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Polynomials in the de Branges spaces of entire functions

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Abstract. We study the problem of density of polynomials in the de Branges spaces $\mathcal{H}(E)$ of entire functions and obtain conditions (in terms of the distribution of the zeros of the generating function E) ensuring that the polynomials belong to the space $\mathcal{H}(E)$ or are dense in this space. We discuss the relation of these results with the recent paper of V. P. Havin and J. Mashreghi on majorants for the shift-coinvariant subspaces. Also, it is shown that the density of polynomials implies the hypercyclicity of translation operators in $\mathcal{H}(E)$.

Introduction

Let E be an entire function satisfying the inequality

(1) $$|E(z)| > |E(\bar{z})|, \quad z \in \mathbf{C}^+,$$

where $\mathbf{C}^+ = \{z : \operatorname{Im} z > 0\}$ is the upper half-plane. We denote the class of such functions (known as the Hermite–Biehler class) by HB. With each function $E \in HB$ we associate a Hilbert space $\mathcal{H}(E)$ which consists of all entire functions F such that F/E and F^*/E belong to the Hardy class $H^2(\mathbf{C}^+)$ (here and later on $F^*(z) = \overline{F(\bar{z})}$). The inner product, which makes $\mathcal{H}(E)$ a Hilbert space, is defined by the formula

$$\langle F, G \rangle_E = \int_{\mathbf{R}} \frac{F(t)\overline{G(t)}}{|E(t)|^2}\, dt.$$

The theory of the spaces $\mathcal{H}(E)$ introduced by L. de Branges [11] has important applications in mathematical physics. At the same time, de Branges spaces are of interest from the point of view of theory of entire functions. As a special case of de Branges spaces the Paley–Wiener space PW_a of entire functions of exponential type at most a, which are square summable on the real axis, may be considered (this space corresponds to the function $E(z) = \exp(-iaz)$, $a > 0$).

De Branges spaces are also closely connected with the shift-coinvariant subspaces K_Θ of the Hardy class $H^2(\mathbf{C}^+)$ known also as model subspaces (these subspaces are discussed in details in the monographs [21], [22]). Let Θ be an inner function in the upper half-plane. Put $K_\Theta = H^2(\mathbf{C}^+) \ominus \Theta H^2(\mathbf{C}^+)$. If E is an entire function satisfying (1), then $\Theta_E = E^*/E$ is inner and the mapping $F \mapsto F/E$ is a unitary operator from $\mathcal{H}(E)$ onto K_{Θ_E}. Conversely, each inner function Θ, meromorphic in the whole complex plane, is of the form E^*/E for some function E of the Hermite–Biehler class.

In the present note we are concerned with the following three related problems. By $\mathcal{P}$ we denote the set of all polynomials.

1. *For which entire functions $E \in HB$ does the function $f \equiv 1$ belong to the space $\mathcal{H}(E)$?*

2. *For which $E \in HB$ does the inclusion $\mathcal{P} \subset \mathcal{H}(E)$ hold?*

3. *For which $E \in HB$ are the polynomials dense in $\mathcal{H}(E)$?*

Density of polynomials is a classical problem for weighted functional spaces. At the same time, there are rather deep motivations for considering these problems for the de Branges spaces in particular. The first question is inspired by the recent papers by V. P. Havin and J. Mashreghi [15], [16] where the admissible majorants for the model subspaces were studied in the spirit of the Beurling–Malliavin theorem (the first problem was also investigated in [23] from a different point of view). By an admissible majorant for the space K_Θ we mean a nonnegative function w on the real line such that $w \ge |f|$ for some nonzero function $f \in K_\Theta$. It turns out that the condition $1 \in \mathcal{H}(E)$ is crucial for the existence of a majorant with the fastest rate of decrease.

An admissible majorant w is said to be minimal if for any other admissible majorant $\widetilde{w}$ such that $\widetilde{w} \le Cw$ we have $\widetilde{w} \asymp w$, that is, $cw \le \widetilde{w} \le Cw$ for some positive constant c.

Theorem. (Havin, Mashreghi [15]) *Let E be a Hermite–Biehler class entire function of zero exponential type with zeros α_n such that*

$$\sum_{n=1}^{\infty} \frac{\log|\alpha_n|}{|\alpha_n|} < \infty. \tag{2}$$

Then, either

(*a*) $1/E \in L^2(\mathbf{R})$ *and* $1/|E|$ *is the minimal majorant for* K_{Θ_E};

or

(*b*) $1/E \notin L^2(\mathbf{R})$ *and there is no positive and continuous minimal majorant for* K_{Θ_E}.

Thus, our results concerning the inclusion $1 \in \mathcal{H}(E)$ produce new classes of model subspaces with minimal majorants. In Section 5 of the present note we obtain a certain refinement of the theorem of Havin and Mashreghi: we show that the statement remains true under the milder (and more natural) assumption $\sum_{n=1}^{\infty}|\alpha_n|^{-1}<\infty$.

The third problem (density of polynomials) is closely connected with the Hamburger moment problem on the line. It was shown in [9] that the answer to this question will lead to a description of all canonical solutions of the indeterminate Hamburger problem. Moreover, in [10] a criterion for the density of polynomials was obtained, which is, however, somewhat implicit and not so easy to apply.

At the same time, there exist very simple geometrical conditions on the zeros of E sufficient for the density of polynomials. It was mentioned by N. I. Akhiezer [1] (see also [2], Addenda and Problems to Chapter 2) that if $E\in HB$ is a canonical product of zero genus with zeros lying in a half-strip $\{z:|\operatorname{Re} z|\le h \text{ and } \operatorname{Im} z<0\}$, then $\mathcal{P}\subset\mathcal{H}(E)$ and $\operatorname{Clos}_E\mathcal{P}=\mathcal{H}(E)$.

A function $E\in HB$ will be referred to as a *symmetric function* if it satisfies the identity $E^*(z)=E(-z)$. In this case the zeros of E are symmetric with respect to the imaginary axis. It was shown by V. P. Gurarii [14] that if E is a symmetric function with zeros in the angle $\{z:-3\pi/4\le\arg z\le-\pi/4\}$, then the polynomials are also dense in $\mathcal{H}(E)$ (again we assume that E is a canonical product of zero genus). Clearly, all the zeros z_n of a Hermite–Biehler class function are in the lower half-plane $\mathbf{C}^-$ and satisfy the Blaschke condition $\sum_{n=1}^{\infty}|\operatorname{Im} z_n|(1+|z_n|^2)^{-1}<\infty$.

It should be noted that Problems 2 and 3 lead to different classes of functions. It is easy to construct an example of the space $\mathcal{H}(E)$ such that $\mathcal{P}\subset\mathcal{H}(E)$, but $\mathcal{P}$ is not dense in $\mathcal{H}(E)$. It is known that if the function $F=\alpha E+\beta E^*$, $\alpha,\beta\in\mathbf{C}$, belongs to $\mathcal{H}(E)$, then it is orthogonal to the domain of the operator of multiplication by z (see [11], Theorem 29). Thus, F is orthogonal to all the polynomials whenever $\mathcal{P}\subset\mathcal{H}(E)$. It should be mentioned that for a function $E\in HB$ of zero type one has $F\in\mathcal{H}(E)$ for some nonzero α and β if and only if the zeros z_n of E satisfy the condition $\sum_{n=1}^{\infty}|\operatorname{Im} z_n|<\infty$ [3].

P. Koosis [18] has constructed much more subtle examples where the polynomials belong to a de Branges space but are not dense in this space.

