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Plurisubharmonic functions characterized by one-variable extremal functions

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1. Introduction

This paper is a short study of certain "extremal-like" functions associated to compact sets in $\mathbf{C}^N$. These functions have been studied previously in [BCL] and [BlLM]. Much of this paper generalizes results in [BCL], and forms part of the author's thesis [M].

Before we begin discussing the main results, we provide some background material that has motivated the study of these functions. Let E be a bounded Borel set in $\mathbf{C}^N$. Define

$$V_E(z) := \sup\{u(z) : u \in L \text{ and } u \leq 0 \text{ on } E\}, \tag{1.1}$$

where

$$L := \{u \text{ is plurisubharmonic on } \mathbf{C}^N : u(z) \leq \log^+|z| + C \text{ for some } C \in \mathbf{R}\}$$

is the class of plurisubharmonic functions of logarithmic growth. Here $|\cdot|$ denotes the standard Euclidean norm: Given $z=(z_1, z_2, ..., z_N)$,

$$|z| := \sqrt{|z_1|^2 + |z_2|^2 + ... + |z_N|^2},$$

and we write $\log^+|z| = \max\{\log|z|, 0\}$. Clearly, if $E \subset F$ then $V_E \geq V_F$. The upper semicontinuous regularization $V_E^*(z) := \limsup_{\zeta \to z} V_E(\zeta)$ is called the Siciak–Zaharjuta extremal function of E. If $V_E^* = V_E$ we say that E is *regular*.

We recall the definition of a pluripolar set, which generalizes the notion of a polar set in one complex variable.

Definition 1.1. Given a set $P \subset \mathbf{C}^N$, suppose that for each $z \in P$ we can find a neighbourhood U_z and a plurisubharmonic function u on U_z such that $U_z \cap P \subset \{z : u(z) = -\infty\}$. Then P is called a *pluripolar set*.

We observe the following properties of pluripolar sets (cf., [BCL], [M]).

Lemma 1.2. *Let $p\colon \mathbf{C}^N \to \mathbf{C}$ be a nonconstant polynomial. Then*
1. *If $E\subset \mathbf{C}^N$ is nonpluripolar then $p(E)$ is nonpolar.*
2. *If $F\subset \mathbf{C}$ is polar then $p^{-1}(F)$ is pluripolar.*

We now recall some standard facts on the Siciak–Zaharjuta extremal function that we will use in this paper (cf., [K1], Chapter 5).

Proposition 1.3.
1. *$V_K^* \equiv V_{K\setminus E}^*$ for any compact set K and pluripolar set E.*
2. *For all compact sets $K\subset \mathbf{C}^N$, $\{z : V_K(z) < V_K^*(z)\}$ is a pluripolar set.*
3. *$V_K^* \equiv +\infty$ if and only if K is pluripolar.*

If $K\subset \mathbf{C}^N$ is compact, then Siciak and Zaharjuta have shown (cf., [K1], Theorem 5.1.7) that V_K can be obtained via the formula

(1.2)
$$V_K(z) = \sup\left\{ \frac{\log^+ |p(z)|}{\deg p} : p \text{ is a nonconstant holomorphic polynomial, } \|p\|_K \le 1 \right\},$$

where $\|p\|_K := \sup_{z\in K} |p(z)|$ denotes the uniform norm. It also turns out that a compact set K is regular if and only if V_K is continuous. The following result is shown in [BCL].

Lemma 1.4. *Let $K\subset \mathbf{C}^N$ be compact and regular. Then for any nonconstant polynomial $p\colon \mathbf{C}^N \to \mathbf{C}$, $p(K)$ is also compact and regular.*

Let $\widehat{K} := \{z\in \mathbf{C}^N : |p(z)| \le \|p\|_K \text{ for all polynomials } p\}$ denote the *polynomial hull* of K. Then from (1.2),
1. $\widehat{K} = \{z\in \mathbf{C}^N : V_K(z) = 0\}$;
2. $V_{\widehat{K}} = V_K$.

A compact set is *polynomially convex* if $\widehat{K} = K$.

Note that given a nonpolar compact set K in $\mathbf{C}$, (i.e., $N=1$), V_K^* is the classical Green function of K with logarithmic pole at infinity. In particular, V_K^* is harmonic on $\mathbf{C}\setminus K$.

In several variables explicit computation of V_E for an arbitrary Borel set E is virtually impossible in general. For some restricted classes of sets, more information about V_E (including some explicit and semi-explicit formulae) is obtained by proving simplifications of formula (1.1) (as, for example, formula (1.2) for compact sets).

However, there is a simple formula for the Siciak–Zaharjuta extremal function associated to a closed ball $B(a,r)=\{z\in\mathbf{C}^N:|z-a|\le r\}$. In direct analogy to the one-variable case,

$$V_{B(a,r)}(z)=\log^+\frac{|z-a|}{r}.$$

For a closed disk in $\mathbf{C}$ we use the notation $\Delta(a,r)=\{z\in\mathbf{C}:|z-a|\le r\}$. We will also write $B:=B(0,1)$ (and $\Delta:=\Delta(0,1)$) to denote the unit ball in $\mathbf{C}^N$ (and unit disk in $\mathbf{C}$).

If $K\subset\mathbf{C}^N$ is compact, and p_d is a polynomial of degree d, then

$$\frac{1}{d}V^*_{p_d(K)}(\zeta)=0 \quad \text{for all } \zeta\in p_d(K)\setminus Z,$$

where $Z:=\{\zeta\in\mathbf{C}\colon V_{p_d(K)}(\zeta)<V^*_{p_d(K)}(\zeta)\}$ is a polar set. Hence by Lemma 1.2 and Proposition 1.3,

$$\frac{1}{d}V^*_{p_d(K)}(p_d(z))\le\frac{1}{d}V^*_{p_d(K)\setminus Z}(p_d(z))\le V^*_{K\setminus p_d^{-1}(Z)}(z)\le V^*_K(z),$$

i.e.,

$$\frac{1}{d}V^*_{p_d(K)}(p_d(z))\le V^*_K(z). \tag{1.3}$$

On the other hand, if $\|p_d\|_K\le 1$, then

$$V^*_{p_d(K)}(w)\ge V_\Delta(w)=\log^+|w| \quad \text{for all } w\in\mathbf{C};$$

in particular $V^*_{p_d(K)}(p_d(z))\ge\log^+|p_d(z)|$. Taking the supremum on both sides over all nonconstant polynomials p_d with $\|p_d\|_K\le 1$, and using (1.2), we have

$$\sup_{p_d}\frac{1}{d}V^*_{p_d(K)}(p_d(z))\ge V_K(z). \tag{1.4}$$

Note that for K regular, (1.3), (1.4) and Lemma 1.4 imply that

$$\sup_{p_d}\frac{1}{d}V_{p_d(K)}(p_d(z))=V_K(z).$$

Fix a positive integer n; then using only the nonconstant polynomials of degree $\le n$, we define

$$V_K^{(n)}(z):=\sup\left\{\frac{V^*_{p(K)}(p(z))}{\deg p}: p \text{ is a holomorphic polynomial and } 1\le\deg p\le n\right\}.$$

Thus if K is regular, $V_K^{(n)}\nearrow V_K$ as $n\to\infty$.

