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On subextension of pluriharmonic and plurisubharmonic functions

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Abstract. The problem of *subextension* of plurisubharmonic functions is considered. Recently it was shown by Cegrell–Zeriahi, that subextension is always possible for negative plurisubharmonic functions in the energy class $\mathcal{F}$. In this paper we construct, for every hyperconvex domain Ω, a negative plurisubharmonic function in the class $\mathcal{E}$ which cannot be subextended. Given any pseudoconvex domain we construct a pluriharmonic function that cannot be subextended.

1. Introduction

We use the notation $\mathcal{PSH}(\Omega)$ and $\mathcal{PH}(\Omega)$ for plurisubharmonic and pluriharmonic functions, respectively, on a domain Ω, and the notation $\mathcal{PSH}^-(\Omega)=\{u\in\mathcal{PSH}(\Omega); u\le 0\}$.

Any smooth bounded domain satisfying a non-degeneracy condition on the Levi-form on the boundary is a domain of existence for plurisubharmonic functions ([C1], [BB]). However, one can certainly argue that inequalities are more natural than equalities for plurisubharmonic functions, and thus the following problem was studied by El Mir [E]: Take domains Ω and Ω' in $\mathbf{C}^n$, with $\Omega\subsetneq\Omega'$ and $\Omega\cap\Omega'\neq\varnothing$. Given a plurisubharmonic function u in Ω, when is it possible to find a $u'\in\mathcal{PSH}(\Omega')$, with $u\not\equiv-\infty$, such that $u'|_\Omega\le u$? The pair (Ω', u') is called a *subextension* of u.

E. Bedford and B. A. Taylor constructed, for any $\mathcal{C}^2$-smooth domain in $\mathbf{C}^n$, $n\ge 2$, a smooth negative plurisubharmonic function that does not subextend [BT]. This improves an example by Fornæss and Sibony [FS].

Recently U. Cegrell and A. Zeriahi [CZ] showed that under the assumption that the total Monge–Ampère mass is bounded and the function has in a sense boundary values zero, i.e. is in the energy class $\mathcal{F}$, one can always find a subextension, without increasing the total mass.

Let us recall a definition from [C3]. Let Ω be a *hyperconvex* domain, i.e. a domain that admits a negative continuous plurisubharmonic exhaustion function, and let $\mathcal{E}_0(\Omega)$ be the set of negative and bounded plurisubharmonic functions on Ω which tends to zero at the boundary and have bounded total Monge–Ampère mass. Let $\mathcal{F}(\Omega)$ be the set of plurisubharmonic functions u such that there is a sequence $(u_j)_{j=1}^{\infty}$ of functions from $\mathcal{E}_0(\Omega)$ such that $u_j \searrow u$ and $\sup_j \int_\Omega (dd^c u_j)^n < \infty$. The set of functions locally in $\mathcal{F}(\Omega)$ is denoted by $\mathcal{E}(\Omega)$.

In Section 2 we show why we cannot expect to have subextension of functions in $\mathcal{E}$ in general, even when we have control over the Monge–Ampère mass. Furthermore we show that subextensions of pluriharmonic functions are not possible on pseudoconvex domains. In Section 3 we construct, for any given hyperconvex domain Ω, a function $u \in \mathcal{E}(\Omega)$ that cannot be subextended. Finally we also prove a positive result.

2. Subextension of pluriharmonic functions

Most examples of functions that are impossible to subextend across a boundary point work because they have to large a singularity at a boundary point. In Theorem 2.6 below the singularity is so large that subextension is impossible due to integrability reasons and in Example 2.4 and in Theorem 3.2 below the functions have to large Lelong number at the boundary. In the counterexamples to subextension mentioned in the introduction ([FS], [BT]) "propagation of Lelong numbers" were used. We will use a simpler idea in Example 2.4 and Theorem 3.2.

Remember that the Lelong number of $u \in \mathcal{PSH}(\Omega)$ at the point $P \in \Omega$ can be defined as

$$\nu(u,P) = \lim_{r\to 0} \frac{\sup\{u(z)\,;\, z\in B(P;r)\}}{\log r}. \tag{1}$$

It is well known that this quantity is finite for any plurisubharmonic function $u \not\equiv -\infty$.

