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Uniqueness of the group topology of some homeomorphism groups

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Abstract. In this paper, we generalize a theorem of Kallman [2, Theorem 1.1] and we resolve the unsettled case there.

Kallman [2, Theorem 1.1] shows that if X is a Hausdorff topological space with a countable basis that does not have exactly two isolated points and if G is a group of homeomorphisms of X such that:

(1) for any non-singleton open set $U \subseteq X$, there exists $g \in G \setminus \{1\}$ such that $g|_{X\setminus U} = 1|_{X\setminus U}$, and

(2) G has a Polish group topology such that $e_x\colon G \to X$ defined by $e_x(g) = gx$ is continuous for all $x \in X$,

then if H is any Polish topological group and if $\theta\colon G \to H$ is any group isomorphism, it follows that θ is a topological isomorphism.

Note that condition (1) is the usual local movability condition for the reconstructibility of topological spaces from their groups of homeomorphisms [4].

In this paper, we generalize this theorem of Kallman by dropping the requirement of the condition on the maps e_x altogether (these functions are not even required to be continuous) and also we replace the requirement that X has a countable basis by the weaker requirement that X has a countable dense set each point of which has a countable fundamental system of neighbourhoods [1, Chapter IX, p. 93, Example 12].

The case where X has exactly two isolated points was left unsettled in [2]. We show that in this case the conclusion of our generalized theorem holds if and only if $G/\langle\{g^2 : g \in G\}\rangle$ is countable.

Recall that a topological space is Souslin if it is a Hausdorff space which is a continuous image of a Polish space (i.e. a completely metrizable separable topological space). The proof of our generalized theorem depends on the Souslin graph theorem [1, Chapter IX, p. 69, Theorem 4] which replaces the theory of functions with the Baire property [3] employed in [2]. The consideration of the case where X has exactly two isolated points depends on a delicate argument in topological groups involving the axiom of choice. This explains why it was left unsettled in [2]. We shall need two lemmas.

Lemma 1. *Let G be a Souslin topological space and let $\{A_i : i \geq 1\}$ be a countable family of Borel subsets of G such that for all $g, g' \in G$ such that $g \neq g'$, there exists some $i \geq 1$ with $g \in A_i$ and $g' \notin A_i$. Then $\{A_i : i \geq 1\}$ generates the σ-algebra of Borel subsets of G.*

Proof. Define $\psi\colon G \to \{0,1\}^N$ by $\psi(g) = (C_i(g))_{i \geq 1}$, where C_i is the characteristic function of A_i for $i \geq 1$. Note that ψ is an injective Borel map [1, Chapter IX, p. 61, Proposition 9] by hypothesis.

Suppose that B is a Borel subset of G and let $j\colon B \to G$ be the canonical injection, then $(\psi \circ j) \times 1\colon B \times \{0,1\}^N \to \{0,1\}^N \times \{0,1\}^N$ is a Borel map and $\Gamma_{\psi \circ j} = ((\psi \circ j) \times 1)^{-1}(\Delta_{\{0,1\}^N})$ is a Borel subset of $B \times \{0,1\}^N$, where $\Delta_{\{0,1\}^N}$ is the diagonal in $\{0,1\}^N \times \{0,1\}^N$. Hence $\Gamma_{\psi \circ j}$ is Souslin and $\psi(B) = pr_2(\Gamma_{\psi \circ j})$ is a Souslin subset of $\{0,1\}^N$. Applying the same argument to $G \setminus B$, it follows that $\{\psi(B), \psi(G \setminus B)\}$ is a partition of $\psi(G)$ into Souslin subsets, hence $\psi(B)$ is a Borel subset of $\psi(G)$ [1, Chapter IX, p. 66, Corollary 1]. Our assertion follows since $\{\psi(A_i) : i \geq 1\}$ generates the σ-algebra of Borel sets of $\psi(G)$ (since $\psi(A_i) = \psi(G) \cap p_i^{-1}(1)$, where $p_i\colon \{0,1\}^N \to \{0,1\}$ is the projection onto the ith factor, hence the σ-algebra generated by $\{\psi(A_i) : i \geq 1\}$ contains a basis of the $\psi(G)$ topology). □

Lemma 2. *Let G, H be Souslin topological groups and let $\psi\colon G \to H$ be a group homomorphism which is a Borel map. Suppose that G is a Baire space. Then ψ is continuous.*

Proof. Note that $\psi \times 1\colon G \times H \to H \times H$ is a Borel map and that Γ_ψ (the graph of ψ) $= (\psi \times 1)^{-1}(\Delta_H)$, where Δ_H is the diagonal of $H \times H$, so that Γ_ψ is Souslin [1, Chapter IX, p. 61, Proposition 10]. The continuity of ψ follows from the Souslin graph theorem [1, Chapter IX, p. 69, Theorem 4]. □

Now we can establish our generalization of Kallman's theorem [2, Theorem 1.1].