Basic properties of the functions $V_K^{(n)}$ have been studied in [BlLM] and [M]; in particular, $V_K^{(n)}$ is continuous if K is a regular compact set. In this paper we will concern ourselves exclusively with the case $n=1$. The original motivation for studying $V_K^{(1)}$ is that

$$V_K^{(1)} \equiv V_K \tag{1.5}$$

holds for certain compact sets. Simple examples for which (1.5) holds are N-fold products of planar compact sets. Of greater interest is the work of Lundin and Baran ([L], [Ba]), which shows that (1.5) holds when K is a compact, convex body (i.e., a set with nonempty interior) in $\mathbf{R}^N \subset \mathbf{C}^N$ which is symmetric with respect to the origin (i.e., $x \in K \Leftrightarrow -x \in K$). The question arises as to what extent (1.5) holds in general. The investigations in [BCL] and [M] address this question in some detail. Define, for a compact set $K \subset \mathbf{C}^N$,

$$R := \{z \in \mathbf{C}^N : V_K^{(1)}(z) = V_K(z)\}.$$

For a (not necessarily symmetric) convex body $K \subset \mathbf{R}^N \subset \mathbf{C}^N$, R contains a set of real dimension $N+1$, which includes all of $\mathbf{R}^N$ ([BCL]). However, it has been shown that in $\mathbf{R}^2$, (1.5) fails for "most" convex sets ([BuLM], [M]). Altogether it appears that in general, the notions of $V_K^{(1)}$ and V_K are fundamentally different in several variables, although in one variable they both reduce to the classical Green function for K with logarithmic pole at infinity.

Thus $V_K^{(1)}$ originally arose as an auxiliary to aid in the study of V_K. In the course of investigating $V_K^{(1)}$ (and its upper semicontinuous regularization $V_K^{(1)*}$), some interesting properties have been discovered. They form the subject of this paper. In Section 2 we describe $K^{(1)}$, the zero set of $V_K^{(1)}$ associated to a regular compact set K; in particular, we show that $K^{(1)}$ is a *lineally convex* set (see Definition 2.1). At the end of that section we construct a counterexample to a result that was claimed in the paper [BlLM] (see also [M], Lemma 1.9); however, it turns out that one does not need this incorrect result to prove the main theorems there. In Section 3 we introduce the *projection capacity* $C(K)$ of an arbitrary compact set K in $\mathbf{C}^N$. This notion was suggested by J.-P. Calvi. We show that $C(K)>0$ if and only if $V_K^{(1)*} \in L$. In Section 4 we take a look at the notion of a C-regular set, which was one of the ingredients used in [BCL] to prove that $V_{S_2}^{(1)} \not\equiv V_{S_2}$, where

$$S_2 := \{(x_1, x_2) \in \mathbf{R}^2 : x_1 \geq 0, x_2 \geq 0 \text{ and } x_1 + x_2 \leq 1\}$$

is the simplex in $\mathbf{R}^2 \subset \mathbf{C}^2$. The simplex happens to be a C-regular set. However, this property is not really special; we prove here that all compact sets K with $C(K)>0$ are C-regular. We then use this to show that for *any* compact set K, the *Robin* *function of* $\left(V_K^{(1)}\right)^*$ (see Definition 4.6) can be obtained as the upper envelope of the Robin functions of its complex linear images $l(K)$ (formula (4.4)). This formula was previously only known to be true for C-regular sets (see [BCL]).

2. Lineally convex sets

By a *complex affine hyperplane in* $\mathbf{C}^N$ we will mean a set of the form

$$H=\{z\in\mathbf{C}^N : l(z)=const.\},$$

where $l\colon \mathbf{C}^N\to\mathbf{C}$ is a linear polynomial.

Definition 2.1. A compact set $E\subset\mathbf{C}^N$ is *lineally convex* if its complement $\mathbf{C}^N\setminus E$ is the union of complex affine hyperplanes in $\mathbf{C}^N$. The *lineally convex hull* $\widetilde{E}$ of a compact set E is the smallest lineally convex compact set containing E.

Denoting by H a complex affine hyperplane in $\mathbf{C}^N$, we have

$$\widetilde{E}=\mathbf{C}^N\setminus\Bigl(\bigcup_{H\cap E=\varnothing} H\Bigr). \tag{2.1}$$

Remark 2.2. Every compact, convex subset $X\subset\mathbf{C}^N$ is lineally convex. This follows easily from the fact that any point $z\in\mathbf{C}^N\setminus X$ lies on a *real* affine hyperplane $\widehat{H}$ that avoids X, and hence lies on a complex affine hyperplane $H\subset\widehat{H}$.

Consider now a compact set $K\subset\mathbf{R}^N\subset\mathbf{C}^N$. Since $\mathbf{R}^N$ is convex in $\mathbf{C}^N$, to show that K is lineally convex we need only to find a complex affine hyperplane H through each point of $\mathbf{R}^N\setminus K$ that does not intersect K.

In two dimensions we make the following observation. Given a point $(a,b)\in\mathbf{R}^2$, the complex hyperplane given by $H:=\{(z_1,z_2)\in\mathbf{C}^2 : z_1-a=i(z_2-b)\}$ intersects $\mathbf{R}^2$ precisely at (a,b). Thus *any* compact subset of $\mathbf{R}^2$ is lineally convex.

Since $V_K^{(1)}$ is defined in terms of the Green functions $V_{l(K)}$, properties of $V_K^{(1)}$ depend fundamentally on properties of K which can be expressed in terms of the collection of images $l(K)$.

Definition 2.3. A compact set $E\subset\mathbf{C}^N$ has the property $\mathcal{P}$ if for all complex linear maps $l\colon\mathbf{C}^N\to\mathbf{C}$, $l(E)$ is polynomially convex.

Lemma 2.4. *Let K be a lineally convex compact subset of $\mathbf{C}^N$. If K has the property $\mathcal{P}$, then K is polynomially convex.*

Proof. Let $z\notin K$. Since K is lineally convex, we can find a linear map l such that $\zeta=l(z)\notin l(K)$. Since $l(K)$ is polynomially convex, we can find a polynomial p such that $|p(\zeta)|>\|p\|_{l(K)}$. Hence

$$|p\circ l(z)|=|p(\zeta)|>\|p\|_{l(K)}=\|p\circ l\|_K,$$

i.e., $z\notin\widehat{K}$. This shows that K is polynomially convex. □

Remark 2.5. One needs additional conditions on lineally convex sets to ensure polynomial convexity. The torus $T=\{z:|z_1|=|z_2|=1\}\subset\mathbf{C}^2$ is not polynomially convex, but it is easy to see that T is lineally convex. If $(a_1,a_2)\notin T$ with $|a_1|\neq 1$, then the complex hyperplane $z_1=a_1$ intersects (a_1,a_2) but not T.

On the other hand, there are regular, compact sets which are both lineally and polynomially convex, but do not have the property $\mathcal{P}$. For example, let

$$A:=\{(x_1,x_2)\in\mathbf{R}^2\subset\mathbf{C}^2:x_1^2+x_2^2\in[1,2]\}.$$

Consider $l(z_1,z_2)=z_1+iz_2$; then $l(A)=\{z\in\mathbf{C}:1\leq|z|\leq 2\}$ which is clearly not polynomially convex.

Recall that for any compact set K, $V_K\equiv V_{\widehat{K}}$, and if K is regular we also have $\widehat{K}=\{z:V_K(z)=0\}$. We have a similar type of result concerning $V_K^{(1)}$.

Proposition 2.6. *Let $K\subset\mathbf{C}^N$ be a regular compact set with the property $\mathcal{P}$. Define*

$$K^{(1)}:=\{z\in\mathbf{C}^N:V_K^{(1)}(z)=0\}.$$

Then $K^{(1)}$ is the lineally convex hull of K, and

$$V_{K^{(1)}}^{(1)}\equiv V_K^{(1)}\,. \tag{2.2}$$

Proof. Denote by $\widetilde{K}$ the lineally convex hull of K; we want to show that $\widetilde{K}=K^{(1)}$.

