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# Sharp integral estimates for the fractional maximal function and interpolation

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**Abstract.** We give sharp estimates for the fractional maximal function in terms of Hausdorff capacity. At the same time we identify the real interpolation spaces between $L_1$ and the Morrey space $\mathcal{L}^{1,\lambda}$. The result can be viewed as an analogue of the Hardy–Littlewood maximal theorem for the fractional maximal function.

## 1. Introduction

Quite a significant role in estimation of various operators in analysis is played by the Hardy–Littlewood maximal function $Mf$. There are a lot of papers devoted to properties of $Mf$, its variants, and their applications.

One of the most important variants of the Hardy–Littlewood maximal function is the so-called fractional maximal function defined by the formula

$$M_\lambda f(x) = \sup_{Q \ni x} \frac{1}{|Q|^{1-\lambda/n}} \int_Q |f(y)|\, dy, \quad 1-\frac{\lambda}{n} \in (0,1]. \tag{1.1}$$

It coincides with the Hardy–Littlewood maximal function $Mf$ if $\lambda=0$, and is intimately related to the Riesz potential operator

$$I_\lambda f(x) = \int \frac{f(y)}{|x-y|^{n-\lambda}}\, dy,$$

(see, for example, [1] and [25]).

Our main goal in this paper is to establish analogues of some important properties of the Hardy–Littlewood maximal function for the fractional maximal function.

Maybe, the most important result for the Hardy–Littlewood maximal function is the Hardy–Littlewood maximal theorem (see [15])

$$\|Mf\|_{L_p} \le c\|f\|_{L_p}, \quad p>1, \tag{1.2}$$

with the constant $c>0$ independent of $f\in L_p$. It may seem that, for the fractional maximal operator, a good analogue of the Hardy–Littlewood maximal theorem is the inequality

$$\|M_\lambda f\|_{L_q} \le c\|f\|_{L_p} \quad \text{for } p\in\left(1,\frac{n}{\lambda}\right) \text{ and } \frac{1}{q}=\frac{1}{p}-\frac{\lambda}{n}, \tag{1.3}$$

which is equivalent to the Sobolev inequality (see [25]). However, in the Hardy–Littlewood maximal theorem, we have equivalence

$$\|Mf\|_{L_p}\approx\|f\|_{L_p}, \quad p>1.$$

In Theorem 5 below we show that there is no equivalence in (1.3). Moreover, in Theorem 4 we shall present a couple of spaces $X$ and $Y$ such that for $p$ and $q$ as above

$$L_p\subset X,\, Y\subset L_q \quad \text{and} \quad M_\lambda\colon X\longrightarrow Y,$$

and the equivalence occurs:

$$\|M_\lambda f\|_Y\approx\|f\|_X.$$

This result is related to our answer to the problem: what is a good analogue of the Hardy–Littlewood maximal theorem for the fractional maximal function? Our approach is based on covering arguments and real interpolation.

## 2. Preliminaries

In this section we recall some well-known notation and results to be used later.

We start with some notation and definitions. We consider $\mathbf{R}^n$ with the norm

$$|x|=\max_{1\le i\le n}|x_i|, \quad \text{where } x=(x_1,\dots,x_n).$$

Below, by $Q=Q(x,r)=\{y\in\mathbf{R}^n \mid |x-y|\le r\}$, $r>0$, we denote the cube (closed cubic interval) with center $x$, radius $r$, and sides parallel to the coordinate axes. As usual, by $\mathring{Q}$ we denote the interior of the cube $Q$ and by $\lambda Q(x,r)$, where $\lambda>0$, the cube $Q(x,\lambda r)$, i.e. the cube with the same centre $x$ and of radius $\lambda r$.

We say that cubes $Q_1$ and $Q_2$ are *disjoint* if $\mathring{Q}_1\cap\mathring{Q}_2=\varnothing$.

By a *packing* $\pi$ we mean a set of disjoint cubes. By $|Q|$ we denote the $n$-dimensional volume of the cube $Q$, i.e. $|Q(x,r)|:=(2r)^n$.

Below we will usually denote by $\Omega$ a set of points and by $\overline{\Omega}$ a set of cubes.

We will use the following covering lemma (see [35, Lemma C′], or the $5r$-covering theorem in [22, p. 23]).

**Lemma 1.** *Let* $\overline{\Omega}=\{Q_j\}_{j\in J}$ *be a set of a cubes such that* $\sup_J |Q_j|<\infty$. *Then there exists a packing* $\pi\subset\overline{\Omega}$ *such that*

$$\bigcup_{Q\in\overline{\Omega}} Q\subset \bigcup_{Q\in\pi} 5Q.$$

*Definition* 1. For $1-\lambda/n\in(0,1]$, we say that a function $f\in L_{1,\mathrm{loc}}(\mathbf{R}^n)$ belongs to the Morrey space $\mathcal{L}^{1,\lambda}$ if $M_\lambda f\in L_\infty$, and

$$\|f\|_{\mathcal{L}^{1,\lambda}}=\|M_\lambda f\|_{L_\infty}=\sup_{Q\subset\mathbf{R}^n}\frac{1}{|Q|^{1-\lambda/n}}\int_Q |f(y)|\,dy<\infty.$$

For $\lambda=0$ the fractional maximal function coincides with the Hardy–Littlewood maximal function and the Morrey space coincides with the space $L_\infty$ with equality of norms(1). In contrast to the paper [6] we consider everywhere below just functions and not Radon measures.

The importance of the Hausdorff measure is well-known (see, for example [10]). In estimating it one often uses the so-called Hausdorff capacity (see, for example [5] and [10]).We recall that the Hausdorff capacity $\Lambda^{(\infty)}_{n-\lambda}$ for the set $\Omega\subset\mathbf{R}^n$ is defined by the expression

$$\Lambda^{(\infty)}_{n-\lambda}(\Omega)=\inf_{\{Q_i\}_{i\in I}}\left(\sum_{i\in I}|Q_i|^{1-\lambda/n}\right),$$

where the infimum is taken over all families $\{Q_i\}_{i\in I}$ of cubes that cover the set $\Omega$, i.e. $\Omega\subset\bigcup_{i\in I}Q_i$ (see [3] and [5]).