Theorem. *Let X be a Hausdorff topological space that does not have exactly two isolated points and that has a countable dense set D each point of which has a countable fundamental system of neighbourhoods. Suppose also that G is a group of homeomorphisms of X such that for any non-singleton open set $U \subseteq X$, there* *exists* $g\in G\setminus\{1\}$ *such that* $g|_{X\setminus U}=1|_{X\setminus U}$. *Suppose that* G *has a Polish group topology. Then for any Polish topological group* H*, any group isomorphism* $\theta\colon G\to H$ *is a topological isomorphism. If* X *has exactly two isolated points, then the same conclusion holds if and only if* $G/\langle\{g^2:g\in G\}\rangle$ *is countable.*

Proof. Let A be the set of isolated points of X. Then $D\supseteq A$ and we set $A=\{w_i:i\in I\}$, where I is at most countable. For all $i,j\in I, i\neq j$, let $(w_i w_j)$ be the element of G that interchanges w_i and w_j and fixes all other points of X. Note that $(D\setminus\bar{A})\cup A\subset X$ is countable and dense, and we let $D\setminus\bar{A}=\{e_i:i\in J\}$, where J is at most countable. By hypothesis, we may assume that $\{V_k(e_i)\subseteq X\setminus\bar{A}:k\geq 1\}$ is a fundamental system of neighbourhoods of e_i for $i\in J$. Define $C=\{g_{ik}:i\in J,k\geq 1\}$ $\subseteq G$ such that $\mathrm{var}(g_{ik})\subseteq V_k(e_i)$ for $i\in J, k\geq 1$, where $\mathrm{var}(g)=\{x\in X:g(x)\neq x\}$ for $g\in G$.

Let C be enumerated as $\{g_i:i\geq 1\}$ and for $g_i,g_j\in C$ define

$$\begin{aligned}
C(g_i,g_j) &= \{g\in G: g(\mathrm{var}(g_i))\cap\mathrm{var}(g_j)=\varnothing\}\\
&= \{g\in G:\mathrm{var}(gg_ig^{-1})\cap\mathrm{var}(g_j)=\varnothing\}\\
&= \{g\in G:[gg_ig^{-1},h]=1 \text{ for all } h\in G \text{ such that } \mathrm{var}(h)\subseteq\mathrm{var}(g_j)\}\\
&= \bigcap\{\{g\in G:[gg_ig^{-1},h]=1\}:h\in G,\mathrm{var}(h)\subseteq\mathrm{var}(g_j)\}.
\end{aligned}$$

To see the third equality, let V be a non-singleton open subset of $\mathrm{var}(gg_ig^{-1})\cap$ $\mathrm{var}(g_j)$ such that $gg_ig^{-1}(V)\cap V=\varnothing$, then there is $h\in G\setminus\{1\}$ such that $\mathrm{var}(h)\subseteq V$ and hence $[gg_ig^{-1},h]\neq 1$.

Therefore $C(g_i,g_j)$ (resp. $\theta(C(g_i,g_j))$) is closed in G (resp. H). Note that if g, $g'\in G$ such that $g|_{X\setminus\bar{A}}\neq g'|_{X\setminus\bar{A}}$, then there is $g_i\in C$ so that $g(\mathrm{var}(g_i))\cap g'(\mathrm{var}(g_i))=\varnothing$ and let $g_j\in C$ such that $\mathrm{var}(g_j)\subseteq g'(\mathrm{var}(g_i))$. Then $g\in C(g_i,g_j)$ and $g'\notin C(g_i,g_j)$.

Case 1. $|A|\leq 1$

In this case $\{C(g_i,g_j):i,j\geq 1\}$ (resp. its image under θ) satisfies the hypotheses of Lemma 1 in G (resp. H), hence both θ and θ^{-1} are Borel maps and the theorem follows from Lemma 2.

Case 2. $|A|=2$

In this case $\{1,(w_1w_2)\}=Z(G)$, since $\{gw_1,gw_2\}=\{w_1,w_2\}$ for all $g\in G$, hence $g(w_1w_2)g^{-1}=(w_1w_2)$ and if $g\in Z(G)$, then $g|_{X\setminus A}=1$ (since if $gx\neq x$ for some $x\in X\setminus A$, then there is $g_i\in C$ such that $g(\mathrm{var}(g_i))\cap\mathrm{var}(g_i)=\varnothing$ and $\varnothing=\mathrm{var}(gg_ig^{-1})\cap$ $\mathrm{var}(g_i)=\mathrm{var}(g_i)$ which is absurd). Let $G_1=\{f\in G:f|_A=1|_A\}$. Note that $\{g^2:g\in G\}$ is Souslin so that $\langle\{g^2:g\in G\}\rangle=\bigcup_{n\geq 1}\{g^2:g\in G\}^n$ is a Souslin subgroup of G.

