1-f_2$, where f_1 and f_2 are both twice continuously differentiable, convex and

$$\|f\|_C=\|f_1\|_C+\|f_2\|_C.$$

Hence (36) is proved. By selecting a suitable sequence $X^{(m)}$ of partitions we conclude the existence of a sequence $\{f_m\}_{m=1}^{\infty}$ of piecewise linear functions in $C(I)$ such that $\|f_m\|_C\leq\|f\|_C$ and $f_m\to f$ uniformly. If $\varphi\in C_0^\infty(0,1)$ with $|\varphi|\leq 1$, then the previous lemma gives that

$$\left|\int_I \varphi'' f^*\,dx\right|=\lim_{m\to\infty}\left|\int_I \varphi'' f_m^*\,dx\right|\leq\limsup_{m\to\infty}\|f_m^*\|_C\leq\|f\|_C.$$

Hence $\|f^*\|_C\leq\|f\|_C$ which shows the lemma. □

Proof of Theorem 1.1. By rescaling we may without loss of generality assume that $I=[0,1]$. Suppose $f\in C(I)$ with $\|f\|_C<\infty$. Then there is a measure μ on $(0,1)$ such that

$$f(x)=(1-x)f(0)+xf(1)-G\mu(x),\quad x\in[0,1]. \tag{37}$$

In addition $\|f\|_C$ equals the total variation of μ. Notice that G is defined for all $x,\xi\in\mathbf{R}$ by (35). From (37) it follows that f can be extended to a function F on $\mathbf{R}$ such that $|\int_I \varphi'' F\,dx|\leq\|\varphi\|_\infty\|f\|_C$ whenever $\varphi\in C_0^\infty(\mathbf{R})$. Let $\varphi\in C_0^\infty(-1,1)$ be nonnegative with $\int_I\varphi\,dx=1$. For $\varepsilon>0$ set

$$\varphi_\varepsilon(x)=\frac{1}{\varepsilon}\varphi\Big(\frac{x}{\varepsilon}\Big).$$

Let $F_\varepsilon=F*\varphi_\varepsilon$ be the convolution of F with φ_ε. Putting $f_\varepsilon=F_\varepsilon|_I$, we have that

$$\|f_\varepsilon\|_C\leq\|f\|_C$$

and $f_\varepsilon\to f$ uniformly on I. If $\varphi\in C_0^\infty(0,1)$ with $|\varphi|\leq 1$, then the last lemma implies that

$$\left|\int_I \varphi'' f^*\,dx\right|=\lim_{\varepsilon\downarrow 0}\left|\int_I \varphi'' f_\varepsilon^*\,dx\right|\leq\|f\|_C.$$

The theorem is proved. □

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