Moreover, if in the above definition of $\Lambda^{(\infty)}_{n-\lambda}$ we take the infimum over all coverings of $\Omega$ by dyadic cubes $Q_i$, then we get the *dyadic* Hausdorff capacity $\tilde{\Lambda}^{(\infty)}_{n-\lambda}$. We will use the easily proved fact that if $\tilde{\Lambda}^{(\infty)}_{n-\lambda}$ is finite then in the definition of the dyadic Hausdorff capacity it is enough to take packings.

The following well-known lemma will be important for us (see, for example [3]).

**Lemma 2.** *Hausdorff capacity and dyadic Hausdorff capacity are equivalent. More precisely,*

$$\Lambda^{(\infty)}_{n-\lambda}(\Omega)\le\tilde{\Lambda}^{(\infty)}_{n-\lambda}(\Omega)\le 2^{n(2-\lambda/n)}\Lambda^{(\infty)}_{n-\lambda}(\Omega).$$

The left hand-side inequality is immediate. For the right hand-side inequality it is enough to let $\{Q_D\}_D$ be the set of all dyadic cubes such that

$$|Q|<|Q_D|\le 2^n|Q|,$$

and observe that $Q$ is covered by a union of $2^n$ cubes from $\{Q_D\}_D$.

(1) Morrey spaces were introduced by C. B. Morrey in 1938 in connection with certain problems in elliptic partial differential equations and calculus of variations (see [24]). Later, Morrey spaces found important applications to Navier–Stokes ([23], [34]) and Schrödinger ([26], [28], [29], [31], [30]) equations, elliptic problems with discontinuous coefficients ([9], [12]), and potential theory ([1], [4]). An exposition of the Morrey spaces can be found in the book [19].

We finish this section with the notion of the distance function. Let $(X_0, X_1)$ be two Banach or quasi-Banach spaces linearly and continuously embedded in some linear topological space $X$. If $f \in X_0 + X_1$, we denote by

$$\rho(f, B_{X_1}(t))_{X_0} = \inf_{\|g\|_{X_1} \leq t} \|f - g\|_{X_0}$$

the distance from $f$ to the ball of $X_1$ of radius $t$ in the metric of $X_0$.

For example, for the couple $(L_1, L_\infty)$ the distance function is evidently equal to

$$\rho(f, B_{L_\infty}(t))_{L_1} = \int_{\mathbf{R}^n} (|f(x)| - t)_+ \, dt.$$

In the theory of real interpolation it is known that the so-called $(\theta, q)$-interpolation spaces can be defined in terms of the distance function (see [8, Theorem 7.1.7, p. 178]). In particular, the norm in the space $(X_0, X_1)_{1-1/p,p}$ can be defined by the expression

$$\|f\|_{(X_0,X_1)_{1-1/p,p}} = \left( \int_0^\infty \rho(f, B_{X_1}(t))_{X_0} t^{p-1} \frac{dt}{t} \right)^{1/p}. \tag{2.1}$$

For example, the space $L_p = (L_1, L_\infty)_{1-1/p,p}$ can be defined by using the distance function $\rho(f, B_{L_\infty}(t))_{L_1}$.

## 3. Wiener–Stein equivalence and Hausdorff capacity

In this section we show that Hausdorff capacity naturally arises in one possible generalization of the so-called Wiener–Stein equivalence for the fractional maximal function. From now on, $1 - \lambda/n \in (0, 1]$ is fixed.

The following definition (see [18]) will be important for us.

*Definition* 2. Let $\overline{\Omega}$ be a set of cubes. We define the $(1-\lambda/n)$-capacity of the set $\overline{\Omega}$ by

$$|\overline{\Omega}|_{1-\lambda/n} = \sup_{\pi \subset \overline{\Omega}} |\pi|_{1-\lambda/n} = \sup_{\pi \subset \overline{\Omega}} \sum_{Q \in \pi} |Q|^{1-\lambda/n},$$

where the supremum is taken over all packings $\pi$ of cubes in $\overline{\Omega}$.

Let $t \in \mathbf{R}_+$, and let $f \in L_1 + \mathcal{L}^{1,\lambda}$. We define the following set of points:

$$\Omega_t = \{x \mid (M_\lambda f)(x) > t\}, \tag{3.1}$$

and the set of cubes:

$$\overline{\Omega}_t = \left\{ Q \,\middle|\, \frac{1}{|Q|^{1-\lambda/n}} \int_Q |f(y)| \, dy > t \right\}. \tag{3.2}$$

It is clear that

$$\Omega_t = \bigcup_{Q\in\overline{\Omega}_t} Q.$$

We shall see that the $(1-\lambda/n)$-capacity of the set $\overline{\Omega}_t$ is intimately related to the quantity

$$\rho(f, B_{\mathcal{L}^{1,\lambda}}(t))_{L_1} = \inf_{\|g\|_{\mathcal{L}^{1,\lambda}}\le\, t} \|f-g\|_{L_1},$$

i.e. to the distance (in the metric of $L_1$) from the function $f$ to the ball of radius $t$ of the Morrey space $\mathcal{L}^{1,\lambda}$.

