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Auslander–Reiten sequences on schemes

Peter Jørgensen

Abstract. Auslander–Reiten sequences are the central item of Auslander–Reiten theory, which is one of the most important techniques for the investigation of the structure of abelian categories.

This note considers X, a smooth projective scheme of dimension at least 1 over the field k, and $\mathcal{C}$, an indecomposable coherent sheaf on X. It is proved that in the category of quasi-coherent sheaves on X, there is an Auslander–Reiten sequence ending in $\mathcal{C}$.

Auslander–Reiten theory is one of the most important techniques for the investigation of the structure of abelian categories. It has been used extensively in the representation theory of finite-dimensional algebras, and makes it possible to understand many module categories both quantitatively and qualitatively. See [2] for a comprehensive introduction.

The purpose of this note is to show that Auslander–Reiten sequences, the central item of Auslander–Reiten theory, frequently exist in categories of quasi-coherent sheaves on schemes. An Auslander–Reiten sequence is a short exact sequence

$$0 \longrightarrow \mathcal{A} \xrightarrow{a} \mathcal{B} \xrightarrow{b} \mathcal{C} \longrightarrow 0 \tag{1}$$

so that a is left almost split and b is right almost split. For a to be left almost split means that it is not a split monomorphism, and that each morphism $\mathcal{A} \rightarrow \mathcal{M}$ which is not a split monomorphism factors through a. Dually, for b to be right almost split means that it is not a split epimorphism, and that each morphism $\mathcal{M} \rightarrow \mathcal{C}$ which is not a split epimorphism factors through b.

The precise result of this note is the following. Let X be a smooth projective scheme of dimension $d \geq 1$ over the field k, and let $\mathcal{C}$ be an indecomposable coherent sheaf on X. Then there exists an Auslander–Reiten sequence (1) in the category of quasi-coherent sheaves on X. Moreover, $\mathcal{A}$ is isomorphic to $(\Sigma^{d-1}\mathcal{C}) \otimes \omega$, where $\Sigma^{d-1}\mathcal{C}$ is the $(d-1)$st syzygy in a minimal injective resolution of $\mathcal{C}$ in the category of quasi-coherent sheaves, and ω is the dualizing sheaf of X.

The sheaves $\mathcal{A}$ and $\mathcal{B}$ are not in general coherent, but only quasi-coherent. This is analogous to ring theory: If C is a finitely presented non-projective R-module with local endomorphism ring, then by [1, Theorem 4] there is an Auslander–Reiten sequence in the category of all R-modules,

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0,$$

but A and B are not in general finitely presented.

However, observe that if X is a curve, then $d{=}1$, and then $\Sigma^{d-1}\mathcal{C}$ is just $\mathcal{C}$ which *is* coherent, so in this case, $\mathcal{A}$ and $\mathcal{B}$ are coherent. So if X is a curve, then we recover the result known from [9] that the category of coherent sheaves on X has Auslander–Reiten sequences.

Proposition 1. *Let* A *be a* k*-linear abelian category with enough injectives over the field* k*. Let* $\mathcal{C}$ *be an object with local endomorphism ring. Let* $\mathcal{A}$ *be another object for which there is a natural equivalence*

$$\mathrm{Hom}(\mathcal{C},-)' \simeq \mathrm{Ext}^d(-,\mathcal{A}) \tag{2}$$

for some $d{\geq}1$*, where the prime denotes dualization with respect to* k*.*

Then there is a short exact sequence

$$0 \longrightarrow \Sigma^{d-1}\mathcal{A} \longrightarrow \mathcal{B} \xrightarrow{b} \mathcal{C} \longrightarrow 0, \tag{3}$$

where b *is right almost split. Here* $\Sigma^{d-1}\mathcal{A}$ *is the* $(d{-}1)$*st syzygy in an injective resolution of* $\mathcal{A}$*.*

Proof. There is a natural equivalence

$$\mathrm{Ext}^d(-,\mathcal{A}) \simeq \mathrm{Ext}^1(-,\Sigma^{d-1}\mathcal{A}),$$

and combined with equation (2) this gives the Auslander–Reiten formula

$$\mathrm{Hom}(\mathcal{C},-)' \simeq \mathrm{Ext}^1(-,\Sigma^{d-1}\mathcal{A}). \tag{4}$$

It is standard that this implies the proposition. Namely, one takes a non-zero element ε in $\mathrm{Hom}(\mathcal{C},\mathcal{C})'$ which vanishes on the Jacobson radical J of the local ring $\mathrm{Hom}(\mathcal{C},\mathcal{C})$. Equation (4) gives that ε corresponds to an element e in $\mathrm{Ext}^1(\mathcal{C},\Sigma^{d-1}\mathcal{A})$, and the short exact sequence (3) can then be obtained as a representative of e. $\square$