Choose $z\notin K^{(1)}$. Then $V_K^{(1)}(z)>0$, so we can find a complex linear function $l\colon\mathbf{C}^N\to\mathbf{C}$ such that $V_{l(K)}(l(z))>0$. Since $l(K)$ is regular (by Lemma 1.4), $l(z)\notin l(K)$. Clearly the complex hyperplane $H:=\{w\in\mathbf{C}^N:l(w)=l(z)\}$ contains z, and $l(K)=\{\zeta\in\mathbf{C}:V_{l(K)}(\zeta)=0\}$. Hence $l(K)\cap\{l(z)\}=\varnothing$, so that $K\cap H=\varnothing$. Hence $z\notin\widetilde{K}$.

Conversely, if $z\notin\widetilde{K}$ there is a complex hyperplane H containing z with $H\cap K=\varnothing$. Choose a complex linear function $l\colon\mathbf{C}^N\to\mathbf{C}$ such that $H=\{w\in\mathbf{C}^N:l(w)=l(z)\}$. Then $l(z)\in l(H)$, so that $l(z)\notin l(K)$, and thus $V_{l(K)}(l(z))>0$ since $l(K)$ is regular and polynomially convex. Hence $V_K^{(1)}(z)>0$, i.e., $z\notin K^{(1)}$.

Altogether, this shows that $\widetilde{K}=K^{(1)}$.

We now show (2.2). When $z\in K^{(1)}$ we get zero on both sides of the equation. Also, clearly $V^{(1)}_{K^{(1)}}\le V^{(1)}_K$, so it remains to show that $V^{(1)}_{K^{(1)}}\ge V^{(1)}_K$. Take $z\notin K^{(1)}$ and let $\varepsilon>0$. Choose a complex linear function l such that $V^{(1)}_K(z)\le V_{l(K)}(l(z))+\varepsilon$. First, we claim that

$$l(K)=l(K^{(1)}). \tag{2.3}$$

Let $\zeta\in l(K^{(1)})$. Then $\zeta=l(z_0)$ for some $z_0\in K^{(1)}=\widetilde{K}$. The set $\{w\in K:l(w)=\zeta\}$ is nonempty, so there exists $z_1\in K$ such that $l(z_1)=\zeta$, i.e. $\zeta\in l(K)$. So $l(K^{(1)})\subset l(K)$. The reverse inclusion is obvious, and thus (2.3) holds. Now

$$V^{(1)}_K(z)\le V_{l(K)}(l(z))+\varepsilon=V_{l(K^{(1)})}(l(z))+\varepsilon\le V^{(1)}_{K^{(1)}}(z)+\varepsilon,$$

where the middle equality follows from (2.3). Letting $\varepsilon\to 0$, we have (2.2). □

Lemma 2.7. *Let K be a regular compact set. Then*

$$K^{(1)}=\bigcap_l l^{-1}(\widehat{l(K)}). \tag{2.4}$$

In particular, $K^{(1)}$ is lineally convex.

Proof. Denote the right-hand side by F. We need to show that $K^{(1)}=F$. If $z\in K^{(1)}$, we have $V_{l(K)}(l(z))=0$ for all l, and hence $l(z)\in\widehat{l(K)}$ for all l. This shows that $K^{(1)}\subset F$. Conversely, if $z\in l^{-1}(\widehat{l(K)})$ for all l, then $l(z)\in\widehat{l(K)}$ for all l, so that $V_{l(K)}(l(z))=0$ for all l. Hence $V^{(1)}_K(z)=0$, i.e., $z\in K^{(1)}$, which shows that $F\subset K^{(1)}$.

To see that $K^{(1)}$ is lineally convex, take $z\notin K^{(1)}$ and choose l such that $l(z)\notin\widehat{l(K)}$. Then the hyperplane $H:=\{w\in\mathbf{C}^N:l(w)=l(z)\}$ satisfies $H\cap l^{-1}\big(\widehat{l(K)}\big)=\varnothing$, so that $H\cap K^{(1)}=\varnothing$. □

Remark 2.8. If A is given as in Remark 2.5, from (2.4) it is easy to see that

$$A^{(1)}=\{(x_1,x_2)\in\mathbf{R}^2:x_1^2+x_2^2\le 2\}.$$

Another corollary of Lemma 2.7 is the following result in [BCL].

Corollary 2.9. *Let K be a regular, polynomially convex, compact set. If $V_K=V^{(1)}_K$, then K is lineally convex.*

Proof. We have $K=\widehat{K}=\{z:V_K(z)=0\}=\{z:V^{(1)}_K(z)=0\}=K^{(1)}$. Since $K^{(1)}$ is lineally convex, the result follows. □

Let Π be the collection of polynomially convex, regular compact sets in $\mathbf{C}^N$. In [K2], Klimek has shown that one can obtain a metric Γ on Π (the *Klimek metric*) via the formula

$$\Gamma(E,F)=\|V_E-V_F\|_{\mathbf{C}^N},\quad E,F\in\Pi.$$

We can do a similar thing with $V^{(1)}$. Let Π^1 denote the collection of regular, lineally convex compact sets in $\mathbf{C}^N$ with the property $\mathcal{P}$. Note that $\Pi^1\subset\Pi$ by Lemma 2.4. Now define

$$\Gamma^{(1)}(E,F):=\|V_E^{(1)}-V_F^{(1)}\|_{\mathbf{C}^N},\quad E,F\in\Pi^1.$$

Proposition 2.10. *$\Gamma^{(1)}$ is a metric on Π^1.*

Proof. Clearly $\Gamma^{(1)}$ is nonnegative and symmetric. To check that $\Gamma^{(1)}$ is positive definite, suppose $\Gamma^{(1)}(E,F)=0$. Then $V_E^{(1)}=V_F^{(1)}$, and hence $E^{(1)}=F^{(1)}$. By Proposition 2.6, this means that $E=F$.

For the triangle inequality, take E_1,E_2,E_3 in Π^1; then given $z\in\mathbf{C}^N$, we have

$$\begin{aligned}|V_{E_1}^{(1)}(z)-V_{E_3}^{(1)}(z)| &\le |V_{E_1}^{(1)}(z)-V_{E_2}^{(1)}(z)|+|V_{E_2}^{(1)}(z)-V_{E_3}^{(1)}(z)|\\ &\le \Gamma^{(1)}(E_1,E_2)+\Gamma^{(1)}(E_2,E_3).\end{aligned}$$

Since z was arbitrary, this gives the triangle inequality. □

It is shown in [K2] that (Π,Γ) is a complete metric space. Let χ be the Hausdorff metric on the collection of compact sets in $\mathbf{C}^N$, i.e.,

$$\chi(E_1,E_2):=\inf\{\varepsilon>0: E_1\subset E_2^\varepsilon \text{ and } E_2\subset E_1^\varepsilon\},$$

where we write $E^\varepsilon:=\{z\in\mathbf{C}^N:|z-w|\le\varepsilon \text{ for some } w\in E\}$.

Remark 2.11. Note that convergence in the Hausdorff metric does not imply convergence in Γ. Given a nonregular set K, for each $\varepsilon_j:=1/j$ the set K^{ε_j} is regular, and $K^{\varepsilon_j}\searrow K$ in the Hausdorff metric as $j\to\infty$. However, since (Π,Γ) is complete and on the other hand K is not regular, the sequence K^{ε_j} cannot converge in Γ to K.

In [BlLM] and [M], the following incorrect result was claimed:

(†) *Let K be a regular compact set. Given $\varepsilon>0$, there exists $\delta>0$ such that if K' is a regular compact set with $K'\subset K^\delta$ and $K\subset(K')^\delta$, then $\|V_K-V_{K'}\|_{\mathbf{C}^N}<\varepsilon$; i.e.,*

$$\chi(K,K')<\delta \quad\Longrightarrow\quad \Gamma(K,K')<\varepsilon.$$

We sketch the following counterexample to (†) in one variable.