**Theorem 1.** *Suppose $t\in\mathbf{R}_+$ and $f\in L_1+\mathcal{L}^{1,\lambda}$ is such that $\rho(f, B_{\mathcal{L}^{1,\lambda}}(t))_{L_1}$ is finite for all $t>0$. Let $\overline{\Omega}_t$ be the set of cubes defined by* (3.2). *Then $t|\overline{\Omega}_t|_{1-\lambda/n}\approx \rho(f, B_{\mathcal{L}^{1,\lambda}}(t))_{L_1}$ in the sense that*

$$5^{-n(1-\lambda/n)}\rho(f, B_{\mathcal{L}^{1,\lambda}}(t))_{L_1} \le t|\overline{\Omega}_t|_{1-\lambda/n} \le 2\rho\left(f, B_{\mathcal{L}^{1,\lambda}}\left(\frac{t}{2}\right)\right)_{L_1}.$$

*Proof.* Let $\varepsilon>0$ be an arbitrary small real number. Then, by the definition of the $(1-\lambda/n)$-capacity, there exists a packing $\pi_0$ of disjoint cubes belonging to $\overline{\Omega}_t$ such that

$$|\pi_0|_{1-\lambda/n} = \sum_{Q\in\pi_0} |Q|^{1-\lambda/n} \ge \frac{1}{1+\varepsilon}|\overline{\Omega}_t|_{1-\lambda/n}, \tag{3.3}$$

and for every cube $Q\in\pi_0$ we have

$$\frac{1}{|Q|^{1-\lambda/n}} \int_Q |f(y)|\,dy > t. \tag{3.4}$$

Let $h$ be a function such that $\|h\|_{\mathcal{L}^{1,\lambda}}\le t/2$. Then, by Definition 1, we have

$$\frac{1}{|Q|^{1-\lambda/n}} \int_Q |h(y)|\,dy \le \frac{t}{2}$$

for every cube $Q$. From this and (3.4), it follows that

$$\begin{aligned}\|f-h\|_{L_1} &\ge \sum_{Q\in\pi_0} \int_Q |f(y)|\,dy - \sum_{Q\in\pi_0} \int_Q |h(y)|\,dy \\ &\ge t \sum_{Q\in\pi_0} |Q|^{1-\lambda/n} - \frac{t}{2} \sum_{Q\in\pi_0} |Q|^{1-\lambda/n} = \frac{t}{2} \sum_{Q\in\pi_0} |Q|^{1-\lambda/n}.\end{aligned}$$

Now, (3.3) implies that

$$\frac{t}{1+\varepsilon}|\overline{\Omega}_t|_{1-\lambda/n} < t \sum_{Q\in\pi_0} |Q|^{1-\lambda/n} \le 2 \inf_{\|h\|_{\mathcal{L}^{1,\lambda}}\le t/2} \|f-h\|_{L_1} = 2\rho\left(f, B_{\mathcal{L}^{1,\lambda}}\left(\frac{t}{2}\right)\right)_{L_1}.$$

Hence, letting $\varepsilon\to 0$, we obtain

$$t|\overline{\Omega}_t|_{1-\lambda/n} \le 2\rho\left(f, B_{\mathcal{L}^{1,\lambda}}\left(\frac{t}{2}\right)\right)_{L_1}. \tag{3.5}$$

Now we prove the reverse inequality. We define a function $g$ by

$$g(x) = f(x)\chi_{\mathbf{R}^n\setminus\Omega_t}(x).$$

Observe that $\|g\|_{\mathcal{L}^{1,\lambda}}\le t$. Indeed, let $Q$ be an arbitrary cube. In the case when $Q\subset\Omega_t$, we have trivially

$$\frac{1}{|Q|^{1-\lambda/n}}\int_Q |g(y)|\,dy = 0.$$

If $Q\cap(\mathbf{R}^n\setminus\Omega_t)\neq\varnothing$, then $Q\notin\overline{\Omega}_t$. Therefore,

$$\frac{1}{|Q|^{1-\lambda/n}}\int_Q |g(y)|\,dy \le \frac{1}{|Q|^{1-\lambda/n}}\int_Q |f(y)| \le t.$$

Thus, $\|g\|_{\mathcal{L}^{1,\lambda}}\le t$ and we see that

$$\rho(f, B_{\mathcal{L}^{1,\lambda}}(t))_{L_1} \le \|f-g\|_{L_1} = \|f\|_{L_1(\Omega_t)}. \tag{3.6}$$

For estimating the last quantity, we use a "limiting cubes" construction, used previously in [18]. Consider the following set of points:

$$C = \{x\in\Omega_t \mid \text{there exists } Q(x,r)\in\overline{\Omega}_t\}.$$

For any element $x\in C$ we construct a "limiting" cube $Q(x,r_x)$ in the following way. First, we define the function

$$\varphi_x(r) = \frac{1}{|Q(x,r)|^{1-\lambda/n}}\int_{Q(x,r)} |f(y)|\,dy, \quad r>0.$$

From the continuity property of the Lebesgue integral (see e.g. [13, Theorem 8(e), p. 323]) it follows that $\varphi_x$ is a continuous function on $\mathbf{R}_+$, and it tends to zero as $r$ tends to infinity (since $f\in L_1+\mathcal{L}^{1,\lambda}$ is such that $\rho(f,B_{\mathcal{L}^{1,\lambda}}(t))_{L_1}$ is finite for all $t>0$). Moreover, from $x\in C$ we immediately obtain

$$\sup_{r>0}\varphi_x(r) > t.$$

Hence, the number

$$r_x = \sup\{r \in \mathbf{R}_+ \mid \varphi_x(r) > t\} \tag{3.7}$$

is positive and finite for any $x \in C$. Therefore, we can consider the family of "limiting" cubes $\{Q(x, r_x)\}_{x\in C}$. From (3.7) we have three important properties:

(i) for all $x \in C$ we have

$$\frac{1}{|Q(x, cr_x)|^{1-\lambda/n}} \int_{Q(x,cr_x)} |f(y)|\, dy \le t \quad \text{for } c \ge 1; \tag{3.8}$$

(ii) let $\varepsilon > 0$, then, for every $x \in C$, there exists $r'_x = r'_x(\varepsilon) > 0$ such that

$$Q(x, r'_x) \in \overline{\Omega}_t \quad \text{and} \quad r'_x < r_x < (1+\varepsilon) r'_x; \tag{3.9}$$

(iii) for each cube $Q \in \overline{\Omega}_t$ there exists a cube $\widetilde{Q} \in \{Q(x, r_x)\}_{x\in C}$ such that $Q \subset \widetilde{Q}$.