Given a function $u\colon \mathbf{C}^n \supset \Omega \to \mathbf{R}\cup\{-\infty\}$ we define a *slice* of u through P and $Q \in \mathbf{C}^n$, $u_{P,Q}$, as: $u_{P,Q}(\zeta) := u(P+\zeta Q), \zeta \in \mathbf{C}$, wherever this expression make sense. Then we have $\nu(u_{P,Q},0) \geq \nu(u,P)$. Given domains Ω and Ω' in $\mathbf{C}^n$, with $\Omega \subsetneq \Omega'$, and $\Omega \cap \Omega' \neq -\infty$, and a plurisubharmonic function u on Ω. Suppose u has a subextension v to Ω', and take $P \in \overline{\Omega} \cap \Omega'$. From equation (1) it is clear that whenever $Q \in \Omega$ we have $\nu(v_{P,Q},0) \geq \nu(u_{P,Q},0)$.

On the other hand we have the well known result.

Lemma 2.1. *Assume that* $u\in\mathcal{PSH}(B(P;R))$, *where* $B(P;R)$ *is the ball with center at* P *and radius* R, *and* $B=B(0;1)$, *as usual. Fix a point* $Q\in B$, *then* $\nu(u_{P,Q},0)=\nu(u,P)$ *for all* $Q\in B\setminus A$, *where* A *is a pluripolar set.*

On the exceptional set A the Lelong number of $u_{P,Q}$ can behave arbitrarily bad, see [CP].

Lemma 2.2. *Let* Ω *be a domain in* $\mathbf{C}^n$, *and* $P\in\partial\Omega$. *Take* $u\in\mathcal{PSH}(\Omega)$, *and choose* R *such that* $\partial B(P;R)\cap\Omega\neq\varnothing$. *If* $\nu(u_{P,Q},0)=+\infty$ *for a non-polar set* $A\subset\partial B(P;R)$, *then we cannot subextend* u *to any set* $\Omega'\ni P$.

Proof. If $v\le u$ on Ω, and v is plurisubharmonic on Ω', where $P\in\Omega'$, then

$$\nu(v_{P,Q},0)\ge\nu(u_{P,Q},0).$$

Lemma 2.1 gives us that $\nu(v,P)=+\infty$, which is impossible. □

To see that subextension is not in general possible in $\mathcal{E}$, consider the following example.

Example 2.3. Let B be the unit ball in $\mathbf{C}^n$. The function

$$u(z_1,...,z_n)=\frac{|z_1|^2-1}{|z_1-1|^2}$$

is pluriharmonic and there is no subextension across the point $(1,\mathbf{0})$. Furthermore $u\in\mathcal{E}(B)$ and $(dd^cu)^n=0$ on B.

Proof. $u(z)=\mathrm{Re}((z_1+1)/(z_1-1))$, so u is pluriharmonic. Assume that u has a subextension v. Then we can estimate the Lelong number of v along slices of the type $l=(1-\zeta,\zeta(p_2,...,p_n))$ with the Lelong number of u along the same slices. Lemma 2.2 shows that subextension is not possible.

Since u is the real part of a holomorphic function and $u\le0$ on B, it is clear that $u\in\mathcal{E}(B)$ and $(dd^cu)^n=0$. □

The one-dimensional case of the example above was suggested to me by F. Wikström ([W]), for which I am very grateful.

From Example 2.3 it is clear that having control of the Monge–Ampère mass, in the sense that there is a constant C such that $\int_K(dd^cu)^n\le C$ for any $K\Subset\Omega$, does not suffice for the existence of a subextension. Comparing this with, e.g., $u\equiv-1\in\mathcal{E}\setminus\mathcal{F}$ we see that for some functions in $\mathcal{E}\setminus\mathcal{F}$ subextension is possible. In a way subextension in $\mathcal{F}$ is possible due to the conditions on the boundary values.