Finally, it should be mentioned that the problem of density of polynomials was recently considered by M. Kaltenbäck and H. Woracek [17] in connection with the problem of the structure of de Branges subspaces of a given de Branges space. In the next section we compare the results of [17] with our results.

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1. Main results

In what follows we assume that E is an infinite product of the form

$$E(z)=\prod_{n=1}^{\infty}\Big(1-\frac{z}{\bar z_n}\Big), \tag{3}$$

where $z_n=x_n+iy_n\in\mathbf{C}^+$ and $\sum_{n=1}^{\infty}|z_n|^{-1}<\infty$ (for the sake of convenience we denote the zeros of E by $\bar z_n$ so that $z_n\in\mathbf{C}^+$). In particular, E belongs to the so-called class A discussed in detail in B. Ya. Levin's monograph [19]. The case when zeros z_n lie in a Stolz angle $\Gamma_\gamma=\{z=x+iy: y\ge\gamma|x|\}$, $\gamma>0$, is of special interest.

In the present note we give some geometric conditions generalizing the results of Akhiezer and Gurarii. In particular, we solve the following problem: to describe the subsets Ω of $\mathbf{C}^-$ such that for any function E of the form (3) with zeros in the set Ω we have the inclusion $\mathcal{P}\subset\mathcal{H}(E)$. In this case the inclusion $\mathcal{P}\subset\mathcal{H}(E)$ holds independently of the distribution of zeros of E in Ω, and we say that the set Ω is a *distribution independent set* (a *DI-set*).

An analogous problem may be considered for the class of symmetric Hermite–Biehler functions: we say that a set $\Omega\subset\mathbf{C}^-$ symmetric with respect to the imaginary axis is a *distribution independent set for symmetric functions from the class HB* (a *DI_s-set*) if $\mathcal{P}\subset\mathcal{H}(E)$ for any symmetric entire function of the form (3) with zeros in Ω.

The results of Akhiezer and Gurarii state that the sets $\{z\in\mathbf{C}^-:|x|\le h\}$, $h>0$, and $\{z\in\mathbf{C}^-:|x|\le|y|\}$ are a DI-set and a DI$_s$-set, respectively. By x and y we denote the real and imaginary parts of the complex variable z.

In Section 3 we obtain an explicit description of the distribution independent sets, which generalizes the theorems of Akhiezer and Gurarii.

Theorem 1. (1) *A set $\Omega\subset\mathbf{C}^-$ is a DI-set if and only if*

$$\limsup_{\substack{z\in\Omega\\|x|\to\infty}}\frac{x^2}{|y|(1+\log|x|)}<\infty. \tag{4}$$

(2) *A set* $\Omega \subset \mathbf{C}^-$ *is a* DI_s*-set if and only if*

$$\limsup_{\substack{z\in\Omega \\ |x|\to\infty}} \frac{|x|-|y|}{(|x|\log|x|)^{1/2}} < \infty. \tag{5}$$

Furthermore, if E *is a function of the form* (3) *with zeros in a DI-set or a symmetric function with zeros in a* DI_s*-set, then* $\mathrm{Clos}_E \mathcal{P} = \mathcal{H}(E)$.

In particular, the set of points lying below any parabola, that is, $\{z: y \le -q|x|^2\}$, $q>0$, is a DI-set, whereas $\{z: y \le -|x|^\alpha\}$ is not a DI-set for any $\alpha<2$. Note also that the angle $\{z\in\mathbf{C}^-: |x|\le|y|\}$ is the widest DI_s-angle.

The following theorem provides another condition sufficient for the density of polynomials. To state it we introduce two quantities which measure the approach of the zeros to the real axis for a general function $E\in HB$ and for a symmetric function respectively. For $R>0$ put

$$S(R) = \sum_{|x_n|>R} \left|\mathrm{Re}\,\frac{1}{z_n}\right| = \sum_{|x_n|>R} \frac{|x_n|}{|z_n|^2}$$

and

$$S_s(R) = \sum_{x_n^2 > y_n^2 + R^2} \frac{|x_n| - y_n}{|z_n|^3}.$$

Theorem 2. *Let* E *be a function of the form* (3) *and* $\inf_n y_n > 0$.
(1) *If there is a constant* $A>0$ *such that* $S(R)\le AR^{-1}$, *then* $\mathrm{Clos}_E \mathcal{P} = \mathcal{H}(E)$.
(2) *If* E *is a symmetric function and* $S_s(R)\le AR^{-2}$, *then* $\mathrm{Clos}_E \mathcal{P} = \mathcal{H}(E)$.

Now, we compare Theorem 2 with the results of M. Kaltenbäck and H. Woracek [17]. In particular, one of the main theorems of [17] states that polynomials are dense in $\mathcal{H}(E)$ for a function of the form (3) if

$$\sum_{n=1}^{\infty} \left(\arg z_n - \frac{\pi}{2}\right)^2 < \infty,$$

where $\arg z$ stands for the main branch of the argument with the values in $(-\pi,\pi]$. Clearly, the latter condition is equivalent to the convergence of the series

$$\sum_{n=1}^{\infty} \left|\frac{x_n}{z_n}\right|^2 < \infty,$$

which, in turn, implies that $S(R)\le AR^{-1}$ (and, moreover, $S(R)=o(R^{-1})$, as $R\to\infty$). Thus, Theorem 2 (1) extends the result of M. Kaltenbäck and H. Woracek.

Analogously, the result of [17] concerning the symmetric functions follows from Theorem 2 (2).

Let the zeros z_n lie outside some Stolz angle Γ_γ. Then $|x_n| \asymp |z_n|$ and the condition $S(R) \le AR^{-1}$ is equivalent to $\sum_{|z_n|>R} |z_n|^{-1} \le CR^{-1}$. Thus, the sequence z_n tends to infinity very fast (say, as a progression ρ^n, $\rho>1$). In Section 4 we obtain certain sharper estimates in the case when all the zeros lie in a Stolz angle. By $n(t)$ we denote the number of the zeros in the disk $\{z : |z| \le t\}$.

Theorem 3. *Let the zeros z_n lie in a Stolz angle Γ_γ. If*

$$\sum_{|z_n|>R} \frac{|x_n|}{|z_n|^2} = o\left(\frac{1}{R}\int_0^R \frac{n(t)}{t}\,dt + R\int_R^\infty \frac{n(t)}{t^3}\,dt\right), \quad \text{as } R \to \infty, \tag{6}$$

then $\mathcal{P} \subset \mathcal{H}(E)$.

Condition (6) is fulfilled for the sequences z_n growing "faster than any power" (say, $|z_n| \asymp \exp(\log^\alpha n)$, $\alpha>1$), but it fails for $|z_n| \asymp n^\alpha$, $\alpha>1$.

Theorem 4. *Let $\{z_n\}_{n=1}^\infty \subset \{z : \gamma_1 x \le y \le \gamma_2 x\}$, $\gamma_1, \gamma_2 > 0$, and*

$$C_1 n^\alpha \le |z_n| \le C_2 n^\alpha.$$

Then there exist exponents α_1 and α_2 (depending on γ_1, γ_2, C_1, and C_2) such that $|E(x)| \to \infty$, as $|x| \to \infty$, and $\mathcal{P} \subset \mathcal{H}(E)$ if $\alpha > \alpha_1$, whereas $|E(x)| \to 0$, as $|x| \to \infty$, if $\alpha < \alpha_2$.