Lemma 2. *Let* A *be an abelian category with a short exact sequence*

$$0 \longrightarrow \mathcal{A} \xrightarrow{a} \mathcal{I} \xrightarrow{i} \Sigma\mathcal{A} \longrightarrow 0,$$

where $\mathcal{A} \xrightarrow{a} \mathcal{I}$ *is an injective envelope and where* $\mathrm{Ext}^1(\mathcal{I},\mathcal{A}){=}0$*.*

If $\mathcal{A}$ *has local endomorphism ring, then so does* $\Sigma\mathcal{A}$*.*

Proof. Any morphism $\Sigma\mathcal{A}\xrightarrow{\sigma}\Sigma\mathcal{A}$ gives rise to a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{A} & \xrightarrow{a} & \mathcal{I} & \xrightarrow{i} & \Sigma\mathcal{A} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\iota} & & \downarrow{\scriptstyle\sigma} & & \\ 0 & \longrightarrow & \mathcal{A} & \xrightarrow[a]{} & \mathcal{I} & \xrightarrow[i]{} & \Sigma\mathcal{A} & \longrightarrow & 0, \end{array}$$

for when σ is given, consider $\mathcal{I}\xrightarrow{\sigma i}\Sigma\mathcal{A}$ which represents an element in $\mathrm{Ext}^1(\mathcal{I},\mathcal{A})$. Since this Ext is 0, the morphism $\mathcal{I}\xrightarrow{\sigma i}\Sigma\mathcal{A}$ must factor through $\mathcal{I}\xrightarrow{i}\Sigma\mathcal{A}$. This gives the morphism $\mathcal{I}\xrightarrow{\iota}\mathcal{I}$, and the morphism $\mathcal{A}\xrightarrow{\alpha}\mathcal{A}$ and hence the diagram follows.

Observe that in the diagram, if α is an isomorphism, then so is ι because $\mathcal{A}\xrightarrow{i}\mathcal{I}$ is an injective envelope, and hence, so is σ.

To show that $\mathrm{End}(\Sigma\mathcal{A})$ is local, we must show that if σ is a non-invertible element, then $\mathrm{id}_{\Sigma\mathcal{A}}-\sigma$ is invertible. That σ is non-invertible means that it is not an isomorphism. Embed σ in a diagram as above. Then α is not an isomorphism, for otherwise σ would be an isomorphism, as observed above. So α is a non-invertible element of $\mathrm{End}(\mathcal{A})$. But then $\mathrm{id}_{\mathcal{A}}-\alpha$ is invertible because $\mathrm{End}(\mathcal{A})$ is local. That is, $\mathrm{id}_{\mathcal{A}}-\alpha$ is an isomorphism. But there is also a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{A} & \xrightarrow{a} & \mathcal{I} & \xrightarrow{i} & \Sigma\mathcal{A} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\mathrm{id}_{\mathcal{A}}-\alpha} & & \downarrow{\scriptstyle\mathrm{id}_{\mathcal{I}}-\iota} & & \downarrow{\scriptstyle\mathrm{id}_{\Sigma\mathcal{A}}-\sigma} & & \\ 0 & \longrightarrow & \mathcal{A} & \xrightarrow[a]{} & \mathcal{I} & \xrightarrow[i]{} & \Sigma\mathcal{A} & \longrightarrow & 0, \end{array}$$

and as observed above, when $\mathrm{id}_{\mathcal{A}}-\alpha$ is an isomorphism, so is $\mathrm{id}_{\Sigma\mathcal{A}}-\sigma$. That is, $\mathrm{id}_{\Sigma\mathcal{A}}-\sigma$ is an invertible element of $\mathrm{End}(\Sigma\mathcal{A})$. $\square$

The following lemma uses the dualizing sheaf of a projective scheme, as described for instance in [4, Section III.7]. The lemma can be found between the lines in several papers, but seems not to be stated explicitly anywhere.

Lemma 3. *Let X be a projective scheme of dimension d with Gorenstein singularities over the field k. Let ω be the dualizing sheaf.*

Let $\mathcal{C}$ be a coherent sheaf which has a bounded resolution of locally free coherent sheaves. Then there are natural equivalences

$$\mathrm{Ext}^i_{\mathsf{QCoh}\,X}(\mathcal{C},-)'\simeq\mathrm{Ext}^{d-i}_{\mathsf{QCoh}\,X}(-,\mathcal{C}\otimes\omega),$$

where $\mathsf{QCoh}(X)$ is the category of quasi-coherent sheaves on X.

