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Endpoint estimates for Riesz transforms of magnetic Schrödinger operators

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Abstract. Let $A=-(\nabla-i\vec{a})^2+V$ be a magnetic Schrödinger operator acting on $L^2(\mathbf{R}^n)$, $n\geq 1$, where $\vec{a}=(a_1,...,a_n)\in L^2_{\mathrm{loc}}$ and $0\leq V\in L^1_{\mathrm{loc}}$. Following [1], we define, by means of the area integral function, a Hardy space H^1_A associated with A. We show that Riesz transforms $(\partial/\partial x_k-ia_k)A^{-1/2}$ associated with A, $k=1,...,n$, are bounded from the Hardy space H^1_A into L^1. By interpolation, the Riesz transforms are bounded on L^p for all $1<p\leq 2$.

1. Introduction

Consider a real vector potential $\vec{a}=(a_1,...,a_n)$ and an electric potential V. In this paper, we assume that

$$a_k\in L^2_{\mathrm{loc}} \quad \text{for all } k=1,...,n, \tag{1.1}$$

$$0\leq V\in L^1_{\mathrm{loc}}. \tag{1.2}$$

Let $L_k=\partial/\partial x_k-ia_k$. We define the form Q by

$$Q(u,v)=\sum_{k=1}^{n}\int_{\mathbf{R}^n} L_k u\overline{L_k v}\,dx+\int_{\mathbf{R}^n} Vu\bar{v}\,dx$$

with domain

$$\mathcal{D}(Q)=\big\{u\in L^2: L_k u\in L^2 \text{ for } k=1,...,n \text{ and } \sqrt{V}u\in L^2\big\}.$$

It is well known that this symmetric form is closed. Note also that it was shown by Simon [21] that this form coincides with the minimal closure of the form given by the same expression but defined on $C_0^\infty(\mathbf{R}^n)$ (the space of C^∞ functions with compact support). In other words, $C_0^\infty(\mathbf{R}^n)$ is a core of the form Q.

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Let us denote by A the self-adjoint operator associated with Q. The domain of A is given by

$$\mathcal{D}(A)=\Big\{u\in\mathcal{D}(Q),\ \text{there is } v\in L^2 \text{ so that } Q(u,\varphi)=\int_{\mathbf{R}^n} v\bar{\varphi}\,dx,\ \text{for all } \varphi\in\mathcal{D}(Q)\Big\},$$

and A is given by the expression

$$Au=\sum_{k=1}^{n} L_k^* L_k u+Vu. \tag{1.3}$$

Formally, we write $A=-(\nabla-i\vec{a})^2+V$. For $k=1,...,n$, the operators $L_kA^{-1/2}$ are called the *Riesz transforms* associated with A. It is easy to check that

$$\|L_k u\|_{L^2}\le\|A^{1/2}u\|_{L^2},\quad u\in\mathcal{D}(Q)=\mathcal{D}(A^{1/2}), \tag{1.4}$$

for any $k=1,...,n$, and hence the operators $L_kA^{-1/2}$ are bounded on L^2. Note that this is also true for $V^{1/2}A^{-1/2}$.

We now remind the reader of some known results of boundedness of Riesz transforms $L_kA^{-1/2}$ associated with A. It was recently proved in [20] (Theorem 11) that by using the finite speed propagation property, for each $k=1,...,n$, the operator $L_kA^{-1/2}$ is of weak type $(1,1)$, and hence, by interpolation, is bounded on L^p for all $1<p\le 2$. See also Shen's result, Theorem 0.5 of [18] for L^p-boundedness of Riesz transforms of certain Schrödinger operators $-\Delta+V$. The results in [18] were extended to the case of magnetic Schrödinger operators in [19]. We note that for $p>2$, the counter example studied in [18] with the potential $V(x)=|x|^{-2+\varepsilon}$ shows that the operator $\nabla(-\Delta+V)^{-1/2}$ is not necessarily bounded on L^p. However, L^p-boundedness of Riesz transforms for large values of p can be obtained if one imposes certain additional regularity conditions on the potential V (see [18]). For information on L^p estimates of Riesz transforms associated with elliptic operators on manifolds or Euclidean domains, see [4], [2] and [17].