We construct an example showing that in this case the behavior of E depends essentially on the zeros' distribution. Namely, for any $\alpha>1$ there exist z_n with $\arg z_n = \pi/4$, $|z_n| \asymp n^\alpha$, and $1 \notin \mathcal{H}(E)$.

If the zeros are distributed regularly along a single ray one may obtain an explicit formula for the limit exponent. Let all the zeros z_n lie on the ray $\{z : y = \gamma x\}$, $\gamma>0$, and assume that there exists the density $\Delta = \lim_{t\to\infty} t^{-1/\alpha} n(t)$, $0<\Delta<\infty$. It is the case, in particular, if $z_n = n^\alpha(1+\gamma i)$. Put

$$\alpha_\gamma = 2 - \frac{2\arctan\gamma}{\pi}.$$

Then $|E(x)| \to \infty$, $|x| \to \infty$, and, moreover, $\mathcal{P} \subset \mathcal{H}(E)$, if $\alpha > \alpha_\gamma$, whereas $|E(x)| \to 0$, as $x \to \infty$, if $\alpha < \alpha_\gamma$.

In Section 6 we discuss the relationship between the density of polynomials and the hypercyclicity phenomenon. Recall that a vector f is said to be *hypercyclic* for a continuous linear operator T in a Fréchet space F if its orbit $\{T^n f\}_{n=0}^\infty$ is dense in F. In this case the operator T is also said to be *hypercyclic.*

The first example of a hypercyclic operator was obtained by G. D. Birkhoff [8] who showed that the translation operators $T_w: f \mapsto f(\,\cdot\,+w)$, $w \in \mathbf{C}$, $w \neq 0$, are hypercyclic in the space of all entire functions with the topology of uniform convergence on compact subsets of the plane. It was shown in [20] that the differentiation operator is also hypercyclic in this space, whereas in [13] this result was extended to all operators commuting with differentiation except the scalar multiples of identity. Thus, spaces of entire functions proved to be an important source of hypercyclic operators.

K. C. Chan and J. H. Shapiro [12] studied the hypercyclicity of translations in the setting of Hilbert spaces of entire functions of "slow growth". In [12] the question was posed whether translations in a "reasonable" space of entire functions are always hypercyclic, and it was shown that this is not true. For example, differentiation and translations in the Paley–Wiener space are bounded but not hypercyclic (in particular, T_w is an isometry of PW_a if $w \in \mathbf{R}$). We show, making use of the results of [12], that translation operators may be hypercyclic on the de Branges spaces. Moreover, translations are hypercyclic as soon as the polynomials are dense in $\mathcal{H}(E)$.

Theorem 5. *Let $\mathcal{H}(E)$ be a de Branges space such that* $\mathrm{Clos}_E \mathcal{P} = \mathcal{H}(E)$ *and the differentiation operator is bounded in $\mathcal{H}(E)$. Then the translation operators T_w, $w \neq 0$, are hypercyclic in $\mathcal{H}(E)$.*

Note that the translations are bounded in $\mathcal{H}(E)$ whenever the differentiation operator $\mathcal{D}$ is bounded, since $T_w = \exp(w\mathcal{D})$. The de Branges spaces $\mathcal{H}(E)$ such that the differentiation is bounded on $\mathcal{H}(E)$ were described by the author in [4]. A sufficient (but not necessary) condition is that $E'/E \in L^\infty(\mathbf{R})$. This is the case, in particular, if E is an entire function of the form (3) with zeros in a Stolz angle.

2. Preliminaries

In this section we consider certain general conditions sufficient for the density of polynomials. Let P be a polynomial with zeros in $\mathbf{C}^-$. Then $P \in HB$ and, clearly, for any $E \in HB$ the function $\widetilde{E} = PE$ is also in HB. Moreover, $\|F\|_{\widetilde{E}} \le C\|F\|_E$ whenever $F \in \mathcal{H}(E)$ and, consequently, $\mathcal{H}(E) \subset \mathcal{H}(\widetilde{E})$.

Lemma 6. *$\mathcal{H}(E)$ is dense in $\mathcal{H}(\widetilde{E})$ if and only if the domain of the operator of multiplication by z is dense in $\mathcal{H}(E)$.*

Proof. We consider the case when P is of degree one; the general case follows by induction. Let $P(z) = z - a$, $a \in \mathbf{C}^-$. Then $\mathcal{H}(E)$ coincides with the domain of multiplication by the independent variable z in $\mathcal{H}(\widetilde{E})$ (indeed, $F \in \mathcal{H}(E)$ if and only if both $(z-a)F$ and $(z-\bar{a})F$ belong to $\mathcal{H}(\widetilde{E})$).

By [11], Theorem 29, the domain of multiplication by z is not dense in $\mathcal{H}(\widetilde{E})$ if and only if there exists a nonzero function of the form $\alpha\widetilde{E}+\beta\widetilde{E}^*$, $\alpha,\beta\in\mathbf{C}$, which belongs to $\mathcal{H}(\widetilde{E})$. Finally, note that

$$\alpha\widetilde{E}+\beta\widetilde{E}^* = (z-a)(\alpha E+\beta E^*)+2i\beta\,\mathrm{Im}\,aE^*,$$

and $E^*\in\mathcal{H}(\widetilde{E})$. Hence, the inclusions $\alpha\widetilde{E}+\beta\widetilde{E}^*\in\mathcal{H}(\widetilde{E})$ and $\alpha E+\beta E^*\in\mathcal{H}(E)$ are equivalent. □

Lemma 7. *The polynomials are dense in $\mathcal{H}(E)$ if and only if they are dense in $\mathcal{H}(\widetilde{E})$.*

Proof. Again it suffices to consider the case when $P(z)=z-a$, $a\in\mathbf{C}^-$. Assume that the polynomials are dense in $\mathcal{H}(\widetilde{E})$. Let $F\in\mathcal{H}(E)$. Then $(z-a)F\in\mathcal{H}(\widetilde{E})$ and there exists a sequence of polynomials P_n such that $P_n\to(z-a)F$ in $\mathcal{H}(\widetilde{E})$. Since the point evaluation functionals are continuous in the de Branges spaces, $P_n(a)\to 0$, as $n\to\infty$. Consider the polynomials $Q_n(z)=(P_n(z)-P_n(a))/(z-a)$. Then

$$\|Q_n-F\|_E = \|P_n-P_n(a)-(z-a)F\|_{\widetilde{E}} \le \|P_n-(z-a)F\|_{\widetilde{E}}+|P_n(a)|\,\|1\|_{\widetilde{E}},$$

and, consequently, $\|Q_n-F\|_E\to 0$, as $n\to\infty$.

The converse statement follows immediately from Lemma 6. Indeed, if the polynomials are dense in $\mathcal{H}(E)$, then the domain of the operator of multiplication by z is dense in $\mathcal{H}(E)$. Hence, by Lemma 6, $\mathcal{H}(E)$ is dense in $\mathcal{H}(\widetilde{E})$ and, therefore, $\mathcal{P}$ is dense in $\mathcal{H}(\widetilde{E})$. □

Recall that the reproducing kernel of the space $\mathcal{H}(E)$ corresponding to the point $w\in\mathbf{C}$ is of the form

$$K(z,w)=\frac{i}{2\pi}\frac{\overline{E(w)}E(z)-\overline{E^*(w)}E^*(z)}{z-\overline{w}}.$$

Theorem 8. *Let E be an entire function of the Hermite–Biehler class such that the domain of the operator of multiplication by z is dense in $\mathcal{H}(E)$. Assume that there is a sequence P_n of polynomials, which converges to the function E uniformly on any compact set and such that*

$$|P_n(x)|\le C(1+|x|)^N|E(x)|,\quad x\in\mathbf{R},\ n\in\mathbf{N}, \tag{7}$$

for some nonnegative integer N and for some $C>0$. Then $\mathcal{P}\subset\mathcal{H}(E)$ and $\mathrm{Clos}_E\mathcal{P}=\mathcal{H}(E)$.

