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Lower estimates for integral means of univalent functions

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Let Ω be a simply connected domain which is a proper subset of the complex plane $\mathbf{C}$. Then by the Riemann mapping theorem there exists a function f which maps the unit disk $\mathbf{D}$ centered at the origin conformally onto Ω. An important characteristic of the boundary properties of Ω is the integral means spectrum (see [1], [5], [6]):

$$\beta_f(t) = \limsup_{r\to 1} \frac{\log \int_0^{2\pi} |f'(re^{i\theta})|^t \, d\theta}{\log[1/(1-r)]}.$$

We also define the universal integral means spectrum [5]:

$$B(t) = \sup_f \beta_f(t),$$

where the supremum is taken over all conformal maps from $\mathbf{D}$ into $\mathbf{C}$.

Good upper estimates for $B(t)$ were given by Pommerenke (see [6]). Recently, Hedenmalm and Shimorin (see [2] and [7]) obtained new upper estimates for $B(t)$. In particular, they showed that

$$B(t) < 0.44t^2 \quad \text{for small } t.$$

On the other hand, Makarov [4] proved that there exists a positive constant c such that $B(t) > ct^2$ for t near the origin. This estimate was improved. Namely, it was shown (see [6], p. 178) that $B(t) \geq 0.117t^2$ for small t.

Carleson and Jones conjectured [1] that the extremal domains for $B(t)$ are the basins of attraction of infinity for quadratic polynomials. Using this idea, Kraetzer [3] obtained experimental inequalities

$$B(t) \geq \frac{t^2}{4}, \quad -2 \leq t \leq 2.$$

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In this paper we analytically show that

$$B(t) > \frac{t^2}{5}, \quad 0 < t \le \frac{2}{5}.$$

In the paper the following notation

$$\sum_{k=0}^{\infty} a_k z^k \gg \sum_{k=0}^{\infty} b_k z^k \quad \text{will be used if } a_k \ge b_k \ge 0, \quad k \ge 0.$$

Let $q \ge 2$ be a natural number.

Theorem. *Suppose that a function f is univalent and bounded in $\mathbf{D}$, and moreover that $\log(zf'(z)/f(z)) \gg 0$. Then there exists a function g univalent and bounded in $\mathbf{D}$ such that*

$$\beta_g(t) \ge \frac{1}{\log q} \sum_{k=1}^{q-1} \log I_0\left(a_k t M^{-k/(q-1)}\right),$$

where a_k are the Taylor coefficients of $\log(zf'(z)/f(z))$,

$$M = \sup_{|z|<1} |f(z)| = f(1)$$

and

$$I_0(x) = \sum_{\nu=0}^{\infty} \frac{(x^2/4)^\nu}{\nu!^2}$$

is the modified Bessel function of order zero.

Proof. First of all it will be shown that for any natural number $q \ge 2$ there exists a function g univalent and bounded in $\mathbf{D}$ such that

$$\log g'(z) \ge \sum_{k=0}^{\infty} \sum_{j=1}^{q-1} a_j \left(\frac{1}{M}\right)^{j/(q-1)} z^{q^k j}.$$

We introduce the functions

$$f_m(z) = \left(\frac{1}{M}\right)^{1/m} \sqrt[m]{f(z^m)}, \quad m = 1, 2, \dots .$$

From $\log(zf'(z)/f(z)) \gg 0$ follows that $zf'(z)/f(z) \gg 0$. By integration of the last inequality one can obtain $\log(f(z)/z) \gg 0$, and hence $f_m(z) \gg 0$. Let us also remark that

$$\log f'_m(z)-\frac{1}{m}\log\left(\frac{1}{M}\right)=\log\left(\frac{z^m f'(z^m)}{f(z^m)}\right)+\frac{1}{m}\log\left(\frac{f(z^m)}{z^m}\right)$$
$$\gg\log\left(\frac{z^m f'(z^m)}{f(z^m)}\right)\gg 0.$$

It is easy to see that the function

$$g(z)=M^{q/(q-1)}\lim_{n\to\infty} f_1\circ f_q\circ f_{q^2}\circ\ldots\circ f_{q^n}$$

is well defined, bounded, and univalent in $\mathbf{D}$, and

$$\log g'(z)=\sum_{k=0}^{\infty}\log f'_{q^k}\left(z\left(\frac{1}{M}\right)^{1/q^{k+1}+1/q^{k+2}+\ldots}+\ldots\right)$$
$$=\sum_{k=0}^{\infty}\log f'_{q^k}\left(z\left(\frac{1}{M}\right)^{1/(q-1)q^k}+\ldots\right)$$
$$\gg\sum_{k=0}^{\infty}\log\frac{w_k^{q^k} f'\left(w_k^{q^k}\right)}{f\left(w_k^{q^k}\right)},\qquad w_k=z\left(\frac{1}{M}\right)^{1/((q-1)q^k)},$$

form which it follows that

$$\log g'(z)\gg\sum_{k=0}^{\infty}\sum_{j=1}^{\infty}a_j\left(\frac{1}{M}\right)^{j/(q-1)}z^{q^k j}.$$