Proof. As X is projective over k, there is a projective morphism $X \xrightarrow{f} \operatorname{Spec}(k)$. This is certainly a separated morphism of quasi-compact separated schemes, so according to [7, Example 4.2] the derived global section functor of X,

$$\mathrm{R}\Gamma\colon \mathsf{D}(\mathsf{QCoh}\, X) \longrightarrow \mathsf{D}(\mathsf{Mod}\, k)$$

(which equals the derived direct image functor $\mathrm{R}f_*$), has a right-adjoint

$$f^!\colon \mathsf{D}(\mathsf{Mod}\, k) \longrightarrow \mathsf{D}(\mathsf{QCoh}\, X).$$

The D's indicate derived categories. It is easy to see that $f^!k \cong \omega[d]$, where $[d]$ indicates the operation of shifting complexes d steps to the left; cf. [7, Remark 5.5]. Observe also that since the singularities of X are Gorenstein, ω is an invertible sheaf.

Now consider the following sequence of natural isomorphisms which is taken from [6, Section 4], where the object $\mathcal{N}$ is in $\mathsf{D}(\mathsf{QCoh}\, X)$,

$$\begin{aligned}
\operatorname{Hom}_{\mathsf{D}(\mathsf{QCoh}\, X)}(\mathcal{C},\mathcal{N})' &\cong \operatorname{Hom}_{\mathsf{D}(\mathsf{Mod}\, k)}(\mathrm{R}\Gamma(\mathrm{R}\mathcal{H}om_X(\mathcal{C},\mathcal{N})),k)\\
&\overset{\text{(a)}}{\cong} \operatorname{Hom}_{\mathsf{D}(\mathsf{QCoh}\, X)}(\mathrm{R}\mathcal{H}om_X(\mathcal{C},\mathcal{N}),f^!k)\\
&\overset{\text{(b)}}{\cong} \operatorname{Hom}_{\mathsf{D}(\mathsf{QCoh}\, X)}\big(\mathrm{R}\mathcal{H}om_X(\mathcal{C},\mathcal{O}_X) \overset{\mathrm{L}}{\otimes}_X \mathcal{N}, f^!k\big)\\
&\overset{\text{(c)}}{\cong} \operatorname{Hom}_{\mathsf{D}(\mathsf{QCoh}\, X)}(\mathcal{N},\mathrm{R}\mathcal{H}om_X(\mathrm{R}\mathcal{H}om_X(\mathcal{C},\mathcal{O}_X),f^!k))\\
&\overset{\text{(d)}}{\cong} \operatorname{Hom}_{\mathsf{D}(\mathsf{QCoh}\, X)}\big(\mathcal{N},\mathcal{C} \overset{\mathrm{L}}{\otimes}_X f^!k\big).
\end{aligned}$$

Here (a) is by adjointness between $\mathrm{R}\Gamma$ and $f^!$ and (c) is by adjointness between $\overset{\mathrm{L}}{\otimes}$ and $\mathrm{R}\mathcal{H}om$, while (b) and (d) hold because they clearly hold if $\mathcal{C}$ is a locally free coherent sheaf, and therefore also hold for the given $\mathcal{C}$ because it has a bounded resolution of locally free coherent sheaves, so is finitely built in $\mathsf{D}(\mathsf{QCoh}\, X)$ from such sheaves.

Now, $f^!k$ is $\omega[d]$ and ω is an invertible sheaf. Hence $\mathcal{C} \overset{\mathrm{L}}{\otimes}_X f^!k$ is just $(\mathcal{C}\otimes\omega)[d]$. Inserting this along with $\mathcal{N}=\mathcal{M}[i]$ with $\mathcal{M}$ in $\mathsf{QCoh}(X)$ gives natural isomorphisms

$$\operatorname{Ext}^i_{\mathsf{QCoh}\, X}(\mathcal{C},\mathcal{M})' \cong \operatorname{Ext}^{d-i}_{\mathsf{QCoh}\, X}(\mathcal{M},\mathcal{C}\otimes\omega),$$

proving the lemma. □

Theorem 4. *Let X be a smooth projective scheme of dimension $d\geq 1$ over the field k. Let ω be the dualizing sheaf.*

Let $\mathcal{C}$ be an indecomposable coherent sheaf. Then there is an Auslander–Reiten sequence in the category of quasi-coherent sheaves,

$$0 \longrightarrow (\Sigma^{d-1}\mathcal{C})\otimes\omega \longrightarrow \mathcal{B} \longrightarrow \mathcal{C} \longrightarrow 0.$$

Here $\Sigma^{d-1}\mathcal{C}$ is the $(d-1)$st syzygy in a minimal injective resolution of $\mathcal{C}$ in the category of quasi-coherent sheaves.