Proof. The inclusion $\mathcal{P}\subset\mathcal{H}(E)$ follows immediately from (7). Put $\widetilde{E}(z)=(z+i)^N E(z)$. By Lemma 6, $\mathcal{H}(E)$ is dense in $\mathcal{H}(\widetilde{E})$. By Lemma 7, the polynomials are dense or not dense in $\mathcal{H}(E)$ and in $\mathcal{H}(\widetilde{E})$ simultaneously. Thus, it is sufficient to show that the closure of the polynomials in $\mathcal{H}(\widetilde{E})$ contains $\mathcal{H}(E)$, which implies that $\mathrm{Clos}_{\widetilde{E}}\mathcal{P}=\mathcal{H}(\widetilde{E})$. Moreover, since the linear span of reproducing kernels $K(\,\cdot\,,s)$, $s\in\mathbf{R}$, is dense in $\mathcal{H}(E)$, we have to verify the inclusion $K(\,\cdot\,,s)\in\mathrm{Clos}_{\widetilde{E}}\mathcal{P}$.

Fix $s\in\mathbf{R}$ and put

$$K_n(z,s)=\frac{i}{2\pi}\,\frac{\overline{P_n(s)}P_n(z)-P_n(s)P_n^*(z)}{z-s}.$$

Clearly, $K_n(\,\cdot\,,s)$ is a polynomial, and the sequence $K_n(\,\cdot\,,s)$ converges to $K(\,\cdot\,,s)$ uniformly on any compact set. Let us show that $K_n(\,\cdot\,,s)$ converge to $K(\,\cdot\,,s)$ in the norm of the space $\mathcal{H}(\widetilde{E})$. Take $A>0$ and split the norm $\|K_n(\,\cdot\,,s)-K(\,\cdot\,,s)\|^2_{\widetilde{E}}$ into two parts:

$$\int_{|t-s|\le A}\left|\frac{K_n(t,s)-K(t,s)}{2\pi\widetilde{E}(t)}\right|^2 dt+\int_{|t-s|>A}\left|\frac{K_n(t,s)-K(t,s)}{2\pi\widetilde{E}(t)}\right|^2 dt=I_1+I_2.$$

Let us estimate the integral I_2. By inequality (7), we have

$$\left|\frac{K_n(t,s)-K(t,s)}{\widetilde{E}(t)}\right|\le\frac{2|P_n(s)P_n(t)|}{|(t-s)(t+i)^N E(t)|}+\frac{2|E(s)|}{|(t-s)(t+i)^N|}\le\frac{C(s)}{|t-s|}.$$

Hence, $I_2\to0$, $A\to\infty$, and, choosing a sufficiently large A, we can make the integral I_2 as small as we wish uniformly with respect to n. Now, when A is fixed, the integral I_1 tends to zero when n tends to infinity. $\square$

Remark. H. Woracek has noted that the density of the domain of multiplication by z is essential in Theorem 8 (personal communications). However, this condition may be omitted if we replace (7) by the estimate $|P_n(x)|\le C(1+|x|)^\alpha|E(x)|$, $x\in\mathbf{R}$, for some $\alpha<1/2$. In this case the above arguments work for E instead of $\widetilde{E}$.

Let us introduce some notation. Note that

$$|E(x)|^2=\prod_{n=1}^{\infty}\frac{(x-x_n)^2+y_n^2}{|z_n|^2}=\prod_{n=1}^{\infty}\left(1-\frac{2xx_n-x^2}{|z_n|^2}\right).$$

We split this product into two parts $\Pi_+(x)$ and $\Pi_-(x)$, where

$$\Pi_-(x)=\prod_{\substack{xx_n>0\\ |x_n|>|x|/2}}\left(1-\frac{2xx_n-x^2}{|z_n|^2}\right).$$

Thus, each factor in the product Π_- is smaller than 1, whereas all the factors in the product Π_+ are greater or equal to 1.

If E is a symmetric entire function, then we have the pairs of symmetric zeros $z_n = x_n + iy_n$ and $\tilde{z}_n = -x_n + iy_n$. Hence,

$$|E(x)|^2 = \prod_{n=1}^{\infty} \left(1 + \frac{x^4 + 2x^2(|z_n|^2 - 2x_n^2)}{|z_n|^4}\right),$$

and again we split it into a "large" part Π_+^s and a "small" part Π_-^s. Put

$$\Pi_-^s(x) = \prod_{x_n^2 > y_n^2 + x^2/2} \left(1 + \frac{x^2(x^2 + 2y_n^2 - 2x_n^2)}{|z_n|^4}\right).$$

Thus, the product $\Pi_-^s(x)$ includes exactly those factors which are smaller than 1.

Corollary 9. *Assume that the domain of multiplication by z is dense in $\mathcal{H}(E)$. If there is $N \geq 0$ and $C > 0$ such that $\Pi_-(x) \geq C(1+|x|)^{-N}$, $x \in \mathbf{R}$, or if E is symmetric and $\Pi_-^s(x) \geq C(1+|x|)^{-N}$, then $\operatorname{Clos}_E \mathcal{P} = \mathcal{H}(E)$.*

Proof. Put

$$P_n(z) = \prod_{|k| \leq n} \left(1 - \frac{z}{\bar{z}_k}\right).$$

Then, clearly, $P_n \to E$ uniformly on compact sets. Note also that $|E(x)/P_n(x)|^2 \geq \Pi_-(x)$ for each $n \in \mathbf{N}$. Now we may apply Theorem 8. $\square$

As we have mentioned in the introduction, for a function E of the form (3) the domain of multiplication by z is not dense in $\mathcal{H}(E)$ if and only if $\sum_{n=1}^{\infty} y_n < \infty$, that is, the zeros z_n approach the real axis (see [3]). In the conditions of Theorems 1 and 2 the zeros are separated from the real axis or even lie in a Stolz angle. Thus, in what follows we will consider only the de Branges spaces where the domain of multiplication by z is dense.

3. Proofs of Theorems 1 and 2

Proof of Theorem 1. (1) *Sufficiency of* (4). Let Ω satisfy the condition (4) and E be an entire function of the form (3) such that $\bar{z}_n \in \Omega$. Then there is $M > 0$ such that $x_n^2 \leq M y_n \log|x_n|$ when $|x_n| > 2$.

Let us estimate the "small" factor $\Pi_-(x)$ for sufficiently large positive x. Note that (4) implies that all the zeros, except maybe a finite set, lie in any Stolz angle Γ_γ.