From (3.9) we obtain

$$\begin{aligned} \sum_{\pi \subset \{Q(x,r_x)\}_{x\in C}} |Q(x, r_x)|^{1-\lambda/n} &\le (1+\varepsilon)^{n(1-\lambda/n)} \sum_{\pi' \subset \overline{\Omega}_t} |Q(x, r'_x)|^{1-\lambda/n} \\ &\le (1+\varepsilon)^{n(1-\lambda/n)} |\overline{\Omega}_t|_{1-\lambda/n}. \end{aligned} \tag{3.10}$$

As from (3.5) it follows that $|\overline{\Omega}_t|_{1-\lambda/n} < \infty$, and thus

$$\sup_{x\in C} |Q(x, r_x)| < \infty.$$

Hence, by applying (iii) and Lemma 1, we obtain a packing $\pi \subset \{Q(x, r_x)\}_{x\in C}$ such that

$$\Omega_t \subset \bigcup_{x\in C} Q(x, r_x) \subset \bigcup_{Q\in\pi} 5Q. \tag{3.11}$$

From this, (3.6), (3.8), and (3.10), it follows that

$$\begin{aligned} \rho(f, B_{\mathcal{L}^{1,\lambda}}(t))_{L_1} &\le \|f\|_{L_1(\Omega_t)} \le \|f\|_{L_1(\bigcup_{Q\in\pi} 5Q)} \le \sum_{Q\in\pi} \int_{5Q} |f(y)|\, dy \\ &\le 5^{n(1-\lambda/n)} t \sum_{Q\in\pi} |Q|^{1-\lambda/n} \le 5^{n(1-\lambda/n)} (1+\varepsilon)^{n(1-\lambda/n)} t |\overline{\Omega}_t|_{1-\lambda/n}. \end{aligned}$$

By letting $\varepsilon \to 0$, we obtain

$$\rho(f, B_{\mathcal{L}^{1,\lambda}}(t))_{L_1} \le 5^{n(1-\lambda/n)} t |\overline{\Omega}_t|_{1-\lambda/n}. \quad \square$$

Now we can prove the main result of this section

**Theorem 2.** *Suppose* $t\in\mathbf{R}_+$ *and* $f\in L_1+\mathcal{L}^{1,\lambda}$ *is such that* $\rho(f,B_{\mathcal{L}^{1,\lambda}}(t))_{L_1}$ *is finite for all* $t>0$. *Then*

$$t\Lambda^{(\infty)}_{n-\lambda}(\{x\,|\,M_\lambda f(x)>t\})\approx\rho(f,B_{\mathcal{L}^{1,\lambda}}(t))_{L_1}$$

*in the sense that*

$$c_1\rho(f,B_{\mathcal{L}^{1,\lambda}}(t))_{L_1}\le t\Lambda^{(\infty)}_{n-\lambda}(\{x\,|\,M_\lambda f(x)>t\})\le c_2\rho\left(f,B_{\mathcal{L}^{1,\lambda}}\left(\frac{t}{2}\right)\right)_{L_1},$$

*where* $c_1=2^{-n(2-\lambda/n)}3^{-n(1-\lambda/n)}$, $c_2=2\cdot 5^{n(1-\lambda/n)}$

*Proof.* Let $\Omega_t$ and $\overline{\Omega}_t$ be the sets defined by (3.1) and (3.2), respectively. From Lemma 2 and Theorem 1, it follows that it suffices to prove the estimates

$$\Lambda^{(\infty)}_{n-\lambda}(\Omega_t)\le 5^{n(1-\lambda/n)}|\overline{\Omega}_t|_{1-\lambda/n} \tag{3.12}$$

and

$$\rho(f,B_{\mathcal{L}^{1,\lambda}}(t))_{L_1}\le 3^{n(1-\lambda/n)}t\tilde{\Lambda}^{(\infty)}_{n-\lambda}(\Omega_t). \tag{3.13}$$

We start with (3.12). From Theorem 1 we have

$$\sup_{Q\in\overline{\Omega}_t}|Q|<\infty;$$

therefore, Lemma 1 yields a packing $\pi$ that consists of cubes belonging to $\overline{\Omega}_t$ and such that

$$\Omega_t=\bigcup_{Q\in\overline{\Omega}_t}Q\subset\bigcup_{Q\in\pi}(5Q).$$

Hence, by the definition of $\Lambda^{(\infty)}_{n-\lambda}$, we obtain (3.12):

$$\Lambda^{(\infty)}_{n-\lambda}(\Omega_t)\le\sum_{Q\in\pi}|5Q|^{1-\lambda/n}=5^{n(1-\lambda/n)}\sum_{Q\in\pi}|Q|^{1-\lambda/n}\le 5^{n(1-\lambda/n)}|\overline{\Omega}_t|_{1-\lambda/n}.$$

Now we prove (3.13). From the finiteness of $\rho(f,B_{\mathcal{L}^{1,\lambda}}(t))_{L_1}$ for all $t>0$ and the theorem proved above it follows that $\Lambda^{(\infty)}_{n-\lambda}(\Omega_t)$ is also finite. So from Lemma 2 and the remark above it, it follows that for any $\varepsilon>0$ we can find a packing $\pi$ of dyadic cubes that covers $\Omega_t$ (i.e. $\Omega_t\subset\bigcup_{Q\in\pi}Q$) and satisfies

$$\sum_{Q\in\pi}|Q|^{1-\lambda/n}\le\tilde{\Lambda}^{(\infty)}_{n-\lambda}(\Omega_t)+\varepsilon. \tag{3.14}$$