Let us define the function h via the following equality

$$\log h'(z)=\sum_{k=0}^{\infty}\sum_{j=1}^{q-1}a_j\left(\frac{1}{M}\right)^{j/(q-1)}z^{q^k j}. \tag{1}$$

It is known [6, p. 191] that

$$\exp(x\cos\theta)=I_0(x)+2\sum_{n=1}^{\infty}I_n(x)\cos(n\theta), \tag{2}$$

where

$$I_n(x)=\left(\frac{x}{2}\right)^n\sum_{\nu=0}^{\infty}\frac{(x^2/4)^\nu}{\nu!(\nu+n)!},\quad n=0,1,2,\ldots,$$

are modified Bessel functions. Due to (1) and (2) one can show that

$$\begin{aligned}\int_0^{2\pi} |h'(re^{i\theta})|^t \, d\theta &= \int_0^{2\pi} \exp\left[t \sum_{k=0}^{\infty} \sum_{j=1}^{q-1} r^{jq^k} a_j \left(\frac{1}{M}\right)^{j/(q-1)} \cos(q^k j\theta)\right] d\theta \\ &= \int_0^{2\pi} \prod_{k=0}^{\infty} \prod_{j=1}^{q-1} \exp\left[t r^{jq^k} a_j \left(\frac{1}{M}\right)^{j/(q-1)} \cos(q^k j\theta)\right] d\theta \\ &= \int_0^{2\pi} \prod_{k=0}^{\infty} \prod_{j=1}^{q-1} \Bigg(I_0\left[t r^{jq^k} a_j \left(\frac{1}{M}\right)^{j/(q-1)}\right] \\ &\quad + 2 \sum_{n=1}^{\infty} I_n \left[t r^{jq^k} a_j \left(\frac{1}{M}\right)^{j/(q-1)}\right] \cos(nq^k j\theta) \Bigg) \, d\theta.\end{aligned}$$

We expand and then integrate term-by-term.

Rohde [6, p. 191] showed that

$$\int_0^{2\pi} \prod_{k=1}^{m} \cos(n_k t) \, dt \geq 0 \quad \text{for } n_1, n_2, \ldots, n_m \in \mathbf{Z}.$$

It means that

$$\int_0^{2\pi} |h'(re^{i\theta})|^t \, d\theta \geq 2\pi \prod_{k=0}^{\infty} \prod_{j=1}^{q-1} I_0\left[t r^{jq^k} a_j \left(\frac{1}{M}\right)^{j/(q-1)}\right].$$

Therefore,

$$\begin{aligned}\beta_h(t) &\geq \limsup_{r\to 1} \frac{1}{\log[1/(1-r)]} \sum_{k=0}^{\infty} \sum_{j=1}^{q-1} \log I_0\left(t r^{jq^k} a_j M^{-j/(q-1)}\right) \\ &= \limsup_{n\to\infty} \frac{1}{\log[q^n(q-1)]} \sum_{k=0}^{n} \sum_{j=1}^{q-1} \log I_0\left(t a_j M^{-j/(q-1)}\right) \\ &= \frac{1}{\log q} \sum_{j=0}^{q-1} \log I_0\left(t a_j M^{-j/(q-1)}\right).\end{aligned}$$

It is clear that $(g')^{t/2} \gg (h')^{t/2}$ because $\log g' \gg \log h'$ and the Taylor coefficients of e^x are positive. Hence by Parseval's identity,

$$\int_0^{2\pi} |g'(re^{i\theta})|^t \, d\theta \geq \int_0^{2\pi} |h'(re^{i\theta})|^t \, d\theta,$$

which implies the inequality $\beta_g(t) \geq \beta_h(t)$. The theorem is proved. □

Corollary 1. *There exists a function g bounded, analytic, and univalent in the unit disk* $\mathbf{D}$ *such that*

$$\beta_g(t) > \frac{t^2}{5}, \quad 0 < t \le \frac{2}{5}.$$

Proof. Let us show that the function

$$f(z) = z\exp\int_0^z \frac{\exp[(a/b)\sinh(bt)] - 1}{t}\, dt. \tag{3}$$

is univalent in $\mathbf{D}$ for $a{=}1.906$ and $b{=}1.24$.