By Lemma 7, we may eliminate any finite set of zeros. Thus, without loss of generality, let $2xx_n - x^2 \le |z_n|^2/2$. Applying the elementary estimate $\log(1-t) \ge -2t$, $t \in \left[0, \frac{1}{2}\right]$, we get

$$\log \Pi_-(x) = \sum_{x_n > x/2} \log\left(1 - \frac{2xx_n - x^2}{|z_n|^2}\right) \ge -2 \sum_{x_n > x/2} \frac{2xx_n - x^2}{|z_n|^2}.$$

Hence,

$$\log \Pi_-(x) \ge -4x \sum_{x_n > x/2} \frac{x_n}{y_n^2} \ge -4Mx \sum_{x_n > x/2} \frac{\log |x_n|}{x_n y_n} \ge -4M \log x \sum_{x_n > x/2} \frac{1}{y_n},$$

since the function $x^{-1}\log x$ decreases for large x. Analogously, for $x<0$,

$$\log \Pi_-(x) \ge -4M \log |x| \sum_{x_n < x/2} \frac{1}{y_n},$$

when $|x|$ is sufficiently large. Hence, $\log \Pi_-(x) = o(\log |x|)$, as $|x| \to \infty$, and E satisfies the conditions of Corollary 9.

(1) *Necessity of* (4). Assume that

$$\limsup_{\substack{z \in \Omega \\ x \to \infty}} \frac{x^2}{y(1+\log x)} = \infty. \tag{8}$$

In this case we will choose sequences of zeros $\bar{z}_n = x_n - iy_n \in \Omega$ and multiplicities k_n such that for the function

$$E(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z}{\bar{z}_n}\right)^{k_n}$$

we get $\lim_{n\to\infty} |E(x_n)| = 0$.

Assume that $z_1, k_1, \dots, z_{n-1}, k_{n-1}$ are already chosen (without loss of generality we assume that $x_j > 2$). Then, by (8), we can choose $\bar{z}_n \in \Omega$ such that $x_n > \max(|z_{n-1}|, 2^n)$, and

$$\frac{x_n^2}{|z_n|} \ge 2^{n+3} \log x_n \sum_{j=1}^{n-1} k_j. \tag{9}$$

Put $k_n = [|z_n|/2^n] + 1$ (by $[s]$ we denote the integer part of the number s). Then $\sum_{n=1}^{\infty} k_n |z_n|^{-1} < \infty$ and the product for E converges.

Let $t\in[x_n-1,x_n]$. We split the product for $E(t)$ into three parts:

$$|E(t)|^2=\left|\frac{t-z_n}{z_n}\right|^{2k_n}\prod_{j=1}^{n-1}\left|\frac{t-z_j}{z_j}\right|^{2k_j}\prod_{j=n+1}^{\infty}\left|\frac{t-z_j}{z_j}\right|^{2k_j}.$$

Clearly, the last product does not exceed 1. Since $x_j>2$ and $|x_n|>|z_j|$, $j<n$, we have

$$\prod_{j=1}^{n-1}\left(\frac{|t-z_j|^2}{|z_j|^2}\right)^{k_j}<\prod_{j=1}^{n-1}\left(\frac{|x_n-z_j|^2}{|z_j|^2}\right)^{k_j}<\prod_{j=1}^{n-1}(2x_n^2)^{k_j}<\prod_{j=1}^{n-1}(x_n^3)^{k_j}.$$

At the same time,

$$\log\left|\frac{t-z_n}{z_n}\right|^{2k_n}\le k_n\log\left(1-\frac{x_n^2-1}{|z_n|^2}\right)\le-\frac{k_nx_n^2}{2|z_n|^2}.$$

Hence,

$$\log|E(t)|^2\le 3\log x_n\sum_{j=1}^{n-1}k_j-\frac{k_nx_n^2}{2|z_n|^2}\le-\log x_n\sum_{j=1}^{n-1}k_j,$$

where the latter inequality follows from (9) and the definition of k_n. Thus, $\sup_{t\in[x_n-1,x_n]}|E(t)|\to0$, as $n\to\infty$, and, consequently, $1/E\notin L^2(\mathbf{R})$.

(2) *Sufficiency of* (5). Without loss of generality let all zeros lie in the angle $\Gamma_{4/5}$. Then

$$\sup_{x_n^2>y_n^2+x^2/2}\frac{x^2(2x_n^2-x^2-2y_n^2)}{|z_n|^4}<1.$$

Therefore, there is a constant $C_1>0$ such that

$$\begin{aligned}\log\Pi_-^s(x)&=\sum_{x_n^2>y_n^2+x^2/2}\log\left(1+\frac{x^2(x^2+2y_n^2-2x_n^2)}{|z_n|^4}\right)\\&\ge-C_1\sum_{x_n^2>y_n^2+x^2/2}\frac{x^2(2x_n^2-2y_n^2-x^2)}{|z_n|^4}.\end{aligned}$$

If (5) is satisfied we may assume that $x_n-y_n\le Mx_n^{1/2}(\log x_n)^{1/2}$ whenever $x_n^2>y_n^2+x^2/2$ and $|x|>2$. Hence,

$$\log\Pi_-^s(x)\ge-C_1\sum_{x_n^2>y_n^2+x^2/2}\frac{x^2(4Mx_n^{3/2}(\log x_n)^{1/2}-x^2)}{|z_n|x_n^3}.$$

Consider the function

$$F(t,x)=\frac{x^2(4Mt^{3/2}(\log t)^{1/2}-x^2)}{t^3}.$$

It is easy to see that there is a constant $C_2=C_2(M)>0$ such that $|F(t,x)|\le C_2\log|x|$ whenever $t>|x|>2$. Thus,

$$\log \Pi_-^s(x)\ge -C_3\log|x|\sum_{x_n^2>y_n^2+x^2/2}\frac{1}{|z_n|},$$

and we may apply Corollary 9.

(2) *Necessity of* (5). The proof is analogous to the proof of the necessity of (4). We choose pairs of symmetric zeros $z_n=x_n+iy_n$ and $\tilde z_n=-x_n+iy_n$ and multiplicities k_n such that $1/E\notin L^2(\mathbf{R})$, where

$$E(z)=\prod_{n=1}^{\infty}\Big(1-\frac{z}{\bar z_n}\Big)^{k_n}\Big(1-\frac{z}{\bar{\tilde z}_n}\Big)^{k_n}.$$

Assume that z_j,k_j, $1\le j\le n-1$, are already chosen (we assume that $x_j>2$). If Ω does not satisfy (5) we can choose $\bar z_n\in\Omega$ such that $x_n>y_n$ and

$$\frac{(x_n-y_n)^2}{x_n\log x_n}\ge 2^{2n}x_{n-1}\sum_{j=1}^{n-1}k_j.$$

Put $k_n=[x_n/2^n x_{n-1}]+1$. Then $\sum_{n=1}^{\infty}k_n|z_n|^{-1}<\infty$ and E is well defined.

Let $t_n^2=x_n^2-y_n^2$. We split the product for $|E(t_n)|^2$ into three parts Π_1, Π_2, and Π_3, corresponding to the zeros $\{z_j\}_{j<n}$, $\{z_j\}_{j>n}$, and to the zero z_n, respectively. It is easily shown that there exist absolute positive constants C_1 and C_2, such that

$$\log \Pi_1(t_n)\le C_1\log x_n\sum_{j=1}^{n-1}k_j,$$

and $\log\Pi_2(t_n)\le C_2$. On the other hand, by the choice of x_n and k_n,

$$\log \Pi_3(t_n)=2k_n\log\left(1-\frac{\big(x_n^2-y_n^2\big)^2}{|z_n|^4}\right)\le -C_3k_n\frac{(x_n-y_n)^2}{x_n}\le -C_3 2^n\log x_n\sum_{j=1}^{n-1}k_j.$$

Therefore, $|E(t_n)|\to 0$, as $n\to\infty$, and, moreover, $|E(t)|=o(1)$, $t\in(t_n-1,t_n)$. Thus, $1\notin\mathcal{H}(E)$. $\square$

Proof of Theorem 2. (1) We show that there exist $l\in\mathbf{N}$ and $C>0$ such that

$$\Pi_-(x)\geq C(\delta/x)^{2l} \tag{10}$$

for sufficiently large $|x|$; here $\delta=\inf_n y_n$. Once the inequality (10) is proved the result follows from Corollary 9.