Moreover, if some dyadic cube $Q$ and all those of its neighbours which belong to the next dyadic cube (i.e. the dyadic cube $\widetilde{Q}$ which contains $Q$ and has volume $2^n|Q|$) belong to $\pi$ then we will replace $Q$ and its neighbours by $\widetilde{Q}$. Clearly the quantity $\sum_{Q\in\pi}|Q|^{1-\lambda/n}$ will not increase in this process. Therefore without loss of generality, we may assume that all cubes $Q$ in $\pi$ are such that not all neighbours of $Q$ belongs to $\pi$ and so for all cubes in $\pi$ we have the property

$$3Q\cap(\mathbf{R}^n\backslash\Omega_t)\neq\varnothing. \tag{3.15}$$

Then, (3.15) and (3.2) imply that

$$\frac{1}{|3Q|^{1-\lambda/n}}\int_{3Q}|f(y)|\,dy\leq t,$$

whence we obtain $\|f\chi_{(\mathbf{R}^n\backslash\Omega_t)}\|_{\mathcal{L}^{1,\lambda}}\leq t$ by (3.1). Therefore,

$$\begin{aligned}\rho(f,B_{\mathcal{L}^{1,\lambda}}(t))_{L_1}&\leq\|f-f\chi_{\mathbf{R}^n\backslash\Omega_t}\|_{L_1}=\|f\chi_{\Omega_t}\|_{L_1}\leq\sum_{Q\in\pi}\int_Q|f(y)|\,dy\\&\leq\sum_{Q\in\pi}\int_{3Q}|f(y)|\,dy\leq t\sum_{Q\in\pi}|3Q|^{1-\lambda/n}\leq 3^{n(1-\lambda/n)}t(\tilde{\Lambda}^{(\infty)}_{n-\lambda}(\Omega_t)+\varepsilon).\end{aligned}$$

Finally, letting $\varepsilon\to0$, we get (3.13). $\square$

*Remark* 1. In the case $\lambda=0$, Theorem 2 gives

$$c_1\rho(f,B_{L_\infty}(t))_{L_1}\leq t|\{x\,|\,Mf(x)>t\}|\leq c_2\rho\left(f,B_{L_\infty}\left(\frac{t}{2}\right)\right)_{L_1}, \tag{3.16}$$

with constants $c_1$ and $c_2$ depending only on the dimension $n$ and $M$ is the usual Hardy–Littlewood maximal function. Moreover, from the exact formula

$$\rho(f,B_{L_\infty}(t))_{L_1}=\int_{\{x\in\mathbf{R}^n\,|\,|f(x)|>t\}}(|f(x)|-t)\,dx$$

it follows that

$$\frac{1}{2}\int_{\{x\in\mathbf{R}^n\,|\,|f(x)|>2t\}}|f(x)|\,dx\leq\rho(f,B_{L_\infty}(t))_{L_1}\leq\int_{\{x\in\mathbf{R}^n\,|\,|f(x)|>t\}}|f(x)|\,dx.$$

Therefore from (3.16) we obtain the well-known Wiener–Stein equivalence (see [35] and [32])

$$c_1\int_{\{x\in\mathbf{R}^n\,|\,|f(x)|>2t\}}|f(x)|\,dx\leq t|\{x\,|\,Mf(x)>t\}|\leq c_2\int_{\{x\in\mathbf{R}^n\,|\,|f(x)|>t/2\}}|f(x)|\,dx.$$

Thus Theorem 2 can be considered as one possible generalization of the Wiener–Stein equivalence for the fractional maximal function.

## 4. An analogue of the Hardy–Littlewood maximal theorem

Since the space $(L_1, \mathcal{L}^{1,\lambda})_{1-1/p,p}$ can be described in terms of the distance function $\rho(f, B_{\mathcal{L}^{1,\lambda}}(t))_{L_1}$ (see (2.1)), Theorem 2 implies the following statement, which can be viewed as an analogue of the Hardy–Littlewood maximal theorem for the fractional maximal function.

**Theorem 3.** *Suppose* $p\in(1,\infty)$ *and* $f\in(L_1, \mathcal{L}^{1,\lambda})_{1-1/p,p}$. *Then*

$$\|f\|_{(L_1,\mathcal{L}^{1,\lambda})_{1-1/p,p}} \approx \|M_\lambda f\|_{L_p(\Lambda^{(\infty)}_{n-\lambda})},$$

*where*

$$\|f\|_{L_p(\Lambda^{(\infty)}_{n-\lambda})} = \left(p\int_{\mathbf{R}_+} \Lambda^{(\infty)}_{n-\lambda}(\{x\in\mathbf{R}^n \mid |f(x)|>t\})t^{p-1}\,dt\right)^{1/p} < \infty.$$

*Proof.* Since $f\in(L_1, \mathcal{L}^{1,\lambda})_{1-1/p,p}$, the equality

$$\|f\|_{(L_1,\mathcal{L}^{1,\lambda})_{1-1/p,p}} = \left(\int_0^\infty \rho(f, B_{\mathcal{L}^{1,\lambda}}(t))_{L_1} t^{p-1}\,\frac{dt}{t}\right)^{1/p} \tag{4.1}$$

shows that $\rho(f, B_{\mathcal{L}^{1,\lambda}}(t))_{L_1}$ is finite for all $t>0$. Therefore, Theorem 2 implies that in (4.1) we can replace $\rho(f, B_{\mathcal{L}^{1,\lambda}}(t))_{L_1}$ by $t\Lambda^{(\infty)}_{n-\lambda}(\{x\,|\,(M_\lambda f)(x)>t\})$, and therefore

$$\|f\|_{(L_1,\mathcal{L}^{1,\lambda})_{1-1/p,p}} \approx \|M_\lambda f\|_{L_p(\Lambda^{(\infty)}_{n-\lambda})}. \quad \square$$

*Remark* 2. In the case $\lambda=0$, Theorem 3 gives

$$\|f\|_{(L_1,L_\infty)_{1-1/p,p}} \approx \|Mf\|_{L_p}.$$

As $(L_1, L_\infty)_{1-1/p,p}=L_p$ we therefore obtain the Hardy–Littlewood maximal theorem with equivalence of norms.