Suppose that $G/\langle\{g^2:g\in G\}\rangle$ is countable, then G_1 (resp. $\theta(G_1)$) is a Souslin subgroup of G (resp. H) [1, Chapter IX, p. 60, Proposition 7] and hence a Borel subgroup of G (resp. H) [1, Chapter IX, p. 66, Corollary 1]. We conclude that G_1 (resp. $\theta(G_1)$) is an open subgroup of G (resp. H) [1, Chapter IX, p. 55, Theorem 1], [1, Chapter IX, p. 69, Lemma 9]. Now $\theta|_{G_1}: G_1 \to \theta(G_1)$ is a topological isomorphism by considering G_1 as a group of homeomorphisms of $X \setminus A$ and appealing to Case 1, and hence θ is a topological isomorphism as well.

Now, suppose that $G/\langle\{g^2 : g \in G\}\rangle$ is uncountable. Let L be a countable dense subset of G_1. Then there exists a dense subgroup G_2 of G_1 of index 2 such that $G_2 \supseteq \{g^2 : g \in G\} \cup L$ (this uses the axiom of choice which establishes the existence of a basis of a vector space over $\mathbf{Z}/2\mathbf{Z}$). There exists a group homomorphism $d: G_1 \to \mathbf{Z}/2\mathbf{Z}$ with $\ker(d) = G_2$. Note that $G = G_1 \times \mathbf{Z}/2\mathbf{Z}$ (as algebraic isomorphism). Define $\psi: G \to G$ by $\psi(g, y) = (g, d(g) + y)$. Then ψ is a group automorphism. Let τ be the group topology on G such that $\psi: G \to G_\tau$ is a topological isomorphism, so that G_τ is a Polish topological group. Using the topological isomorphism ψ, observe that for all $x_0 \in G_1 \setminus G_2$ we have $((x_n, 0))_{n \geq 1} \subset G_2 \times \{0\}$ converges to $(x_0, 0) \in (G_1 \setminus G_2) \times \{0\}$ if and only if $((x_n, 0))_{n \geq 1}$ τ-converges to $(x_0, d(x_0))$ and $d(x_0) \neq 0$, so that neither $\mathrm{id}: G \to G_\tau$ nor $\mathrm{id}: G_\tau \to G$ is continuous.

Case 3. $3 \leq |A| < \infty$

In this case, let $G_1 = \{f \in G : f|_A = 1|_A\} = C_G(\{(w_i w_j) : i, j \in I, i \neq j\})$. Then G_1 (resp. $\theta(G_1)$) is closed in G (resp. H). If $n = |A| \geq 3$, let $S_n = \{s \in G : s|_{X \setminus A} = 1|_{X \setminus A}\}$ so that $|S_n| = n!$. Now the family $\{G_1 s : s \in S_n\} \cup \{C(g_i, g_j) : i, j \geq 1\}$ (resp. its image under θ) satisfies the hypotheses of Lemma 1 in G (resp. H), hence θ and θ^{-1} are Borel maps and the theorem follows from Lemma 2.

Case 4. $|A| = \infty$

In this case we define, for distinct $i, j \in I$ and for distinct $i', j', k' \in I$,

$$\begin{aligned} M(\{i,j\},\{i',j',k'\}) &= \{g \in G : g(\{w_i, w_j\}) \cap \{w_{i'}, w_{j'}, w_{k'}\} = \varnothing\} \\ &= \{g \in G : [g(w_i w_j) g^{-1}, (w_r w_s)] = 1 \text{ for } r, s \in \{i', j', k'\}\} \\ &= \bigcap_{r, s \in \{i', j', k'\}} \{g \in G : [g(w_i w_j) g^{-1}, (w_r w_s)] = 1\}. \end{aligned}$$

We claim that the family

$$\begin{gathered} \{M(\{i,j\},\{i',j',k'\}) : i, j \in I \text{ distinct and } i', j', k' \in I \text{ distinct}\} \\ \cup \{C(g_i, g_j) : i, j \geq 1\} \text{ (resp. its image under } \theta) \end{gathered}$$

satisfies the hypotheses of Lemma 1 in G (resp. H). Certainly, it is a family of closed sets in G (resp. H). Suppose that $g, g' \in G$ are distinct. If $g|_{X \setminus \bar{A}} \neq g'|_{X \setminus \bar{A}}$, then there exists $g_i, g_j \in C$ such that $g \in C(g_i, g_j)$ and $g' \notin C(g_i, g_j)$. On the other hand, if $g|_{X \setminus \bar{A}} = g'|_{X \setminus \bar{A}}$ assume $g w_i \neq g' w_i = w_{i'}$ and we may assume $w_i \neq w_{i'}$. Let $i \neq j \in I$ and choose $j', k' \in I$ such that i', j', k' are distinct and $g(\{w_i, w_j\}) \cap \{w_{i'}, w_{j'}, w_{k'}\} = \varnothing$, then $g \in M(\{i,j\}, \{i', j', k'\})$ and $g' \notin M(\{i,j\}, \{i', j', k'\})$. It follows from Lemma 1 that θ and θ^{-1} are Borel maps and the theorem follows again from Lemma 2. □

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