Let $K=[0,2]$, and let Ω be a fixed neighbourhood of K (say $\Omega=\Delta(0,3)$). Fix $\varepsilon\in\left(0,\frac{1}{2}\|V_{[1,2]}-V_{[0,2]}\|_{\mathbf{C}\setminus\Omega}\right)$. Given $\delta>0$ we choose a finite set of points $\{x_j\}_j$ in the interval $[0,1]$ such that

$$\bigcup_j [x_j-\delta, x_j+\delta]\supset[0,1].$$

Now take

$$K':=[1,2]\cup\left(\bigcup_j [x_j-\alpha, x_j+\alpha]\right),$$

where $\alpha>0$ is chosen so small that $\|V_{K'}-V_{[1,2]}\|_{\mathbf{C}\setminus\Omega}<\varepsilon$. Then K' is regular, $K'\subset K^\delta$, and $K\subset(K')^\delta$, but

$$\|V_K-V_{K'}\|_{\mathbf{C}}\geq\|V_K-V_{K'}\|_{\mathbf{C}\setminus\Omega}\geq\|V_{[1,2]}-V_{[0,2]}\|_{\mathbf{C}\setminus\Omega}-\|V_{K'}-V_{[1,2]}\|_{\mathbf{C}\setminus\Omega}>2\varepsilon-\varepsilon=\varepsilon.$$

Thus we have shown that given $\varepsilon>0$ sufficiently small, then for any $\delta>0$ we can construct a set K' such that $\chi(K,K')<\delta$ but $\Gamma(K,K')>\varepsilon$.

The main theorems of [BlLM] and [M] made use of results that were proved using the incorrect claim (†). We will now reprove these results without using (†), thus showing that the main theorems in those papers remain true. We need to use a couple of standard properties of the Siciak–Zaharjuta extremal function for a compact set.

Given a compact set K, we have

$$V_K(z)=V_{a+T(K)}(a+T(z)) \tag{2.5}$$

for any $a,z\in\mathbf{C}^N$ and invertible complex linear map $T\colon\mathbf{C}^N\to\mathbf{C}^N$. This is a special case of a more general result due to Klimek ([K1], Theorem 5.3.6), but (2.5) can also be proved easily using (1.1) and the fact that the class L is invariant under affine transformations. Note that (2.5) implies, in particular, that $T(K)$ is regular if and only if K is. Recall that in one complex variable it is well-known that the complex Green function is invariant under a conformal map.

For regular compact sets E and F in $\mathbf{C}^N$, we have the well-known property

$$\|V_E-V_F\|_{\mathbf{C}^N}=\|V_E-V_F\|_{E\cup F}=\max\{\|V_E\|_F,\|V_F\|_E\}. \tag{2.6}$$

We now introduce the following standard notation: If $T\colon\mathbf{C}^N\to\mathbf{C}^M$ is a linear transformation then $\|T\|:=\sup_{|z|=1}|T(z)|$ is the standard operator norm. We also write $(\mathbf{C}^N)^*$ to denote the space of complex linear functions $l\colon\mathbf{C}^N\to\mathbf{C}$.

Proposition 2.12. *Let $K\subset\mathbf{C}^N$ be a regular compact set. Given $\varepsilon>0$ there exists $\delta>0$ such that if $T\colon\mathbf{C}^N\to\mathbf{C}^N$ is a complex linear map with $\|T-I\|<\delta$ and $\|T^{-1}-I\|<\delta$, then $\|V_K-V_{T(K)}\|<\varepsilon$.*

Proof. Given $\delta\in(0,1)$, let T be a complex linear mapping with $\|T-I\|<\delta$ and $\|T^{-1}-I\|<\delta$. Since K is regular, V_K is uniformly continuous on a large ball $B_M:=B(0,M)$; let ω denote the modulus of continuity on B_M. For points $z\in B(0,M-\delta)$, we have

$$|V_K(z)-V_{T(K)}(z)|=|V_K(z)-V_K(T^{-1}(z))|\leq\omega(\|T^{-1}-I\||z|)\leq\omega(\delta M). \tag{2.7}$$

Since K is compact, $K\subset B(0,R)$ for some $R>0$. Now $\|T-I\|<\delta<1$ implies that $T(K)\subset B(0,2R)$; thus if we have chosen $M>2R+\delta$, then $T(K)\cup K\subset B(0,M-\delta)$, so that, by (2.7),

$$\|V_K-V_{T(K)}\|_{\mathbf{C}^N}=\|V_K-V_{T(K)}\|_{K\cup T(K)}\leq\omega(\delta M).$$

For a fixed $\varepsilon>0$ we can take δ small enough so that the right-hand side of the above inequality is less than ε. □

Lemma 2.13. *Let $K,K'\subset\mathbf{C}^N$ be regular compact sets. For any nonconstant holomorphic polynomial $p\colon\mathbf{C}^N\to\mathbf{C}$,*

$$\frac{1}{\deg p}\|V_{p(K)}-V_{p(K')}\|_{\mathbf{C}}\leq\|V_K-V_{K'}\|_{\mathbf{C}^N}. \tag{2.8}$$

Proof. First note that $p(K)$ and $p(K')$ are regular in $\mathbf{C}$. Hence using (2.6), we need only estimate $|V_{p(K)}(w)-V_{p(K')}(w)|$ at points $w\in p(K)\cup p(K')$. Fix $w\in p(K')$. Then $V_{p(K')}(w)=0$, and writing $w=p(z)$ for some $z\in K'$, we have

$$\frac{1}{\deg p}V_{p(K)}(w)=\frac{1}{\deg p}V_{p(K)}(p(z))\leq V_K(z)\leq\|V_K\|_{K'},$$

where the fact that $(V_{p(K)}\circ p)/\deg p\in L$ and $(V_{p(K)}\circ p)/\deg p\leq 0$ on K implies that $(V_{p(K)}\circ p)/\deg p\leq V_K$. Similarly, if $w\in p(K)$ we obtain $V_{p(K')}(w)/\deg p\leq\|V_{K'}\|_K$. Applying (2.6) to these two inequalities yields (2.8). □

Proposition 2.14. *Let $K\subset\mathbf{C}^N$ be a regular compact set. Given $\varepsilon>0$ there exists $\delta\in\left(0,\frac{1}{2}\right)$ such that, if $l_1,l_2\colon\mathbf{C}^N\to\mathbf{C}$ are complex linear functions with $\|l_1\|=1$ and $\|l_1-l_2\|<\delta$, then $\|V_{l_1(K)}-V_{l_2(K)}\|_{\mathbf{C}}<\varepsilon$.*

Proof. Recall that $(\mathbf{C}^N)^*$ is the vector space of complex linear functions from $\mathbf{C}^N$ to $\mathbf{C}$. Given l_1 and l_2, let $T^*\colon(\mathbf{C}^N)^*\to(\mathbf{C}^N)^*$ be a complex linear transformation such that $T^*(l_1)=l_2$. By elementary functional analysis there exists a complex linear transformation $T\colon\mathbf{C}^N\to\mathbf{C}^N$ with the property that $l(T(z))=(T^*(l))(z)$ for any $l\in(\mathbf{C}^N)^*$ and $z\in\mathbf{C}^N$. In particular, $l_1(T(z))=l_2(z)$ for all $z\in\mathbf{C}^N$. Using this and (2.8), we have

$$\|V_{l_1(K)}-V_{l_2(K)}\|_{\mathbf{C}}=\|V_{l_1(K)}-V_{l_1(T(K))}\|_{\mathbf{C}}\leq\|V_K-V_{T(K)}\|_{\mathbf{C}^N}. \tag{2.9}$$

Now if $\|l_1\|=1$ and $\|l_1-l_2\|<\delta$, we claim that we can always find a complex linear transformation T^* such that $T^*(l_1)=l_2$, $\|T^*-I\|<2\delta$ and $\|(T^*)^{-1}-I\|<2\delta$ as follows. For simplicity, we rotate our coordinates so that in our new coordinates, $l_1(z):=z_1$; then in the dual coordinates $(z_1^*, ..., z_N^*)$ we have $l_1=(1,0,...,0)$ and $l_2=(1+\delta_1,\delta_2,...,\delta_N)$ for some numbers $\{\delta_j\}_{j=1}^N$ that satisfy $\sum_{j=1}^N|\delta_j|^2\leq\delta^2$. If we set

$$T^* := I+E$$

such that E is given by the matrix of column vectors $[\alpha\, 0 \dots 0]$ with $\alpha=(\delta_1,\delta_2,...,\delta_N)$, then clearly $T^*(l_1)=l_2$, and $\|T^*-I\|=\|E\|\leq\delta$. Now $\|(T^*)^{-1}-I\|\leq\delta/(1-\delta)<2\delta$. This proves the claim.