Proof. The proof will give slightly more than stated: Let X be a projective scheme over k of dimension $d\geq 1$ with Gorenstein singularities, and let $\mathcal{C}$ be an indecomposable coherent sheaf which has a bounded resolution of locally free coherent sheaves. Then we will prove that the indicated Auslander–Reiten sequence exists. In the smooth case, each coherent sheaf has a resolution as required by [4, Exercise III.6.5].

The category of quasi-coherent sheaves $\mathsf{QCoh}(X)$ is clearly k-linear, and by [5, Lemma 3.1], it is a Grothendieck category, that is, a cocomplete abelian category with a generator and exact filtered colimits. Hence it has injective envelopes by [8, Theorem 10.10], and in particular, it has enough injectives.

Also, the category of coherent sheaves is an abelian category and so has split idempotents. It also has finite-dimensional Hom sets, as follows e.g. from Serre finiteness, [4, Theorem III.5.2(a)]. Hence the endomorphism ring of the indecomposable sheaf $\mathcal{C}$ is local; cf. [10, p. 52].

Finally, if we set $i=0$ in Lemma 3, then we get the natural equivalence

$$\mathrm{Hom}(\mathcal{C},-)' \simeq \mathrm{Ext}^d(-,\mathcal{C}\otimes\omega).$$

This is equation (2) of Proposition 1, with $\mathcal{A}=\mathcal{C}\otimes\omega$.

So all the conditions of Proposition 1 are satisfied, and hence, there is a short exact sequence in $\mathsf{QCoh}(X)$,

$$0 \longrightarrow \Sigma^{d-1}(\mathcal{C}\otimes\omega) \xrightarrow{a} \mathcal{B} \xrightarrow{b} \mathcal{C} \longrightarrow 0,$$

where b is right almost split. Since ω is invertible, this sequence equals the one in the theorem if we construct $\Sigma^{d-1}(\mathcal{C}\otimes\omega)$ by means of a minimal injective resolution of $\mathcal{C}\otimes\omega$. So to finish the proof, we must show that a is left almost split. Since b is right almost split, it is enough by classical Auslander–Reiten theory to show that $\mathrm{End}(\Sigma^{d-1}(\mathcal{C}\otimes\omega))$ is a local ring, see e.g. [2, Proposition V.1.14].

For this, note that the minimal injective resolution gives short exact sequences

$$0 \longrightarrow \Sigma^{l}(\mathcal{C}\otimes\omega) \longrightarrow \mathcal{I}^{l} \longrightarrow \Sigma^{l+1}(\mathcal{C}\otimes\omega) \longrightarrow 0$$

for $l=0, ..., d-2$, where $\Sigma^l(\mathcal{C}\otimes\omega)\to\mathcal{I}^l$ is an injective envelope. Hence, starting with the knowledge that $\operatorname{End}(\mathcal{C}\otimes\omega)\cong\operatorname{End}(\mathcal{C})$ is local, successive uses of Lemma 2 will prove that all the rings

$$\operatorname{End}(\mathcal{C}\otimes\omega), ..., \operatorname{End}(\Sigma^{d-1}(\mathcal{C}\otimes\omega))$$

are local, provided that we can show

$$\operatorname{Ext}^1(\mathcal{I}^l, \Sigma^l(\mathcal{C}\otimes\omega)) = 0$$

for $l=0, ..., d-2$, that is,

$$\operatorname{Ext}^{l+1}(\mathcal{I}^l, \mathcal{C}\otimes\omega) = 0 \quad \text{for } l = 0, ..., d-2. \tag{5}$$

However, if $\mathcal{I}$ is any injective, then Lemma 3 gives

$$\operatorname{Ext}^j(\mathcal{I}, \mathcal{C}\otimes\omega) \cong \operatorname{Ext}^{d-j}(\mathcal{C}, \mathcal{I})' = 0$$

for $j=1, ..., d-1$, and this implies (5). $\square$

The theorem has the following immediate corollary which was already known from [9].

Corollary 5. *Let X be a smooth projective curve over the field k. Then the category of coherent sheaves has right and left Auslander–Reiten sequences.*

Indeed, if $\mathcal{C}$ and $\mathcal{A}$ are indecomposable coherent sheaves, then there are Auslander–Reiten sequences of coherent sheaves,

$$0 \longrightarrow \mathcal{C}\otimes\omega \longrightarrow \mathcal{B} \longrightarrow \mathcal{C} \longrightarrow 0$$

and

$$0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{B}' \longrightarrow \mathcal{A}\otimes\omega^{-1} \longrightarrow 0.$$

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Peter Jørgensen
Department of Pure Mathematics
University of Leeds
Leeds LS2 9JT
United Kingdom
popjoerg@maths.leeds.ac.uk

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