The Taylor coefficients of the function (3) are real and hence it is enough to show that f is univalent in D_+ and $f(D_+)\cap\mathbf{R}=\varnothing$, where $D_+=\{z{:}z\in\mathbf{D}, \Im z>0\}$. Due to (3)

$$\frac{d}{d\theta}\log f(e^{i\theta}) = i\exp\left[\frac{a}{b}\sinh(be^{i\theta})\right],$$

which means that

$$\begin{aligned}\frac{d|f|}{d\theta} &= -|f|\Im\exp\left[\frac{a}{b}\sinh(be^{i\theta})\right]\\ &= -|f|\exp\left[\frac{a}{b}\sinh(b\cos\theta)\cos(b\sin\theta)\right]\sin\left[\frac{a}{b}\cosh(b\cos\theta)\sin(b\sin\theta)\right].\end{aligned}$$

It is evident that for $\theta\in(0,\pi)$ and fixed above parameters a and b the following inequality holds

$$0 < \frac{a}{b}\cosh(b\cos\theta)\sin(b\sin\theta) \le \frac{a}{b}\cosh b\sin b = 2.72... < \pi.$$

Hence

$$\frac{d|f|}{d\theta} < 0, \quad 0 < \theta < \pi.$$

Thus, the injectivity of the function (3) in D_+ is proved.

Consider now the equation

$$\frac{d\arg f(e^{i\theta})}{d\theta} = \exp\left[\frac{a}{b}\sinh(b\cos\theta)\cos(b\sin\theta)\right]\cos\left[\frac{a}{b}\cosh(b\cos\theta)\sin(b\sin\theta)\right] = 0$$

which holds if

$$\frac{a}{b}\cosh(b\cos\theta)\sin(b\sin\theta) = \frac{\pi}{2}.$$

The last equation has four roots $\theta_1{=}0.61469...$, $\theta_2{=}1.98524...$, $\theta_3{=}\pi-\theta_2$ and $\theta_4{=}\pi-\theta_1$ on the interval $(0,\pi)$. It is clear that the inequalities $\Im f(e^{i\theta_1}){>}0$, $\Im f(e^{i\theta_2}){>}0$ imply that $\Im f(z){>}0$ for all $z\in D_+$. Calculations show that $\Im f(e^{i\theta_1}){=}0.00170...{>}0$ and $\Im f(e^{i\theta_2}){=}0.00726...{>}0$.

Therefore the function f is univalent in $\mathbf{D}$.

The Taylor coefficients of $\log(zf'(z)/f(z))$ are positive because

$$\log\left(\frac{zf'(z)}{f(z)}\right)=\frac{a}{b}\sinh(bz)=a\sum_{k=0}^{\infty}\frac{b^{2k}z^{2k+1}}{(2k+1)!}\gg 0.$$

Let $q=30$. From the proved theorem follows that there exists a univalent function g such that

$$\beta_g(t)\geq\frac{1}{\log 30}\sum_{k=0}^{15}\log I_0\left(\frac{ab^{2k}t}{(2k+1)!}M^{-(2k+1)/29}\right),$$

where $M=f(1)=72.8884...$. Computations show that $\beta_g(t)>0.2007t^2$ for $t\in\left(0,\frac{2}{5}\right]$. This concludes the proof of Corollary 1. □

Makarov [4] showed that $B(t)>3t-1$ for $t\in\left(0,\frac{1}{3}+\varepsilon\right]$ for some small $\varepsilon>0$.

Corollary 2. $B(t)>3t-1$ *for* $t\leq(15-\sqrt{205})/2=0.341...$.

The proof immediately follows from Corollary 1.

It is interesting to remark that there is a conjecture that $B(t)>3t-1$ for $t<6-4\sqrt{2}=0.343...$, and $B(t)=3t-1$ for $t\geq 6-4\sqrt{2}$.

Remark. Our theorem shows that the first Taylor coefficient a_1 of $\log(zf'(z)/f(z))$ plays very important role for lower estimates of $B(t)$. It is known that $|a_1|\leq 2$. Unfortunately, there are no bounded functions in the disk $\mathbf{D}$ for which $a_1=2$ because the equality $a_1=2$ immediately implies that $f(z)=z/(1-z)^2$. The function f constructed in Corollary 1 realizes our attempt to obtain quite big a_1 (actually close to 2) with not so big M. Of course, it is possible to improve our lower estimates for $B(t)$ by constructing other examples. However, in our opinion, it would be quite difficult to reach the experimental estimates $B(t)\geq t^2/4$ by using our theorem.

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