Fix $\varepsilon>0$. Using Proposition 2.12 we choose $\delta>0$ such that if $T\colon \mathbf{C}^N\to\mathbf{C}$ is a complex linear map with $\|T-I\|<2\delta$ and $\|T^{-1}-I\|<2\delta$ then

$$\|V_K-V_{T(K)}\|_{\mathbf{C}^N}<\varepsilon. \tag{2.10}$$

Now for such a δ, if $\|l_1\|=1$ and $\|l_2-l_1\|<\delta$, we obtain, by the previous argument, a linear transformation T^* with $\|T^*-I\|\leq2\delta$ and $\|(T^*)^{-1}-I\|\leq2\delta$. Using the fact that the dual transformation T on $\mathbf{C}^N$ satisfies $\|T^*\|=\|T\|$ and $\|(T^*)^{-1}\|=\|T^{-1}\|$, we have, by Proposition 2.12, that (2.10) holds for T. Applying this to (2.9) yields the result. □

These results show that in certain cases we can estimate $\Gamma(K,K')$ in terms of $\chi(K,K')$ if K and K' have additional nice properties. Siciak has observed in [S] that a sequence of compact sets $\{K_j\}_{j=1}^\infty$ converging in χ to a compact set K also converges to K in Γ if and only if the family $\{V_{K_j}\}_{j=1}^\infty$ is equicontinuous.

3. The projection capacity

For a linear polynomial $l(z)=a_0+a_1z_1+a_2z_2+...+a_Nz_N$, it follows from (2.5) that

$$V_{l(K)}(l(z))=V_{tl(K)+c}(tl(z)+c)\quad \text{for any } c,t\in\mathbf{C} \text{ with } t\neq0.$$

Hence we may assume that the supremum in the definition of $V_K^{(1)}$ is taken over a normalized set of linear polynomials; in particular, those with no constant term (i.e., $a_0=0$), and with

$$|a_1|^2+|a_2|^2+...+|a_N|^2=1.$$

This allows us to rewrite the the definition of $V_K^{(1)}$ as follows:

$$V_K^{(1)}(z):=\sup\{V_{l(K)}(l(z)) : l\in(\mathbf{C}^N)^* \text{ and } \|l\|=1\}. \tag{3.1}$$

We now recall the logarithmic capacity, of one-variable complex potential theory. Let $X \subset \mathbf{C}$ be compact. The *Robin constant* ρ_X is given by

$$\rho_X = \limsup_{|t|\to\infty} (V_X(t) - \log|t|) = \lim_{|t|\to\infty} (V_X(t) - \log|t|).$$

The *logarithmic capacity* of X is given by $\operatorname{cap}(X) = e^{-\rho_X}$. A compact set E is polar if and only if $\operatorname{cap}(X)=0$. There are other characterizations of the logarithmic capacity in terms of *transfinite diameter* and *Chebyshev constants* (see [R]), which provide the basis for several applications of complex potential theory.

Definition 3.1. Given a compact set $K \subset \mathbf{C}^N$, we define the *projection capacity of* K by the formula

$$C(K) := \inf\{\operatorname{cap}(l(K)) : l \in (\mathbf{C}^N)^* \text{ and } \|l\| = 1\},$$

where, as before, $\operatorname{cap}(E)$ denotes the logarithmic capacity of $E \subset \mathbf{C}$. It follows immediately from the definition that $C(K_1) \le C(K_2)$ if $K_1 \subset K_2$, i.e., we have *monotonicity* on compact sets.

We also have continuity under decreasing limits.

Proposition 3.2. *Let $\{K_n\}_{n=1}^\infty$ be a decreasing sequence of compact sets $K_{n+1} \subset K_n$, and let $K := \bigcap_{n=1}^\infty K_n$. Then*

$$\lim_{n\to\infty} C(K_n) = C(K).$$

The above result follows from the fact that $\operatorname{cap}(\,\cdot\,)$ is continuous under decreasing limits, together with the fact that for a monotone decreasing sequence of sets $\{K_n\}_{n=1}^\infty$ and a linear map l, we have $\bigcap_{n=1}^\infty l(K_n) = l(\bigcap_{n=1}^\infty K_n)$. For details, see [M].

Remark 3.3. Following Brelot ([Br]), one can define the *inner capacity* C_* and the *outer capacity* C^* for an arbitrary set A as follows:

$$C_*(A) := \sup\{C(K) : K \text{ is compact and } K \subset A\},$$
$$C^*(A) := \inf\{C_*(U) : U \text{ is open and } U \supset A\}.$$

A is called *capacitable* if $C_*(A) = C^*(A)$. Both the open sets and compact sets are capacitable. The capacitability of open sets is a simple consequence of the definition, and the capacitability of compact sets uses Proposition 3.2 (cf., [M]). Thus the projection capacity can be extended to a larger class; however, in what follows we will continue to work exclusively with compact sets. It is not yet known whether or not C^* or C_* are Choquet capacities.

The projection capacity is closely related to the $(V^{(1)})^*$ function. Before proving the main result of this section we recall the following important property of upper envelopes of functions in the class L (cf., [K1], Proposition 5.2.1).

Proposition 3.4. *Let $\mathcal{U}\subset L$ be a nonempty family, and define $u(z):=\sup_{v\in\mathcal{U}} v(z)$. Then either $u^*\in L$ or $u^*\equiv+\infty$.*

Theorem 3.5. *For a compact set $K\subset\mathbf{C}^N$, $C(K)>0$ if and only if $(V_K^{(1)})^*\in L$.*

Proof. Suppose $C(K)>0$. For any $R>0$ such that $K\subset B(0,R)$, we have, for $z\in B(0,R)$,

$$\begin{aligned}V^*_{l(K)}(l(z)) &= \int \log|l(z)-t|\,d\mu_{l(K)}(t)-\log\operatorname{cap}(l(K))\\ &\leq \log 2R-\log\operatorname{cap}(l(K))\leq\log 2R-\log C(K)\end{aligned}$$

if $\|l\|=1$. Hence $V_K^{(1)}(z)\leq\log 2R-\log C(K)$ for all $z\in B(0,R)$, so by Proposition 3.4, $(V_K^{(1)})^*\in L$.