**Lemma 3.** *Let* $1-\lambda/n\in(0,1)$. *Then*

$$\mathcal{L}^{1,\lambda} \supset L_{n/\lambda}, \tag{4.2}$$

$$L_p(\Lambda^{(\infty)}_{n-\lambda}) \subset L_{p/(1-\lambda/n)} \quad \text{for } p\in(0,\infty). \tag{4.3}$$

*Proof.* The first embedding is a consequence of the Hölder inequality. Indeed,

$$\|f\|_{\mathcal{L}^{1,\lambda}} = \sup_Q \frac{|Q|^{\lambda/n}}{|Q|}\int_Q |f(x)|\,dx \le \sup_Q \frac{|Q|^{\lambda/n}}{|Q|^{\lambda/n}}\left(\int_Q |f(x)|^{n/\lambda}\,dx\right)^{\lambda/n} \le \|f\|_{L_{n/\lambda}}.$$

In order to prove (4.3), it suffices to analyze only the case $p=1$, i.e. to show that

$$\|f\|_{L_{1/(1-\lambda/n)}} \le \|f\|_{L_1(\Lambda^{(\infty)}_{n-\lambda})} \tag{4.4}$$

for all $f\in L_{1/(1-\lambda/n)}$. Indeed, (4.3) immediately follows from (4.4):

$$\|f\|_{L_{p/(1-\lambda/n)}} = \||f|^p\|^{1/p}_{L_{1/(1-\lambda/n)}} \le \||f|^p\|^{1/p}_{L_1(\Lambda^{(\infty)}_{n-\lambda})} = \|f\|_{L_p(\Lambda^{(\infty)}_{n-\lambda})}.$$

To verify (4.4), observe that $1-\lambda/n\in(0,1)$ implies that

$$|a+b|^{1-\lambda/n} \le |a|^{1-\lambda/n} + |b|^{1-\lambda/n}$$

for any $a,b\in\mathbf{R}_+$. Therefore, if we consider a covering of the set

$$E_t = \{x\in\mathbf{R}^n \mid |f(x)|>t\}$$

by a family $\{Q_i\}_{i\in I}$ of cubes, i.e. $E_t\subset\bigcup_{i\in I} Q_i$, then

$$|E_t|^{1-\lambda/n} \le \inf_{\{Q_i\}_{i\in I}} \Big|\bigcup_{i\in I} Q_i\Big|^{1-\lambda/n} \le \inf_{\{Q_i\}_{i\in I}} \sum_{i\in I} |Q_i|^{1-\lambda/n} = \Lambda^{(\infty)}_{n-\lambda}(E_t),$$

where the infimum is taken over all coverings $\{Q_i\}_{i\in I}$ for $E_t$. Since the Lebesgue measure of $E_t$, i.e. $|E_t|$, is a positive monotone decreasing function of $t$, we have (see [21] or [16, p. 100])

$$\bigg(\int_0^\infty |E_t|\,dt^{1/(1-\lambda/n)}\bigg)^{1-\lambda/n} \le \int_0^\infty |E_t|^{1-\lambda/n}\,dt, \quad \text{when } 1-\frac{\lambda}{n}\in(0,1).$$

Therefore,

$$\begin{aligned}\|f\|_{L_{1/(1-\lambda/n)}} &= \bigg(\int_0^\infty |E_t|\,dt^{1/(1-\lambda/n)}\bigg)^{1-\lambda/n} \\ &\le \int_0^\infty |E_t|^{1-\lambda/n}\,dt \le \int_0^\infty \Lambda^{(\infty)}_{n-\lambda}(E_t)\,dx = \|f\|_{L_1(\Lambda^{(\infty)}_{n-\lambda})}. \quad \square\end{aligned}$$

Now we are ready to prove the main result of this section. But first we observe that the conditions $p\in(1,n/\lambda)$ and $1/q=1/p-\lambda/n$ imply that $(1-\lambda/n)q\in(1,\infty)$. As $(1-\lambda/n)q>1$ therefore, we can consider the interpolation space

$$X = (L_1, \mathcal{L}^{1,\lambda})_{1-1/(1-\lambda/n)q,(1-\lambda/n)q}.$$

**Theorem 4.** *Suppose* $1-\lambda/n\in(0,1)$, $p\in(1,n/\lambda)$, *and* $1/q=1/p-\lambda/n$. *We let* $X=(L_1,\mathcal{L}^{1,\lambda})_{1-1/(1-\lambda/n)q,(1-\lambda/n)q}$ *and* $Y=L_{(1-\lambda/n)q}(\Lambda^{(\infty)}_{n-\lambda})$. *Then*

$$L_p\subset X,\quad Y\subset L_q, \tag{4.5}$$

*and*

$$\|M_\lambda f\|_Y\approx\|f\|_X \tag{4.6}$$

*with constants of equivalence not depending on* $f$.

*Proof.* We start with (4.5). Since $(1-\lambda/n)q\in(1,\infty)$, the second embedding of Lemma 3 implies that

$$Y\subset L_q$$

and the first one gives the embedding

$$(L_1,L_{n/\lambda})_{1-1/(1-\lambda/n)q,(1-\lambda/n)q}\subset X. \tag{4.7}$$

A simple direct calculation shows that

$$(L_1,L_{n/\lambda})_{1-1/(1-\lambda/n)q,(1-\lambda/n)q}=L_{p,(1-\lambda/n)q},$$

where $L_{p,(1-\lambda/n)q}$ is a Lorentz space. Therefore, the trivial inequality

$$\frac{1}{p}-\frac{1}{q}=1-\left(1-\frac{\lambda}{n}\right)\geq\frac{1}{(1-\lambda/n)q}-\frac{1-\lambda/n}{(1-\lambda/n)q}$$

yields $(1-\lambda/n)q\geq p$, which gives

$$L_{p,p}\subset L_{p,(1-\lambda/n)q}=(L_1,L_{n/\lambda})_{1-1/(1-\lambda/n)q,(1-\lambda/n)q}. \tag{4.8}$$

Hence, by (4.7) and (4.8) we obtain that $L_p\subset X$.

Finally, observe that (4.6) follows from Theorem 3. □

Note that (1.3) is an immediate consequence of Theorem 4.

We finally present direct calculations, which in particular give a proof of the fact that (1.3) cannot be reversed.