The aim of this paper is to study the endpoint estimates of the Riesz transforms $L_kA^{-1/2}$ from the Hardy space H_A^1 into L^1, where H_A^1 is a new class of Hardy spaces associated with A ([1], see Section 2 below). The following is the main result of this paper.

Theorem 1.1. *Assume that* (1.1) *and* (1.2) *hold. Then for each* $k=1,...,n$, *the Riesz transform* $L_kA^{-1/2}$ *and* $V^{1/2}A^{-1/2}$ *are bounded from* H_A^1 *into* L^1, *i.e., there exists a constant* $c>0$ *such that*

$$\|V^{1/2}A^{-1/2}u\|_{L^1}+\sum_{k=1}^{n}\|L_kA^{-1/2}u\|_{L^1}\le c\|u\|_{H_A^1}.$$

Hence by interpolation, $V^{1/2}A^{-1/2}$ *and* $L_kA^{-1/2}$ *are bounded on* L^p *for all* $1<p\le 2$.

The paper is organized as follows. In Section 2, we recall the definition of the Hardy space H^1_A associated with an operator A and state its molecular characterization as in [1]. Our main result, Theorem 1.1, will be proved in Section 3. In Section 4 we show that for the Schrödinger operator $A=-\Delta+V$ with potential satisfying the reverse Hölder inequality, the classical Hardy space H^1 is a proper subspace of the space H^1_A associated with A. It then follows from Theorem 1.1 that the Riesz transform $\nabla A^{-1/2}$ is bounded from H^1 into L^1.

Throughout, the letter "c" will denote (possibly different) constants that are independent of the essential variables.

2. Hardy spaces associated with operators

Let $A=-(\nabla-i\vec{a})^2+V$ be the magnetic Schrödinger operator in (1.3). By the well-known diamagnetic inequality (see Theorem 2.3 of [21] and [6]) we have the pointwise inequality

$$|e^{-tA}f(x)|\le e^{t\Delta}|f|(x) \quad \text{for all } t\ge 0, \quad f\in L^2(\mathbf{R}^n).$$

This inequality implies in particular that the semigroup e^{-tA} maps L^1 into L^∞ and that the kernel $p_t(x,y)$ of e^{-tA} satisfies

$$|p_t(x,y)|\le \frac{1}{(4\pi t)^{n/2}}e^{-|x-y|^2/4t} \tag{2.1}$$

for all $t>0$ and almost all $x,y\in\mathbf{R}^n$.

For any $(x,t)\in\mathbf{R}^n\times(0,+\infty)$, we let

$$P_tf(x)=e^{-tA}f(x) \quad\text{and}\quad Q_tf(x)=-t\frac{dP_t}{dt}f(x)=tAe^{-tA}f(x) \tag{2.2}$$

for $f\in L^p$, $1\le p<\infty$. Consider the area integral function $\mathcal{S}_A(f)$ associated with an operator A, given by

$$\mathcal{S}_A(f)(x)=\biggl(\int_0^\infty\int_{|x-y|<t}|Q_{t^2}f(y)|^2\,\frac{dy\,dt}{t^{n+1}}\biggr)^{1/2}.$$

Since A is a selfadjoint operator, it has a bounded H_∞-calculus in L^2 (see [16]), and hence $\mathcal{S}_A(f)$ is bounded on $L^2(\mathbf{R}^n)$. By using the upper bound (2.1), it was proved in [1] (Theorem 6) that $\mathcal{S}_A(f)$ is bounded on $L^p(\mathbf{R}^n)$ for $1<p<\infty$. This means that for each p, $1<p<\infty$, there exist constants c_1 and c_2 (depending on p) such that $0<c_1\le c_2<\infty$ and

$$c_1\|f\|_p\le\|\mathcal{S}_A(f)\|_p\le c_2\|f\|_p. \tag{2.3}$$

See Theorem 6 of [1].