Without loss of generality let $x>0$. Denote by $\mathcal{N}(x)$ the set of n such that $2xx_n-x^2\geq|z_n|^2/2$ and let $|\mathcal{N}(x)|$ be the number of elements in $\mathcal{N}(x)$. Then

$$\frac{1}{2}|\mathcal{N}(x)|\leq\sum_{x_n>x/2}\frac{2xx_n-x^2}{|z_n|^2}\leq 2x\sum_{|x_n|>x/2}\frac{|x_n|}{|z_n|^2}=2xS(x/2)\leq 4A,$$

which implies that there is $l\in\mathbf{N}$ such that $|\mathcal{N}(x)|\leq l$ for any $x>0$. Note also that $|z_n|\leq 4x$ whenever $n\in\mathcal{N}(x)$. Hence,

$$\prod_{n\in\mathcal{N}(x)}\frac{(x-x_n)^2+y_n^2}{|z_n|^2}\geq\left(\frac{\delta}{4x}\right)^{2l}.$$

Finally,

$$\log\prod_{\substack{n\notin\mathcal{N}(x)\\ x_n>x/2}}\left(1-\frac{2xx_n-x^2}{|z_n|^2}\right)\geq-2\sum_{\substack{n\notin\mathcal{N}(x)\\ x_n>x/2}}\frac{2xx_n-x^2}{|z_n|^2}$$

$$\geq-2\sum_{x_n>x/2}\frac{2xx_n}{|z_n|^2}\geq-4xS(x/2)\geq-8A.$$

Combining the last two inequalities we obtain the estimate (10).

The proof of (2) is analogous. □

Remark. It follows from Corollary 9 and the proof of Theorem 2 that in the case when the zeros lie in a Stolz angle a milder condition $S(R)\leq AR^{-1}\log R$ also implies the density of the polynomials in $\mathcal{H}(E)$.

4. Functions with zeros in an angle

In this section we find in a sense sharp asymptotic of the growth on the real axis for an entire function $E\in HB$ with zeros in a Stolz angle.

Proposition 10. *Let $z_n\in\Gamma_\gamma$. Then there exist functions $A_j(x)$, $j=1,...,4$, such that*

$$0<m_j\leq A_j(x)\leq M_j,\quad x\in\mathbf{R},$$

and for $x\in\mathbf{R}$ *we have*

$$
\begin{aligned}
\log|E(3x)|^2 = A_1\int_0^{|x|}\frac{n(t)}{t}\,dt+A_2x^2\int_{|x|}^{\infty}\frac{n(t)}{t^3}\,dt \\
+A_3|x|\sum_{\substack{|z_n|\ge|x|\\ x_nx<0}}\frac{|x_n|}{|z_n|^2}-A_4|x|\sum_{\substack{|z_n|\ge|x|\\ x_nx>0}}\frac{|x_n|}{|z_n|^2}.
\end{aligned}
\tag{11}
$$

Here the constants m_j *and* M_j *may depend on* γ*.*

Proof. We use the following elementary estimates: $-t/\delta\le\log(1-t)\le -t$, $t\in[0,1-\delta]$, and $t/(K+1)\le\log(1+t)\le t$, $t\in[0,K]$.

Without loss of generality let $x>0$. Then

$$
|E(x)|^2=\prod_{|z_n|<x/3}\frac{|x-z_n|^2}{|z_n|^2}\prod_{\substack{|z_n|\ge x/3\\ x_n<x/2}}\frac{|x-z_n|^2}{|z_n|^2}\prod_{x_n\ge x/2}\frac{|x-z_n|^2}{|z_n|^2}.
$$

Denote the products in the latter formula by Π_j, $j=1,2,3$, respectively. Note that since $z_n\in\Gamma_\gamma$ we have

$$
\frac{2xx_n-x^2}{|z_n|^2}\le\frac{1}{1+\gamma^2}
$$

whenever $x_n\ge x/2$. Hence,

$$
-\frac{\gamma^2+1}{\gamma^2}\sum_{x_n\ge x/2}\frac{2xx_n-x^2}{|z_n^2|}\le\log\Pi_3(x)\le-\sum_{x_n\ge x/2}\frac{2xx_n-x^2}{|z_n^2|}.
\tag{12}
$$

Analogously,

$$
C_1\sum_{\substack{|z_n|\ge x/3\\ x_n<x/2}}\frac{x^2-2xx_n}{|z_n^2|}\le\log\Pi_2(x)\le C_2\sum_{\substack{|z_n|\ge x/3\\ x_n<x/2}}\frac{x^2-2xx_n}{|z_n^2|}
\tag{13}
$$

for some absolute positive constants C_1 and C_2.

Next, we find a rough asymptotic for the product Π_1. Since $|z_n|<x/3$ we get $2x/3\le|x-z_n|\le 2x$ and

$$
2^{2n(x/3)}\frac{(x/3)^{2n(x/3)}}{|z_1|^2|z_2|^2...|z_{n(x/3)}|^2}\le\Pi_1(x)\le 2^{2n(x/3)}\frac{(x/3)^{2n(x/3)}}{|z_1|^2|z_2|^2...|z_{n(x/3)}|^2}.
$$

Hence,

$$
\log\Pi_1(x)=2\log\frac{(x/3)^{n(x/3)}}{|z_1z_2...z_{n(x/3)}|}+A(x)n(x/3)=\int_0^{x/3}\frac{n(t)}{t}\,dt+A(x)n(x/3),
\tag{14}
$$

where $A(x)\asymp 1$. Combining the estimates (12)–(14) and replacing x by $3x$ we get the formula (11). We also used the fact that

$$n(R)+R^2\sum_{|z_n|>R}\frac{1}{|z_n|^2}=R^2\int_R^\infty\frac{n(t)}{t^3}\,dt. \quad\square$$

Thus, only the last term in (11) is responsible for the possible smallness of the function E on the real axis. If this term is asymptotically smaller than the positive summands in (11), as stated in Theorem 3, then E tends to infinity along $\mathbf{R}$ faster than any polynomial, and Theorem 3 follows immediately.

Proof of Theorem 4. Now let $z_n\in\{z:\gamma_1x\le y\le\gamma_2x\}$ and $C_1n^\alpha\le|z_n|\le C_2n^\alpha$. In this case $n(t)\asymp t^{1/\alpha}$, as $t\to\infty$, and we have the following asymptotic for the summands in the formula (11) (all the constants involved depend on γ_i and C_i, but do not depend on R and α):

$$\int_0^R\frac{n(t)}{t}\,dt\asymp\int_0^R t^{-1+1/\alpha}\,dt\asymp\alpha R^{1/\alpha},$$
$$R^2\int_R^\infty\frac{n(t)}{t^3}\,dt\asymp R^2\int_R^\infty t^{-3+1/\alpha}\,dt\asymp\frac{\alpha}{2\alpha-1}R^{1/\alpha}$$

and, finally,

$$R\sum_{|z_n|>R}\frac{x_n}{|z_n|^2}\asymp R\sum_{n=[R^{1/\alpha}]}^\infty\frac{1}{n^{1/\alpha}}\asymp\frac{R^{1/\alpha}}{\alpha-1}.$$

Note that $\alpha+\alpha/(2\alpha-1)=o(1/(\alpha-1))$, as $\alpha\to1$, and $1/(\alpha-1)=o(\alpha)$, as $\alpha\to\infty$. Hence, the coefficient at $R^{1/\alpha}$ in the formula for $\log|E(3R)|$ is positive when α is sufficiently large and negative when α is close to 1. $\square$

In the case of regular distribution along a single ray we obtain an explicit formula for the limit exponential.