**Theorem 5.** *Suppose* $1-\lambda\in(0,1)$ *and* $\varepsilon>0$. *Then there exists a function* $f$ *such that*

(i) $(M_\lambda f)^*(t)\leq\varepsilon\cdot\sup_{t\leq\tau<\infty}1/\tau^{1-\lambda}\int_0^\tau f^*(s)\,ds$ *for some* $t>0$, *here by* $f^*$ *we denote the decreasing rearrangement of the function* $f$ *with respect to the Lebesgue measure*;

(ii) $\|M_\lambda f\|_{L_q}\leq\varepsilon\|f\|_{L_p}$ *for* $p$ *and* $q$ *such that* $p\in(1,1/\lambda)$ *and* $1/q=1/p-\lambda$.

*Proof.* We fix a natural $N>0$ and consider measurable sets on $\mathbf{R}$ defined as follows:

$$\Omega_{N,k}=\left\{x\in\mathbf{R}\,\middle|\,k+\frac{k-1}{N}\le x\le k+\frac{k}{N}\right\}\quad\text{and}\quad\Omega_N=\bigcup_{k=1}^{N}\Omega_{N,k}. \tag{4.9}$$

Next, we introduce the function

$$f_N(x)=\chi_{\Omega_N}(x),$$

and the following sets:

$$\begin{aligned}I_N&=\Omega_N,\\ II_N&=\{x\in\mathbf{R}\,|\,\operatorname{dist}(x,\Omega_N)\le\tfrac12\},\\ III_N&=\{x\in\mathbf{R}\,|\,\operatorname{dist}(x,\Omega_N)>\tfrac12\}.\end{aligned} \tag{4.10}$$

Clearly,

$$\|f_N\|_{L_p}=1, \tag{4.11}$$

and, since $1-\lambda\in(0,1)$, we have

$$\sup_{t\le\tau<\infty}\frac{1}{\tau^{1-\lambda}}\int_0^\tau f_N^*(s)\,ds=\begin{cases}1 & \text{for } t\in(0,1],\\ t^{\lambda-1} & \text{for } t>1.\end{cases} \tag{4.12}$$

Now, we observe that

$$M_\lambda f_N(x)\le\begin{cases}\dfrac{1}{N(\operatorname{dist}(x,I_N)+1/N)^{1-\lambda}}+\dfrac{2}{(1+N)^{1-\lambda}} & \text{for } x\in I_N\cup II_N,\\[2ex] \dfrac{2}{(\operatorname{dist}(x,II_N)+1+N)^{1-\lambda}} & \text{for } x\in III_N.\end{cases} \tag{4.13}$$

The estimate (4.13) follows from the fact that the function

$$\sup_{Q\ni x}\frac{1}{|Q|^{1-\lambda}}\int_Q|f(y)|\,dy$$

attains its maximum only at intervals that contain an entire set $\Omega_{N,k}$. In other words, if the closed interval $Q$ touches the set $\Omega_{N,k}$, defined by (4.9), for some $k$, then for the smallest possible interval $\widetilde{Q}$ such that $\widetilde{Q}\supset Q\cup\Omega_{N,k}$ we have

$$\frac{1}{|\widetilde{Q}|^{1-\lambda}}\int_{\widetilde{Q}}|f(y)|\,dy\ge\frac{1}{|Q|^{1-\lambda}}\int_Q|f(y)|\,dy.$$

Now we can prove the theorem. For (i) it suffices to choose $N$ so as to have

$$\frac{1}{N^{\lambda}}+\frac{2}{(1+N)^{1-\lambda}}<\varepsilon. \tag{4.14}$$

Indeed, (4.13) and (4.14) imply that

$$M_{\lambda}f_N(x)\leq\varepsilon$$

for all $x\in\mathbf{R}$. At the same time, together with (4.12), this gives (i) for all $t\in(0,1]$.

In the case of (ii), direct but tedious calculations show the following estimate:

$$\|M_{\lambda}f_N\|_{L_q}\leq c\biggl(\frac{1}{N^{\lambda}}+\frac{1}{N^{(1-\lambda-1/q)}}+\frac{1}{(1+N)^{1-\lambda}}+\frac{1}{(1+N)^{(1-\lambda)q-1}}\biggr), \tag{4.15}$$

where $c$ is a constant depending only on $q$ and $1-\lambda$. Observe that $p\in(1,1/\lambda)$ and the interrelation between $p$ and $q$ implies that $(1-\lambda)q>1$. Therefore, by (4.11), (4.15) and the fact that $1-\lambda\in(0,1)$, we can take $N$ sufficiently large so as to ensure (ii). $\square$

*Remark* 3. In the paper [11], it was suggested to view the inequality

$$(M_{\lambda}f)^*(t)\leq c\sup_{t\leq\tau<\infty}\frac{1}{\tau^{1-\lambda/n}}\int_0^{\tau}f^*(s)\,ds,$$

as an analogue of the Riesz–Herz equivalence (see below) for the fractional maximal function. From Theorem 5(i) it follows that the converse inequality is not true in general.

## 5. Riesz–Herz equivalence

The Wiener–Stein equivalence is related to the so-called (see, e.g. [7]) Riesz–Herz equivalence

$$t(Mf)^*(t)\approx\int_0^t f^*(s)\,ds.$$

Since $\int_0^t f^*(s)\,ds$ is equal to Peetre's $K$-functional(2) for the couple $(L_1,L_\infty)$ (see [8]), the last equivalence can be rewritten in the form

$$t(Mf)^*(t)\approx K(t,f;L_1,L_\infty).$$

Below we consider an analogue of this equivalence for the fractional maximal function. We define the rearrangement of $f$ with respect to the Hausdorff capacity $\Lambda^{(\infty)}_{n-\lambda}$

(2) Recall that Peetre's $K$-functional for the couple $(X_0,X_1)$ is defined by $K(t,f;X_0,X_1)=\inf_{g\in X_1}(\|f-g\|_{X_0}+t\|g\|_{X_1})$.

to be a monotone nonincreasing and right-continuous function $f^*_{n-\lambda}$ on $(0,\infty)$ such that

$$\Lambda^{(\infty)}_{n-\lambda}(\{x\in\mathbf{R}^n \mid |f(x)|>t\})=|\{s\in\mathbf{R}_+ \mid |f^*_{n-\lambda}(s)|>t\}|. \tag{5.1}$$

Here, on the right-hand side, $|\cdot|$ means the Lebesgue measure of the set.