Conversely, if $(V_K^{(1)})^*\in L$, then for all $z\in\mathbf{C}^N$, $(V_K^{(1)})^*(z)\leq\log|z|+C$ for some $C\in\mathbf{R}$, and hence $V^*_{l(K)}(l(z))\leq\log|z|+C$ for all l. Take $l\in(\mathbf{C}^N)^*$ with $\|l\|=1$. Given $t\in\mathbf{C}$, set $z_1=t\bar{a}_1, ..., z_N=t\bar{a}_N$ (here $a_1, ..., a_N$ are the coefficients of l). Then $l(z)=t$ and $|z|=|t|$. Thus for such z and t with $|z|=|t|\geq1$, we have

$$V^*_{l(K)}(t)=V^*_{l(K)}(l(z))\leq C+\log|z|=C+\log|t|.$$

Letting $|t|\to\infty$, we get $\operatorname{cap}(l(K))\geq e^{-C}$. Thus $C(K)\geq e^{-C}>0$. □

Remark 3.6. It follows immediately that $C(K)=0$ if and only if $(V_K^{(1)})^*\equiv+\infty$. Hence if $C(K)=0$, then $V_K^*\equiv+\infty$ too, so K is *pluripolar* (Proposition 1.3). However, we give here an example of a pluripolar set in $\mathbf{C}^2$ with positive projection capacity. Let

$$K:=\{(z,0):z\in[0,1]\}\cup\{(0,w):w\in[0,1]\}.$$

Then K is pluripolar, being the union of two pluripolar sets. K is also lineally convex. If $l(z,w)=az+bw$ with $|a|^2+|b|^2=1$, then $l(K)$ contains the line segment joining 0 and a, and the line segment joining 0 and b in $\mathbf{C}$. So

$$\operatorname{cap}(l(K))\geq\max\left\{\frac{|a|}{4},\frac{|b|}{4}\right\}\geq\frac{1}{4\sqrt{2}},$$

where we have used the fact that a line segment in $\mathbf{C}$ of length l has logarithmic capacity $l/4$ ([R]).

For $a=b=1/\sqrt{2}$, the lower bound is attained and hence $C(K)=1/4\sqrt{2}$.

4. *C*-regular sets

We close the paper with a discussion of C-regularity, which was first defined in [BCL]. For convenience, from now on all linear maps $l\in(\mathbf{C}^N)^*$ that we consider will have the normalization $\|l\|=1$.

Definition 4.1. A compact set $K\subset\mathbf{C}^N$ is C-regular if there exists $b_0\in(0,1)$ and $R\geq 1$ such that for all $|\eta|>R$,

$$\sup_l V^*_{l(K)}(b\eta) \leq \inf_l V^*_{l(K)}(\eta) \quad \text{for all } 0<|b|<b_0.$$

The main aim here is to show that compact sets with positive projection capacity are C-regular. First we need some lemmas. Recall from the previous section that $\operatorname{cap}(X)=e^{-\rho_X}$ for any compact set $X\subset\mathbf{C}$.

Lemma 4.2. *Let $X\subset\Delta(0,R_0)\subset\mathbf{C}$ be a nonpolar compact set. Then*

$$V^*_X(\zeta)=\rho_X+\log|\zeta|+O\left(\frac{1}{|\zeta|}\right), \tag{4.1}$$

where $O(1/|\zeta|)$ depends only on R_0.

Proof. First, for $|\zeta|>R_0$, we have

$$\begin{aligned} V^*_X(\zeta) &= \int_X \log|\zeta-t|\,d\mu_X(t)+\rho_X \\ &\leq \int_X \log(1+|t|/|\zeta|)\,d\mu_X(t)+\log|\zeta|+\rho_X \leq \log(1+R_0/|\zeta|)+\log|\zeta|+\rho_X \end{aligned}$$

Similarly, for $|\zeta|>R_0$, we can also get the estimate

$$V^*_X(\zeta)\geq \log(1-R_0/|\zeta|)+\log|\zeta|+\rho_X.$$

Since $\log(1\pm\delta)=O(\delta)$ for $\delta>0$ sufficiently small, (4.1) follows. □

The following two lemmas exploit the fact that the Green function for a compact set X in classical potential theory is harmonic away from the set.

Lemma 4.3. *Let $K\subset\mathbf{C}^N$ be a compact set with $C(K)>0$, and let R_0 be such that $l(K)\subset\Delta(0,R_0)$ for all l. Suppose that there exists $R>R_0$ such that $(V_K^{(1)})^*(0)<\inf_l V^*_{l(K)}(\eta)$ for all $|\eta|=R$. Then there exists $b_0>0$ such that for all b with $|b|\leq b_0$,*

$$\sup_l V^*_{l(K)}(b\eta)\leq \inf_l V^*_{l(K)}(\eta) \quad \text{for all } |\eta|=R.$$

Proof. For convenience, we set $F(\eta):=\inf_l V^*_{l(K)}(\eta)$. Suppose the conclusion of the lemma is false. Then we can find sequences $\{\tilde{l}_j\}_{j=1}^\infty$, $\{l_j\}_{j=1}^\infty$, $\{\eta_j\}_{j=1}^\infty$ and $\{b_j\}_{j=1}^\infty$ such that $\eta_j\to\tilde{\eta}$ (with $|\eta_j|=|\tilde{\eta}|=R$) and $b_j\to 0$, and such that

$$V^*_{l_j(K)}(\eta_j) < V^*_{\tilde{l}_j(K)}(b_j\eta_j) \quad \text{for all } j.$$

Pick $\varepsilon\in(0,R-R_0)$. For all j, the functions $u_j:=V^*_{l_j(K)}$ are harmonic on

$$U_\varepsilon := \{\eta : |\eta| \in (R-\varepsilon, R+\varepsilon)\}$$

and uniformly bounded there, using (4.1) (the term ρ_X in (4.1) is uniformly bounded above by $-\log C(K)$). Hence we can find a subsequence j' of j and a harmonic function u on U_ε such that $u_{j'}\to u$ locally uniformly on U_ε. In particular,

$$\|u_{j'}-u\|_{\{\eta:|\eta|=R\}} \longrightarrow 0 \quad \text{as } j'\longrightarrow\infty,$$

so that

$$\lim_{j'\to\infty} u_{j'}(\eta_{j'}) = \lim_{j'\to\infty} u(\eta_{j'}) + \lim_{j'\to\infty} (u_{j'}(\eta_{j'}) - u(\eta_{j'})) = u(\tilde{\eta}).$$

Since $u_j(\eta)\geq F(\eta)$ for all j and $|\eta|=R$, we have $u\geq F$ on $\{\eta:|\eta|=R\}$. Note that $\tilde{l}_j=a_{j_1}z_1+...+a_{j_N}z_N$ for some coefficients $a_{j_1},...,a_{j_N}$. Setting $a_j:=(a_{j_1},...,a_{j_N})$, we have $|a_j|=1$, and

$$\begin{aligned} F(\tilde{\eta}) \leq u(\tilde{\eta}) &= \lim_{j'\to\infty} u_{j'}(\eta_{j'}) \leq \limsup_{j\to\infty} V^*_{\tilde{l}_j(K)}(b_j\eta_j) \\ &= \limsup_{j\to\infty} V^*_{\tilde{l}_j(K)}(\tilde{l}_j(b_j\eta_j\bar{a}_j)) \leq \limsup_{j\to\infty} V^{(1)}_K(b_j\eta_j\bar{a}_j) \leq (V^{(1)}_K)^*(0), \end{aligned}$$

(where $\bar{a}_j=(\bar{a}_{j_1},...,\bar{a}_{j_N})$ for each j), i.e., $\inf_l V^*_{l(K)}(\tilde{\eta})\leq (V^{(1)}_K)^*(0)$, contradicting the hypothesis. $\square$

Lemma 4.4. *Let R_0 be such that $l(K)\subset\Delta(0,R_0)$ for all l. Suppose that for some $b\in\mathbf{C}\setminus\{0\}$ and $R>R_0$ the inequality*

$$\sup_l V^*_{l(K)}(b\eta) \leq \inf_l V^*_{l(K)}(\eta) \tag{4.2}$$

holds for all $|\eta|=R$ and is finite on both sides. Then (4.2) *holds for all $|\eta|\geq R$.*