Given an open set Ω, and a dense subset Λ of the boundary of Ω. As soon as we can construct a function $u_x\in\mathcal{PSH}^-(\Omega)$, such that u_x cannot be subextended over x, for any point $x\in\Lambda$, we can construct a function that is not possible to subextend over any point in the boundary. Take for example the function

$$\Psi = \sum_{\lambda\in\Lambda} \varepsilon_\lambda u_\lambda, \tag{2}$$

where ε_λ is chosen such that the sum converges, is not possible to subextend across any point. This method is efficient if one wants to construct a continuous function which is not possible to subextend. Of course as soon as we have a continuous example Richberg's approximation theorem gives a smooth example.

Example 2.4. There is a $u\in\mathcal{E}(B)$ such that u does not have a subextension to any larger domain $U\supsetneq B$ with $\partial B\cap U\neq\varnothing$.

Proof. Take any point x in $\partial\Omega$. After rotation we might as well assume that $x=\mathbf{e_1}=(1,0,...,0)$. Then the function $g(z)$ in Example 2.3 is in $\mathcal{PSH}^-(B)$. Form the sum as in equation (2) in the remark above. □

To see why subextension of pluriharmonic functions is impossible we begin with a lemma about plurisubharmonic functions with an integrability condition on the boundary.

Lemma 2.5. *Suppose Ω is a pseudoconvex domain in $\mathbf{C}^n$, let $\mathcal{PH}(\Omega)$ be the Fréchet space of pluriharmonic functions with topology generated by the supremum norm on compact set. Take $x\in\partial\Omega$ and define, for $k\in\mathbf{N}$, $r>0$:*

$$L(k,x,r) := \left\{ u\in\mathcal{PH}(\Omega)\,;\int_{\Omega\cap B(x;r)} \max(u,0)\,dV \leq k \right\}. \tag{3}$$

Then $L(k,x,r)$ has no inner points and is closed in $\mathcal{PH}(\Omega)$.

Proof. First we show that $L(k,x,r)$ is closed.

Take u in the closure of $L(k,x,r)$, then there is a sequence $\{u_j\}_{j=1}^{\infty}\subset L(k,x,r)$ such that $u_j\to u$ in $\mathcal{PH}$.

We have that $u_j\to u$ uniformly on compact subsets of Ω, and thus

$$\lim_{j\to\infty} \max(u_j,0) = \max(u,0)$$

on Ω. Fatou's lemma implies that

$$\begin{aligned}\int_{\Omega\cap B(x;r)} \max(u,0)\,dV &= \int_{\Omega\cap B(x;r)} \lim_{j\to\infty} \max(u_j,0)\,dV \\ &\leq \lim_{j\to\infty} \int_{\Omega\cap B(x;r)} \max(u_j,0)\,dV \leq k,\end{aligned}$$

and we have $u\in L(k,x,r)$ as required.

We show that $L(k,x,r)$ has no inner points by reductio ad absurdum. Thus assume that $L(k,x,r)$ has an inner point u_0. In that case there is a $K_0 \Subset \Omega$ such that $u \in \mathcal{PH}$ and

$$\int_{K_0} |u-u_0|\,dV<\varepsilon \quad \Longrightarrow \quad u\in L(k,x,r).$$

Take $z_0 \in \Omega \cap B(x,r) \setminus K_0$. Since Ω is pseudoconvex, and thus a domain of holomorphy, there is f, holomorphic on Ω, such that $\sup_{K_0}|f|<\varepsilon$ with $|f(z_0)|$ as large as we like. By multiplying with a suitable complex number we get

$$|f(z_0)|=\operatorname{Re} f(z_0)=f(z_0)\geq \frac{k\cdot n!}{\pi^n \operatorname{dist}(z_0,\complement\Omega)^n}-u_0. \tag{4}$$