**Theorem 6.** *Suppose $t\in\mathbf{R}_+$ and $f\in L_1$. Then*

$$t(M_\lambda f)^*_{n-\lambda}(t)\approx K(t,f,L_1,\mathcal{L}^{1,\lambda}),$$

*with constants of equivalence not depending on $f$ and $t$.*

*Proof.* Let $\varepsilon>0$ be an arbitrary small number. Then, from the definition of $f^*_{n-\lambda}$, we obtain

$$\Lambda^{(\infty)}_{n-\lambda}(\{x\in\mathbf{R}^n \mid M_\lambda f(x)>(M_\lambda f)^*_{n-\lambda}(t)+\varepsilon\})\le t \tag{5.2}$$

and

$$t\le\Lambda^{(\infty)}_{n-\lambda}(\{x\in\mathbf{R}^n \mid M_\lambda f(x)>(M_\lambda f)^*_{n-\lambda}(t)-\varepsilon\}). \tag{5.3}$$

Indeed, to verify (5.2) and (5.3), it suffices to observe that

$$\begin{aligned}|\{s\in\mathbf{R}_+ \mid (M_\lambda f)^*_{n-\lambda}(s)>(M_\lambda f)^*_{n-\lambda}(t)+\varepsilon\}|&\le t,\\ |\{s\in\mathbf{R}_+ \mid (M_\lambda f)^*_{n-\lambda}(s)>(M_\lambda f)^*_{n-\lambda}(t)-\varepsilon\}|&\ge t.\end{aligned}$$

From Theorem 2 we have

$$\rho(f,B_{\mathcal{L}^{1,\lambda}}(s))_{L_1}\le c_1 s\Lambda^{(\infty)}_{n-\lambda}(\{x \mid M_\lambda f(x)>c_2 s\}) \tag{5.4}$$

and

$$s\Lambda^{(\infty)}_{n-\lambda}(\{x \mid M_\lambda f(x)>s\})\le c_3\rho(f,B_{\mathcal{L}^{1,\lambda}}(c_4 s))_{L_1}. \tag{5.5}$$

We put

$$s_+=\frac{(M_\lambda f)^*_{n-\lambda}(t)+\varepsilon}{c_2}\quad\text{and}\quad s_-=(M_\lambda f)^*_{n-\lambda}(t)-\varepsilon.$$

Then, by using (5.2) and (5.4) we obtain

$$\rho(f,B_{\mathcal{L}^{1,\lambda}}(s_+))_{L_1}\le c_1 s_+ t. \tag{5.6}$$

Moreover, by (5.3) and (5.5) we get

$$ts_-\le c_3\rho(f,B_{\mathcal{L}^{1,\lambda}}(c_4 s_-))_{L_1}. \tag{5.7}$$

Observe that (5.6) yields

$$K(t,f,L_1,\mathcal{L}^{1,\lambda}) \le \inf_{\|g\|_{\mathcal{L}^{1,\lambda}}\le s_+} (\|f-g\|_{L_1}+t\|g\|_{\mathcal{L}^{1,\lambda}}) \le \rho(f,B_{\mathcal{L}^{1,\lambda}}(s_+))_{L_1}+ts_+ \le (c_1+1)ts_+. \tag{5.8}$$

On the other hand, if $\|g\|_{\mathcal{L}^{1,\lambda}}\le c_4 s_-$, then (5.7) implies

$$\inf_{\|g\|_{\mathcal{L}^{1,\lambda}}\le c_4 s_-} (\|f-g\|_{L_1}+t\|g\|_{\mathcal{L}^{1,\lambda}}) \ge \rho(f,B_{\mathcal{L}^{1,\lambda}}(c_4 s_-))_{L_1} \ge \frac{1}{c_3} ts_-.$$

But if $\|g\|_{\mathcal{L}^{1,\lambda}}>c_4 s_-$, we have trivially

$$\inf_{\|g\|_{\mathcal{L}^{1,\lambda}}> c_4 s_-} (\|f-g\|_{L_1}+t\|g\|_{\mathcal{L}^{1,\lambda}}) \ge c_4 ts_-.$$

Therefore,

$$K(t,f,L_1,\mathcal{L}^{1,\lambda}) \ge \min\left(\frac{1}{c_3}, c_4\right) ts_-. \tag{5.9}$$

Finally, letting $\varepsilon\to 0$, from (5.8) and (5.9), we deduce that

$$c_5 t (M_\lambda f)^*_{n-\lambda}(t) \le K(t,f,L_1,\mathcal{L}^{1,\lambda}) \le c_6 t (M_\lambda f)^*_{n-\lambda}(t),$$

where the constants $c_5, c_6>0$ depend only on $1-\lambda/n$ and the dimensional $n$. $\square$

The Hardy–Littlewood maximal theorem follows easily from the Riesz–Herz equivalence. Indeed

$$\begin{aligned}\|f\|_{L_p} &\approx \|f\|_{(L_1,L_\infty)_{1-1/p,p}} = \left(\int_0^\infty (t^{1/p-1}K(t,f,L_1,L_\infty))^p \,\frac{dt}{t}\right)^{1/p} \\ &\approx \left(\int_0^\infty (t^{1/p}(Mf)^*(t))^p\,\frac{dt}{t}\right)^{1/p} = \left(\int_{\mathbf{R}^n} (Mf(x))^p\,dt\right)^{1/p} = \|Mf\|_{L_p}.\end{aligned}$$

In the same way we can give a second proof of our analogue of the Hardy–Littlewood maximal theorem for the fractional maximal function.

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