Proof. Fix l_1 and l_2. Since $V^*_{l_1(K)}$ is harmonic on $\mathbf{C}\setminus\Delta(0,R)$, the function

$$u(\eta) := V^*_{l_2(K)}(b\eta) - V^*_{l_1(K)}(\eta)$$

is subharmonic on $\mathbf{C}\setminus\Delta(0,R)$, $u\leq 0$ on $\{\eta:|\eta|=R\}$, and using (4.1),

$$|u(\eta)| \leq |\rho_{l_2(K)}| + |\rho_{l_1(K)}| + \left(1+\frac{1}{|b|}\right) O\left(\frac{1}{|\eta|}\right),$$

as $|\eta|\to\infty$, which is bounded. Hence by the extended maximum principle for subharmonic functions (cf., [R]), $u\leq 0$ on $\{\eta:|\eta|\geq R\}$. Since l_1 and l_2 were arbitrary, this proves that (4.2) holds for all $|\eta|\geq R$. $\square$

Proposition 4.5. *If K is a compact set with $C(K)>0$, then K is C-regular.*

Proof. First, $K\subset B(0,R_0)$ for some $R_0>0$, and so for all l, $l(K)\subset\Delta(0,R_0)$. Now $(V_K^{(1)})^*\in L$ by Theorem 3.5, and so $(V_K^{(1)})^*(0)<+\infty$. Using (4.1), we can choose $R>R_0$ sufficiently large so that

$$(V_K^{(1)})^*(0)<\inf_l V^*_{l(K)}(\zeta) \quad \text{for } |\zeta|\geq R.$$

By Lemma 4.3, there exists $b_0>0$ such that for all b with $|b|<b_0$ and $|\eta|=R$,

$$\sup_l V^*_{l(K)}(b\eta)\leq \inf_l V^*_{l(K)}(\eta). \tag{4.3}$$

By Lemma 4.4, (4.3) holds for all $|\eta|\geq R$. This proves that K is C-regular. □

Definition 4.6. The Robin function ρ_u associated to a function $u\in L$ is defined for $z\in\mathbf{C}^N$ by

$$\rho_u(z):=\limsup_{|\lambda|\to\infty}(u(\lambda z)-\log|\lambda|).$$

Directly from the definition, we see that ρ_u is *logarithmically homogeneous*, i.e., $\rho_u(\lambda z)=\rho_u(z)+\log|\lambda|$ for any $\lambda\in\mathbf{C}\setminus\{0\}$. Also, we will use the fact that ρ_u is known to be a plurisubharmonic function ([Bl]). Note that in $\mathbf{C}$, the well-known *Robin constant* is the value of the Robin function on the unit circle.

The Robin function is an important tool in the study of functions in the class L. We will use the notion of C-regularity to prove the following result about the Robin function of $(V_K^{(1)})^*$.

Theorem 4.7. *Let $K\subset\mathbf{C}^N$ be compact. Then for all $z\in\mathbf{C}^N\setminus\{0\}$,*

$$\rho_{(V_K^{(1)})^*}(z)=\Big[\sup_l \rho_{V^*_{l(K)}}(l(z))\Big]^*. \tag{4.4}$$

To prove the above theorem we need the following result about C-regular sets.

Lemma 4.8. *Suppose that $K\subset\mathbf{C}^N$ is compact and C-regular. Then there exists $a\in\left(0,\frac{1}{2}\right)$ and $R'\geq 1$ such that for all $|\lambda|>R'$ and $|z|=1$, if l_1 and l_2 satisfy $|l_1(z)|\leq a$ and $|l_2(z)|\geq 1-a$, then*

$$V^*_{l_1(K)}(\lambda l_1(z))\leq V^*_{l_2(K)}(\lambda l_2(z)).$$

Proof. Let $a:=b_0/(b_0+1)$ and $R':=R/(1-a)$, where b_0 and R are as in Definition 4.1. Fix λ and z with $|\lambda|\geq R'$ and $|z|=1$. Then

$$|\lambda l_2(z)|\geq R'(1-a)=R.$$

Also, $l_1(z)=l_2(bz)$ where $b:=l_1(z)/l_2(z)$. Note that

$$|b|=\left|\frac{l_1(z)}{l_2(z)}\right|\leq\frac{a}{1-a}=b_0,$$

so from the definition of C-regularity,

$$V^*_{l_1(K)}(\lambda l_1(z))=V^*_{l_1(K)}(b\lambda l_2(z))\leq V^*_{l_2(K)}(\lambda l_2(z)).\quad\square$$

Proof of Theorem 4.7. We consider two cases, namely, $C(K)>0$ and $C(K)=0$.
Case I. Suppose $C(K)>0$. By Proposition 4.5, K is C-regular. Fix $z\in\mathbf{C}^N\setminus\{0\}$. Then for any l, we have

$$\begin{aligned}\rho_{V^*_{l(K)}}(l(z))&=\limsup_{|\lambda|\to\infty}(V^*_{l(K)}(\lambda l(z))-\log|\lambda|)\\&\leq\limsup_{|\lambda|\to\infty}((V_K^{(1)})^*(\lambda z)-\log|\lambda|)=\rho_{(V_K^{(1)})^*}(z).\end{aligned}$$

Hence $\rho_{(V_K^{(1)})^*}(z)\geq\sup_l\rho_{V^*_{l(K)}}(l(z))$. Since $\rho_{(V_K^{(1)})^*}(z)$ is a plurisubharmonic function, it also follows that

$$\rho_{(V_K^{(1)})^*}(z)\geq\Big[\sup_l\rho_{V^*_{l(K)}}(l(z))\Big]^*.\tag{4.5}$$

For the reverse inequality, let a and R' be as in Lemma 4.8. Fix a linear map l_1. If $|l_1(z)|<a|z|$, let l_2 be any map satisfying $|l_2(z)|>(1-a)|z|$. If $|l_1(z)|\geq a|z|$ then take $l_2=l_1$. For any $|\lambda|>R'/|z|$, we have, using Lemma 4.8,

$$V^*_{l_1(K)}(\lambda l_1(z))-\log|\lambda|\leq V^*_{l_2(K)}(\lambda l_2(z))-\log|\lambda|,$$

with $|l_2(z)|\geq a|z|$. Thus

$$\begin{aligned}V^*_{l_1(K)}(\lambda l_1(z))-\log|\lambda|&\leq V^*_{l_2(K)}(\lambda l_2(z))-\log|\lambda|\\&=V^*_{l_2(K)}(\lambda l_2(z))-\log|\lambda l_2(z)|+\log|l_2(z)|\\&=\rho_{l_2(K)}+\log|l_2(z)|+O\bigg(\frac{1}{|\lambda l_2(z)|}\bigg)\\&=\rho_{V^*_{l_2(K)}}(l_2(z))+O\bigg(\frac{1}{|\lambda l_2(z)|}\bigg)\\&\leq\sup_l\rho_{V^*_{l(K)}}(l(z))+O\bigg(\frac{1}{a|\lambda||z|}\bigg),\end{aligned}$$

where the equality in the third line above follows from (4.1). Note that since l_1 was arbitrary and the last expression above is independent of l_1, we have for $|\lambda|>R'/|z|$,

$$V_K^{(1)}(\lambda z)-\log|\lambda|\le\sup_l \rho_{V^*_{l(K)}}(l(z))+O\Big(\frac{1}{a|\lambda||z|}\Big). \tag{4.6}$$

Now (4.6) holds for all $z\neq 0$, so taking the upper semicontinuous regularization of both sides, we get

$$(V_K^{(1)})^*(\lambda z)-\log|\lambda|\le\Big[\sup_l \rho_{V^*_{l(K)}}(l(z))\Big]^*+O\Big(\frac{1}{a|\lambda||z|}\Big).$$

Letting $|\lambda|\to\infty$ on both sides of the above equation yields

$$\rho_{(V_K^{(1)})^*}(z)\le\Big[\sup_l \rho_{V^*_{l(K)}}(l(z))\Big]^*,$$

which together with (4.5), yields (4.4).