Since $\operatorname{Re} f+u_0 \in \mathcal{PH}(\Omega)$ and

$$\int_{K_0} |\operatorname{Re} f+u_0-u_0|\,dV \leq \int_{K_0}|f|<\varepsilon$$

we get $\operatorname{Re} f \in L(k,x,r)$. In particular, for $\rho=\operatorname{dist}(z,\complement\Omega)<r$,

$$\begin{aligned}
(f+u_0)(z_0) &= \operatorname{Re} f(z_0)+u_0(z_0)\\
&= \frac{1}{\operatorname{vol}(B(z_0;\rho))}\int_{B(z_0;\rho)} (\operatorname{Re} f+u_0)\,dV\\
&\leq \frac{1}{\operatorname{vol}(B(z_0;\rho))}\int_{\Omega\cap B(z_0;\rho)} \max(\operatorname{Re} f+u_0,0)\,dV\\
&\leq \frac{1}{\operatorname{vol}(B(z_0;\rho))}\int_{\Omega\cap B(z_0;r)} \max(\operatorname{Re} f+u_0,0)\,dV\\
&\leq \frac{k\cdot n!}{\pi^n\rho^n}=\frac{k\cdot n!}{\pi^n \operatorname{dist}(z,\complement\Omega)^n}<f(z_0)+u_0(z_0),
\end{aligned}$$

where the very last inequality comes from the choice of f in equation (4). Thus we have reached a contradiction and therefore $L(k,x,r)$ has no inner points. □

Let us recall that a set E in a topological space S is called *meager* if $E=\bigcup_{n=1}^{\infty} E_n$, where $\overline{E}_n$ has no inner points.

Theorem 2.6. *Let Ω be a pseudoconvex domain in $\mathbf{C}^n$. Then there is a pluriharmonic function u on Ω such that u cannot be subextended to any domain $\Omega' \supsetneq \Omega$.*

Proof. Take a dense sequence $\{x_j\}_{j=1}^{\infty}$ of points in $\partial\Omega$, and let r_s be the rational numbers $0<r_s<1$.

Then every $L(k,x_j,r_s) \subset \mathcal{PH}(\Omega)$ is closed and has no inner points. Therefore

$$\bigcup_{k,j,s} L(k,x_j,r_s)=L$$

is meager in $\mathcal{PH}(\Omega)$ according to Baire's theorem. Take $u\in\mathcal{PH}\setminus L$. Suppose $-u$ has a subextension to the domain $\widetilde{\Omega}=\Omega\cap B(x;r)$. Then there is $v\in\mathcal{PSH}(\widetilde{\Omega})$ such that $v\leq -u$ on Ω. Since

$$|v|\geq \max(-v,0)\geq \max(u,0)$$

on Ω the definition of L gives that $v\notin L^1_{\mathrm{loc}}(\Omega)$. Thus $-u$ cannot be subextended across any point on the boundary. □

3. Subextension of plurisubharmonic functions

Proposition 3.1. *Let Ω be a pseudoconvex domain, then there is a plurisubharmonic function that cannot be subextended across any point of the boundary.*

Proof. As in the standard proof that holomorphically convex domains are domains of holomorphy (see e.g. [G]) construct a holomorphic function $f(z)$ that has a zero of total order n at the point p_n, where the sequence of points $\{p_n\}_{n=1}^{\infty}$ have a cluster point at the boundary. Clearly $u(z)=\log|f(z)|$ is plurisubharmonic and have no subextension, since the Lelong number is too large near every boundary point. □

Unfortunately this construction does not give a function $u\in\mathcal{PSH}^-(\Omega)$.

Let Ω be a hyperconvex set in $\mathbf{C}^n$. The *pluricomplex Green function* for Ω with pole at $w\in\Omega$ was introduced by Zahariuta [Z] and Klimek [K]. It is defined as

$$g(z,w)=\sup\{u(z)\,;u\in\mathcal{PSH}^-(\Omega) \text{ and } \nu(u,w)\geq 1\}.$$

Theorem 3.2. *Let Ω be a hyperconvex domain, then there is a function $u\in\mathcal{E}(\Omega)$ such that u has no subextension.*