Example 11. Let all the zeros z_n lie on the ray $\{z:y=\gamma x\}$, $\gamma>0$, and assume that for some $\alpha>1$ the density $\Delta=\lim_{t\to\infty}t^{-1/\alpha}n(t)$, $0<\Delta<\infty$ exists. Then $|E(x)|\to\infty$, as $|x|\to\infty$, if $\alpha>2-2\arctan(\gamma)/\pi$, and $|E(x)|\to0$, as $x\to\infty$, if $\alpha<2-2\arctan(\gamma)/\pi$.

Proof. Here we may apply the Levin–Pfluger theory of entire functions of completely regular growth. By Theorem 25 of [19], Chapter 1,

$$\lim_{R\to\infty}\frac{\log|E(R)|}{R^{1/\alpha}}=H_\alpha=\frac{\pi\Delta}{\sin(\pi/\alpha)}\cos\frac{\pi-\arctan\gamma}{\alpha}.$$

Thus, $H_\alpha>0$ ($H_\alpha<0$) if and only if $\alpha>2-2\arctan(\gamma)/\pi$ ($\alpha<2-2\arctan(\gamma)/\pi$). Note also that $\log|E(-R)|\asymp R^{1/\alpha}$, as $R\to\infty$, for any $\alpha>1$. $\square$

Remark. Note that $\alpha_\gamma = 2 - 2\arctan(\gamma)/\pi \to 2$, as $\gamma \to 0$. Consider the limit case $\gamma = 0$, that is, the case when $z_n = n^\alpha + i$, $n \in \mathbf{N}$. Then it is easily shown that $|E(x)| \to \infty$ (moreover, $\log|E(x)| \asymp |x|^{1/\alpha}$), as $|x| \to \infty$, when $\alpha > 2$, whereas for $\alpha \le 2$ one has $\lim_{x\to\infty} |E(x)| = 0$.

We conclude this section with an example illustrating the subtlety of the growth of a function with power growth of zeros.

Example 12. For any $\alpha > 1$ there exist z_n such that $\arg z_n = \pi/4$, $|z_n| \asymp n^\alpha$ and $1 \notin \mathcal{H}(E)$.

Proof. Take an integer $K > 2$ and let E be the entire function of the form (3) with zeros at the points $K^{\alpha l} e^{-i\pi/4}$ and with multiplicities equal to $K^l - K^{l-1}$, $l \in \mathbf{N}$. Now, if $z_1, z_2, ..., z_n, ...$ are zeros of E repeated according to the multiplicities, then $|z_n| \asymp n^\alpha$ (actually, we have replaced the group of zeros $n^\alpha e^{-i\pi/4}$, $K^{l-1} < n \le K^l$, by a single zero of the corresponding multiplicity).

Let $N \in \mathbf{N}$ and put $t_N = K^{\alpha N - 1/2}$. We show that one can choose a sufficiently large K such that

$$\lim_{N\to\infty} |E(3t_N)| = 0. \tag{15}$$

Then $1/E \notin L^2(\mathbf{R})$, since $E'/E \in L^\infty(\mathbf{R})$, and (15) implies that $\log|E|$ is negative on the intervals $(3t_N - \delta, 3t_N + \delta)$ for some $\delta > 0$ and for all sufficiently large N.

By Proposition 10,

$$\begin{aligned} \log|E(3t_N)|^2 = B_1 \int_0^{|t_N|} \frac{n(t)}{t}\,dt + B_2 n(t_N) \\ + B_3 t_N^2 \sum_{|z_n| > t_N} \frac{1}{|z_n|^2} - B_4 |t_N| \sum_{|z_n| \ge t_N} \frac{|x_n|}{|z_n|^2}, \end{aligned} \tag{16}$$

where $B_1(t_N) + B_2(t_N) + B_3(t_N) \le C$ and $B_4(t_N) \ge c$ for some absolute positive constants C and c. Now we estimate the summands in the formula (16). Note that $K^{\alpha(N-1)} < t_N < K^{\alpha N}$ and $n(t_N) = K^N - K$. Hence,

$$\begin{aligned} \int_0^{|t_N|} \frac{n(t)}{t}\,dt &= \log \frac{t_N^{n(t_N)}}{|z_1 z_2 ... z_{n(t_N)}|} \\ &= \left[(K^N - K)\left(\alpha N - \frac{1}{2}\right) - \alpha \sum_{l=1}^{N-1} l K^l (K-1) \right] \log K < 2\alpha K^N \log K. \end{aligned}$$

Finally, it is easy to see that the third positive summand in (16) does not exceed $2K^N$. Let us estimate the negative part of $\log|E(3t_N)|^2$:

$$t_N \sum_{|z_n|\ge t_N} \frac{|x_n|}{|z_n|^2} = K^{\alpha N-1/2} \sum_{l=N}^{\infty} \frac{K^l(K-1)}{\sqrt{2}K^{\alpha l}} \ge \frac{K^{n+1/2}}{\sqrt{2}}.$$

Thus,

$$\log|E(3t_N)|^2 \le C(3K^N+2\alpha K^N \log K) - \frac{c}{\sqrt{2}} K^{N+1/2},$$

and, taking a sufficiently large K, we can make $\log|E(3t_N)| \to -\infty$, as $N\to\infty$. □

5. Existence of minimal majorants

In this section we generalize the theorem of Havin and Mashreghi by showing that condition (2) may be replaced by a weaker one.

Theorem 13. *Let E be an entire function of the form* (3) *and* $\sum_{n=1}^{\infty} |z_n|^{-1} < \infty$. *Then, either*

(*a*) $1/E \in L^2(\mathbf{R})$ *and* $1/|E|$ *is the minimal majorant for* K_{Θ_E};

(*b*) $1/E \notin L^2(\mathbf{R})$ *and there is no positive and continuous minimal majorant for* K_{Θ_E}.

Statement (b) is proved in [15] without use of (2). We state it here only for the sake of completeness. Also, it is shown in [15] (Theorem 3.8) that for a function E of zero exponential type the function $1/|E|$ is the minimal majorant for K_{Θ_E} as soon as $1/E \in K_{\Theta_E}$. Thus, it remains to prove the inclusion $1/E \in K_{\Theta_E}$ which is, clearly, equivalent to $1/E \in H^2$.

To prove the latter inclusion we make use of the following condition, which is sufficient for a function f to belong to the Hardy class ([11], Theorems 11 and 12). Here $\log^+ t = \max(\log t, 0)$.

Lemma 14. *Let f be a function analytic in $\mathbf{C}^+$ and continuous in $\overline{\mathbf{C}}^+$. Assume that*

$$\limsup_{y\to\infty} \frac{\log|f(iy)|}{y} \le 0 \tag{17}$$

and

$$\liminf_{r\to\infty} \frac{1}{r^2} \int_0^{\pi} \log^+|f(re^{i\varphi})| \sin\varphi \, d\varphi = 0. \tag{18}$$

If, moreover, $f \in L^2(\mathbf{R})$, then $f \in H^2$.