Case II. Suppose $C(K)=0$. Then $(V_K^{(1)})^*\equiv+\infty$, and hence $\rho_{(V_K^{(1)})^*}=+\infty$. Hence we only need to show that the right-hand side of (4.4) is $+\infty$. Let

$$u(z):=\sup_l \rho_{V^*_{l(K)}}(l(z)).$$

Note that for each l, either $\rho_{V^*_{l(K)}}\circ l\in L$, or $\rho_{V^*_{l(K)}}\circ l\equiv+\infty$. We need to show that $u^*\equiv+\infty$. By Proposition 3.4, it is sufficient to show that $u^*\notin L$. To this end, take a sequence $\{a_j\}_{j=1}^\infty\subset\mathbf{C}^N$ with $|a_j|=1$ for all j, such that $\operatorname{cap}(l_j(K))\le 1/j$, where $l_j(z)=a_{j1}z_1+...+a_{jN}z_N$. We may take a convergent subsequence j' of j such that $a_{j'}\to a$ for some a with $|a|=1$; hence $l_{j'}(z)\to l(z)=a_1z_1+...+a_Nz_N$. Now $l(\bar a)=|a_1|^2+...+|a_N|^2=1$, and for all $z\in B\big(\bar a,\frac12\big)$, writing $\eta=z-\bar a$, we have

$$|l(z)|=|l(\bar a+\eta)|=|1+l(\eta)|\ge 1-\|l\||\eta|=1-|\eta|\ge\tfrac12.$$

Since $a_{j'}\to a$, we can find j_0 such that

$$|l_{j'}(z)|\ge\tfrac14\quad\text{for all } z\in B\big(\bar a,\tfrac12\big) \text{ and } j'>j_0,$$

and hence

$$\rho_{V^*_{l_{j'}(K)}}(l_{j'}(z))=\rho_{l_{j'}(K)}+\log|l_{j'}(z)|\ge\log j'-\log 4.$$

Thus for all $z\in B\big(\bar a,\frac12\big)$,

$$u(z)\ge\rho_{V^*_{l_{j'}(K)}}(l_{j'}(z))\ge\log j'-\log 4$$

for all $j'>j_0$; clearly $u^*\notin L$, and so by Proposition 3.4, $u^*\equiv+\infty$. □

For regular sets, (4.4) simplifies as follows.

Corollary 4.9. *Let $K \subset \mathbf{C}^N$ be a regular compact set. Then*

$$\rho_{V_K^{(1)}}(z) = \sup_l \rho_{V_{l(K)}}(l(z)). \tag{4.7}$$

The proof of Corollary 4.9 is identical to the proof of Theorem 4.7, but with no need to take any upper semicontinuous regularizations. Since K is regular, we can use the fact that $V_K^{(1)}$ is continuous ([BLM]), and for each l, $V_{l(K)}$ is continuous (Lemma 1.4).

Remark 4.10. In [BCL], (4.7) is proved for regular compact sets in essentially the same way. In that paper C-regularity is an additional assumption, but here it is seen to be superfluous. Since K is regular, and hence nonpluripolar, $C(K)>0$ (Remark 3.6) and therefore K is C-regular (Proposition 4.5).

We close this paper by using (4.7) to construct an example of a sequence of functions $\{u_j\}_{j=1}^{\infty} \subset L$ that increase to $(V_K^{(1)})^*$ for a regular compact set K, with the property that $\rho_{u_j} \nearrow \rho_{V_K^{(1)}}$.

Example 4.11. Let $K \subset \mathbf{C}^N$ be compact and regular, and let $\{l_k\}_{k=1}^{\infty}$ be an enumeration of the normalized linear maps with rational coefficients (i.e., if $l_k(z) = a_1 z_1 + \ldots + a_N z_N$ then for each $i=1,\ldots,N$, $\Re(a_i)$ and $\Im(a_i)$ are rational). For each $k=1,2,\ldots$, define the increasing sequence of functions

$$u_k(z) := \max_{j \le k} V_{l_j(K)}(l_j(z)).$$

Clearly $u_k \le V_K^{(1)}$, and hence $\rho_{u_k} \le \rho_{V_K^{(1)}}$ for all k.

On the other hand, fixing $z \in \mathbf{C}^N \setminus \{0\}$ and $\varepsilon > 0$, we take $\tilde{l}$ such that $\rho_{V_K^{(1)}}(z) \le \rho_{V_{\tilde{l}(K)}}(\tilde{l}(z)) + \varepsilon/3$. Using the fact that $\{l_k\}_{k=1}^{\infty}$ is a dense subset we can find an l_j in the sequence so that

$$\big|\log|l_j(z)| - \log|\tilde{l}(z)|\big| < \varepsilon/3 \quad \text{and} \quad \|V_{l_j(K)} - V_{\tilde{l}(K)}\|_{\mathbf{C}} < \varepsilon/3,$$

where the second inequality follows from Proposition 2.14. Note that the second inequality also implies that $|\rho_{\tilde{l}(K)} - \rho_{l_j(K)}| < \varepsilon/3$. Using this, we have

$$\rho_{V_K^{(1)}}(z) \le \rho_{\tilde{l}(K)} + \log|\tilde{l}(z)| + \varepsilon/3 \le \rho_{l_j(K)} + \log|l_j(z)| + \varepsilon = \rho_{V_{l_j(K)}}(l_j(z)) + \varepsilon,$$

and hence $\rho_{V_K^{(1)}}(z) \le \rho_{u_k}(z) + \varepsilon$ for all $k \ge j$. Since ε was arbitrary, this shows that $\rho_{u_k} \nearrow \rho_{V_K^{(1)}}$ as $k \to \infty$.

Note that in general, the pointwise convergence of an increasing sequence of functions in L does not imply convergence of the corresponding Robin functions to the Robin function of the limit. The following counterexample is based on ideas of Siciak.

First we construct a compact set $K\subset\mathbf{C}$ such that 0 is an irregular point of this set. Let $\varepsilon>0$, and set

$$K:=\{0\}\cup\biggl(\bigcup_{k=1}^{\infty}\biggl[\frac{1}{2^k},\frac{1}{2^k}+\frac{1}{2^{k^{2+\varepsilon}}}\biggr]\biggr).$$

Every nonzero point of K is regular, and 0 is an irregular point by Wiener's criterion in $\mathbf{C}$ (see e.g., [R]).

For each $j=1,2,...$, consider the set $K_j:=\{z\in\mathbf{C}:\operatorname{dist}(z,K)\le\frac{1}{j}\}$. The sets K_j are compact and regular, so the functions

$$u_j(z)=\begin{cases} V_{K_j}\Bigl(\dfrac{1}{z}\Bigr)+\log|z|, & \text{if } z\neq 0,\\ \rho_{K_j}, & \text{if } z=0,\end{cases}$$

are continuous in $\mathbf{C}$, and for each $z\in\mathbf{C}$ we have $u_j(z)\nearrow u(z)$, where u is given by

$$u(z):=\begin{cases} V_K^*\Bigl(\dfrac{1}{z}\Bigr)+\log|z|, & \text{if } z\neq 0,\\ \rho_K, & \text{if } z=0,\end{cases}$$

and is clearly continuous in $\mathbf{C}$. We have $\lim_{j\to\infty}\rho_{u_j}(1)=\lim_{j\to\infty}V_{K_j}(0)=0$, but on the other hand $\rho_u(1)=V_K^*(0)>0$ by the irregularity of K at 0.

Remark 4.12. For a decreasing sequence of functions in L, pointwise convergence to a limit does imply pointwise convergence of the corresponding Robin functions to the Robin function of the limit.

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