Proof. Take $x\in\partial\Omega$, and let $\{w_j\}_{j=1}^{\infty}$ be a sequence such that $w_j\in\Omega$ and $w_j\to x$, as $j\to\infty$. According to [CCW] we have

$$\varlimsup_{j\to\infty} g(z,w_j)=0$$

for all $z\in\Omega\setminus E$, where the exceptional set E is pluripolar. Take $p\notin E$, and choose a suitable subsequence of w_j such that $g(p,w_j)>-j^{-3}$. Call that subsequence $\{w_j\}_{j=1}^{\infty}$ as well. Define

$$u(z):=\sum_{j=1}^{\infty} jg(z,w_j). \tag{5}$$

Since $g(z,w_j)\in\mathcal{PSH}^-(\Omega)$ the partial sums $\sum_{j=1}^{N} jg(z,w_j)$ is a decreasing sequence of plurisubharmonic functions. And since

$$u(p)=\sum_{j=1}^{\infty} jg(p,w_j)>\sum_{j=1}^{\infty} -j^{-2}>-\infty,$$

$u\in\mathcal{PSH}^-(\Omega)$ and $u\not\equiv-\infty$.

Note that $\nu(u,w_j)\geq j$. Suppose v is a subextension to a domain $\Omega'\supsetneq\Omega$ such that $x\in\Omega'$. Since $v\leq u$ we have that $\nu(v,w_j)\geq\nu(u,w_j)\geq j$. Thus $\lim_{j\to\infty}\nu(v,w_j)\geq +\infty$. Since the Lelong number is finite on any plurisubharmonic function we have a contradiction, and thus u has no subextension across $x\in\partial\Omega$. Since $u\leq 0$ we can proceed as in the remark following Example 2.3.

To this point we only know that $u\leq 0$, so we need to check that $u\in\mathcal{E}$. According to a result by Cegrell (Proposition 3.1 in [C2]) it suffices to show that there is $h\in\mathcal{E}_0(\Omega), h\not\equiv 0$, and a sequence $u_j\in\mathcal{E}_0(\Omega)$, with $u_j\searrow u$ as $j\to\infty$, such that $\sup_j\int_\Omega -h(dd^cu_j)^n<+\infty$.

Since Ω is hyperconvex there is a negative exhaustion function $h\in\mathcal{PSH}^-(\Omega)\cap\mathcal{C}(\overline{\Omega})$. After thinning out w_j even further we can take w_j such that $h(w_j)>-j^{-2n-1}$. Let $u_p=\sum_{j=1}^{p} jg(\,\cdot\,,w_j)$. Since

$$\biggl(\int -h\,(dd^c(\varphi_1+\varphi_2))^n\biggr)^{1/n}\leq\biggl(\int -h\,(dd^c\varphi_1)^n\biggr)^{1/n}+\biggl(\int -h\,(dd^c\varphi_2)^n\biggr)^{1/n}$$

(see Lemma 2.5 in [CW]) we have

$$\begin{aligned}\biggl(\int -h\,(dd^cu_p)^n\biggr)^{1/n}&\leq\sum_{j=1}^{p}\biggl(\int -h\,(dd^cu_j)^n\biggr)^{1/n}\leq\sum_{j=1}^{p}\biggl(\int -(2\pi)^n jh\,\delta_{w_j}\biggr)^{1/n}\\&\leq\sum_{j=1}^{p}\biggl(\frac{(2\pi)^n}{j^{2n}}\biggr)^{1/n}\leq\sum_{j=1}^{p}\frac{2\pi}{j^2}<\frac{\pi^3}{3}.\end{aligned}$$

Thus $u\in\mathcal{E}$. $\square$

Thus for functions bounded above we do not in general have subextension by the theorem above. On the other hand, for functions bounded from below on pseudoconvex domains subextension is always possible.

Proposition 3.3. *Let Ω be a bounded pseudoconvex domain, and Ω' be a hyperconvex domain such that $\Omega\Subset\Omega'\Subset\mathbf{C}^n$. If $u\in\mathcal{PSH}(\Omega)$, and for some $a<0$*

$$\varliminf_{z\to\zeta\in\partial\Omega} u(z)\geq a,$$

for all points ζ on the boundary, then there exist $v\in\mathcal{PSH}(\Omega')$, such that $v\leq u$ on Ω.