Proof of Theorem 13. Let $E_n(z)=1-z/\bar{z}_n$ and, thus, $E=\prod_{n=1}^{\infty} E_n$. Note that $|E_n(iy)|>1$, $y>0$. Hence, the function $f=1/E$ satisfies (17).

Now we verify the property (18) for $f=1/E$. We need some technical remarks. First of all,

$$|E_n(re^{i\varphi})| \ge \left|1-\frac{r}{|z_n|}\right|. \tag{19}$$

On the other hand, for $\varphi\in[0,\pi]$ we have

$$|E_n(re^{i\varphi})|^2 = 1+\frac{2ry_n\sin\varphi-2rx_n\cos\varphi+r^2}{|z_n|^2} \ge 1-2\frac{r|\cos\varphi|}{|z_n|}+\frac{r^2}{|z_n|^2}. \tag{20}$$

Put

$$\Sigma_1(re^{i\varphi}) = \sum_{r/2\le|z_n|\le 2r} \log^+ \frac{1}{|E_n(re^{i\varphi})|},$$

$$\Sigma_2(re^{i\varphi}) = \sum_{|z_n|>2r} \log^+ \frac{1}{|E_n(re^{i\varphi})|}.$$

Clearly, $|E_n(re^{i\varphi})|>1$ whenever $r>2|z_n|$. Therefore,

$$\log^+ \frac{1}{|E(re^{i\varphi})|} \le \sum_{n=1}^{\infty} \log^+ \frac{1}{|E_n(re^{i\varphi})|} \le \Sigma_1+\Sigma_2.$$

Note that $\log^+ s\le|\log t|$ whenever $s\le t$. Hence, applying the estimate (19) and the elementary inequality $\log(1-u)\ge -2u$, $u\in\left(0,\frac{1}{2}\right)$, we get

$$\Sigma_2(re^{i\varphi}) \le \sum_{|z_n|>2r} \left|\log\left(1-\frac{r}{|z_n|}\right)\right| \le \sum_{|z_n|>2r} \frac{2r}{|z_n|} \le C_1 r.$$

Analogously, by (20),

$$\Sigma_1(re^{i\varphi}) \le \frac{1}{2} \sum_{r/2\le|z_n|\le 2r} \left|\log\left(1-2\frac{r\cos\varphi}{|z_n|}+\frac{r^2}{|z_n|^2}\right)\right|.$$

It is easy to see that there is an absolute constant C_2 such that for any $\delta\in\left[\frac{1}{2},2\right]$ we have

$$\int_0^{\pi} |\log(1-2\delta|\cos\varphi|+\delta^2)| \sin\varphi \, d\varphi = 2\int_0^1 |\log(1-2\delta t+\delta^2)| \, dt \le C_2.$$

Hence,

$$\int_0^\pi \Sigma_1(re^{i\varphi}) \sin\varphi \, d\varphi \le C_2 n(2r)$$

(note that $n(t)=o(t)$, as $t\to\infty$). The latter estimate together with the inequality $\Sigma_2(re^{i\varphi})\le C_1 r$ imply (18). The proof is completed. $\square$

6. Hypercyclic operators in the de Branges spaces

To prove Theorem 5 we need an auxiliary class of spaces introduced in [12]. Let $\gamma(z)=\sum_{n=0}^\infty \gamma_n z^n$ be an entire function such that $\gamma_n>0$ for each $n\ge 0$ and the sequence $n\gamma_n/\gamma_{n-1}$ decreases when n tends to infinity (in this case we say that γ is an admissible comparison function). Consider the space $E^2(\gamma)$ of all entire functions $f(z)=\sum_{n=0}^\infty f_n z^n$ for which

$$\|f\|_{2,\gamma}^2 = \sum_{n=0}^\infty \gamma_n^{-2} |f_n|^2 < \infty.$$

Clearly, $E^2(\gamma)$ endowed with the norm $\|\cdot\|_{2,\gamma}$ is a Hilbert space. Moreover, it is easy to see that the condition $\sup_n n\gamma_n/\gamma_{n-1}<\infty$ implies that the differentiation operator $\mathcal{D}$ is bounded on $E^2(\gamma)$. K. C. Chan and J. H. Shapiro have shown that translations are hypercyclic in $E^2(\gamma)$; they also have obtained a much more general result.

Theorem. (Chan, Shapiro [12]) *Suppose that X is a Fréchet space of entire functions with the following properties*:

1. $\mathcal{P}\subset X$ *and* $\mathrm{Clos}_X \mathcal{P}=X$;

2. *the topology of X is stronger than the topology of uniform convergence on compact subsets of the plane*;

3. *the operator T_w is continuous on X*;

4. *$E^2(\gamma)\subset X$ for some admissible comparison function γ.*

Then T_w, $w\ne 0$, is hypercyclic on X.

To apply the Chan–Shapiro theorem to the de Branges spaces satisfying the conditions of Theorem 5 (that is, $\mathrm{Clos}_E \mathcal{P}=\mathcal{H}(E)$ and $\mathcal{D}$ is bounded in $\mathcal{H}(E)$) we need to verify only the last condition. Let $r_n=\|z^n\|_{\mathcal{H}(E)}$ and take a sequence $\gamma_n>0$ such that the sequence $n\gamma_n/\gamma_{n-1}$ decreases and $\{r_n\gamma_n\}_{n=0}^\infty\in l^2(\mathbf{Z}_+)$. Then $\gamma(z)=\sum_{n=0}^\infty \gamma_n z^n$ is an admissible comparison function.

We show that $E^2(\gamma)\subset\mathcal{H}(E)$. Let $f\in E^2(\gamma)$, $f(z)=\sum_{n=0}^{\infty} f_n z^n$. Then, for any $y>0$, we have

$$\left\|\frac{f}{E}(\,\cdot\,+iy)\right\|_2 \le \sum_{n=0}^{\infty}|f_n|\left\|\frac{(\,\cdot\,+iy)^n}{E(\,\cdot\,+iy)}\right\|_2 \le \sum_{n=0}^{\infty}|f_n|r_n,$$

since the functions z^n/E are in the Hardy class $H^2(\mathbf{C}^+)$. Hence,

$$\sup_{y>0}\left\|\frac{f}{E}(\,\cdot\,+iy)\right\|_2 \le \|f\|_{2,\gamma}\|r_n\gamma_n\|_{l^2},$$

and $f/E\in H^2(\mathbf{C}^+)$. Analogously, $f^*/E\in H^2(\mathbf{C}^+)$. Thus, we get the inclusion $E^2(\gamma)\subset\mathcal{H}(E)$, and Theorem 5 is proved.

Remark. As shown in [12], an interesting feature of the spaces $E^2(\gamma)$ is that they provide examples of hypercyclic operators which are compact or even Schatten–von Neumann class perturbations of the identity operator I. Indeed, choosing the sequence γ_n tending to zero rapidly one can ensure that the differentiation operator in $E^2(\gamma)$ belongs to all Schatten–von Neumann classes and so does the operator T_w-I.

The same is true also for "larger" de Branges spaces. In [5] a number of examples are constructed showing that the operator $\mathcal{D}$ in $\mathcal{H}(E)$ may be compact or belong to all Schatten–von Neumann classes. In particular, $\mathcal{D}$ is always compact if the zeros of E lie in a Stolz angle.

Note added in proof. One more proof of Theorem 13 (a) is given in [6], where it is also shown that minimal (but not necessarily positive and continuous) admissible majorants exist for any model subspace K_Θ. Minimal admissible majorants for the de Branges spaces are discussed in detail in a recent preprint [7].

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