Proof. Take $p\in\partial\Omega$. Since $\underline{\lim}_{z\to p} u(z)\geq a$, there is $\delta>0$ such that $u\geq a-1$ on $\Omega\cap B(p;\delta)$. Let $\Omega_r=\{z\in\Omega;\operatorname{dist}(z,\complement\Omega)>r\}$. By compactness there is $r>0$ such that $u\geq a-1$ on $\Omega\setminus\Omega_r$.

Since Ω is pseudoconvex it can be exhausted by smooth, strict pseudoconvex domains. Let $\Omega_1\Subset\Omega$ be a strict pseudoconvex domain with smooth boundary such that $\Omega_1\Supset\Omega_r$. Clearly $u\in L^\infty(\Omega_1\cap\Omega_r)$.

Observe that the extremal function

$$u_1:=\sup\{\phi\in\mathcal{PSH}(\Omega_1)\,;\phi\leq 0 \text{ and } \phi\leq u \text{ on } \Omega_r\}$$

is in $\mathcal{F}(\Omega_1)$.

It is known (Theorem 2.2, [CZ]) that there is a plurisubharmonic function v_1 on Ω' such that $v_1\leq u_1$ on Ω_1. Take $v=v_1+a-1$. □

References

[BB] Bedford, E. and Burns, D., Domains of existence for plurisubharmonic functions, *Math. Ann.* **238** (1978), 67–69.

[BT] Bedford, E. and Taylor, B. A., Smooth plurisubharmonic functions without subextension, *Math. Z.* **198** (1988), 331–337.

[CCW] Carlehed, M., Cegrell, U. and Wikström, F., Jensen measures, hyperconvexity and boundary behaviour of the pluricomplex Green function, *Ann. Polon. Math.* **71** (1999), 87–103.

[C1] Cegrell, U., On the domains of existence for plurisubharmonic functions, in *Complex analysis* (*Warsaw, 1979*), Banach Center Publ. **11**, pp. 33–37, PWN, Warsaw, 1983.

[C2] Cegrell, U., Two examples in pluripotential theory, *Technical Report* **14**, Department of Mathematics, Mid Sweden University, 2000.

[C3] Cegrell, U., The general definition of the complex Monge–Ampère operator, *Ann. Inst. Fourier* (Grenoble) **54** (2004), 159–179.

[CW] Cegrell, U. and Wiklund, J., Monge–Ampère norm for delta-plurisubharmonic functions, *Math. Scand.* **97** (2005), 201–216.

[CZ] Cegrell, U. and Zeriahi, A., Subextension of plurisubharmonic functions with bounded Monge–Ampère mass, *C. R. Acad. Sci. Paris Sér. I Math.* **336** (2003), 305–308.

[CP] Celik, H. I. and Poletsky, E. A., Fundamental solutions of the complex Monge–Ampère equation, *Ann. Polon. Math.* **67** (1997), 103–110.

[E] El Mir, H., Fonctions plurisousharmoniques et ensembles polaires, in *Séminaire Pierre Lelong–Henri Skoda* (*Analyse*). *Années 1978/79*, Lecture Notes in Math. **822**, pp. 61–76, Springer, Berlin, 1980.

[FS] Fornæss, J. E. and Sibony, N., Plurisubharmonic functions on ring domains, in *Complex analysis* (*University Park, PA, 1986*), pp. 111–120, Springer, Berlin, 1987.

[G] Gunning, R. C., *Introduction to Holomorphic Functions of Several Variables*, Vol. **I**, Wadsworth and Brooks/Cole, Pacific Grove, CA, 1990.

[K] Klimek, M., Extremal plurisubharmonic functions and invariant pseudodistances, *Bull. Soc. Math. France* **113** (1985), 231–240.

[W] Wikström, F., Personal communication, 2002.

[Z] Zahariuta, V. P., *Spaces of Analytic Functions and Maximal Plurisubharmonic Functions*, Doc. Sci. thesis, Rostov State University, Rostov-na-Donu, 1984 (Russian).

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