Definition 2.1. ([1]) We say that $f\in L^1$ belongs to the Hardy space H^1_A associated with the semigroup $\{e^{-tA}\}_{t\geq 0}$ of A if $\mathcal{S}_A(f)\in L^1$. If this is the case, we define its norm by

$$\|f\|_{H^1_A}=\|\mathcal{S}_A(f)\|_{L^1}.$$

For any function $f(y,t)$ defined on $\mathbf{R}^{n+1}_+$, let

$$\mathcal{G}(f)(x)=\biggl(\int_0^\infty\int_{|x-y|<t}|f(y,t)|^2\,\frac{dy\,dt}{t^{n+1}}\biggr)^{1/2}.$$

The *tent space* T^p_2 is defined as the space of functions f such that $\mathcal{G}(f)\in L^p$, when $p<\infty$. The resulting equivalence classes are then equipped with the norm, $\|f\|_{T^p_2}=\|\mathcal{G}(f)\|_p$. Thus, $f\in H^1_A$ if and only if $Q_{t^2}f\in T^1_2$ (or $\mathcal{G}(Q_{t^2}f)=\mathcal{S}_A(f)\in L^1$).

Next, a function $a(t,x)$ is called a T^1_2-*atom* if

(i) there exists some ball $B\subset\mathbf{R}^n$ such that $a(t,x)$ is supported in $\widehat{B}$; and

(ii) $$\int_{\widehat{B}}|a(t,x)|^2\,\frac{dx\,dt}{t}\leq\frac{1}{|B|},$$

where $\widehat{B}=\{(y,t)\in\mathbf{R}^{n+1}_+:y\in B \text{ and } B(y,t)\subset B\}$ and $B(y,t)$ is the open ball with center y and radius t.

We now describe an A-molecular characterization for these Hardy spaces. Following [1], a function $\alpha(x)$ is called an A-*molecule* if

$$\alpha(x)=\pi_A(a)(x)=\frac{72}{5}\int_0^\infty Q_{t^2}(\mathcal{I}-P_{t^2})(a(t,\cdot))(x)\,\frac{dt}{t}, \tag{2.4}$$

where $a(t,x)$ is a usual T^1_2-atom. Consider the following identity:

$$\frac{5}{72}=\int_0^\infty(t^2ze^{-t^2z})(1-e^{t^2z})(t^2ze^{-t^2z})\,\frac{dt}{t},$$

which is valid for all $z\neq 0$ in a sector S^0_μ with $\mu\in(0,\pi)$, where S^0_μ is the interior of $S_\mu=\{z\in\mathbf{C}:|\arg z|\leq\mu\}$. As a consequence of H_∞-functional calculus of the operator A ([16]), one has the following identity which is valid for all $f\in H^1_A\cap L^2$,

$$f(x)=\frac{72}{5}\int_0^\infty Q_{t^2}(\mathcal{I}-P_{t^2})\bigl(Q_{t^2}f\bigr)(x)\,\frac{dt}{t}. \tag{2.5}$$

By using identity (2.5) and the atomic decomposition of $Q_{t^2}f\in T^1_2$ (see Theorem 1 of [3]), an A-molecular decomposition of f in the space H^1_A is obtained in Theorem 7 of [1] as follows.

Proposition 2.1. *Let $f\in H^1_A$. There exist A-molecules $\alpha_k(x)$ and numbers λ_k for $k=0,1,2,...$ such that*

$$f(x)=\sum_{k=0}^{\infty}\lambda_k\alpha_k(x). \tag{2.6}$$

The sequence λ_k satisfies $\sum_{k=0}^{\infty}|\lambda_k|\le c\|f\|_{H^1_A}$. Conversely, the decomposition (2.6) *satisfies $\|f\|_{H^1_A}\le c\sum_{k=0}^{\infty}|\lambda_k|$.*

The proof of this proposition is based on estimates of area integrals and tent spaces as in [3]. For the details, we refer the reader to Theorem 7 of [1]. See also [11].

Note. It is proved in [1] that if T is a linear operator which is bounded from L^p to L^p for some p, $1<p\le\infty$, and also bounded from H^1_A to L^1, then T is bounded on L^q for all $1<q<p$. For more properties of the Hardy spaces H^1_A and related topics, we refer to [1], [10] and [11].

3. Proof of Theorem 1.1

Recall that $p_t(x,y)$ is the kernel of the semigroup e^{-tA}. We first note that Gaussian upper bounds carry over from heat kernels to their time derivatives. That is, for each $k\in\mathbf{N}$, there exist two positive constants c_k and C_k such that the time derivatives of p_t satisfy

$$\left|\frac{\partial^k p_t}{\partial t^k}(x,y)\right|\le C_k t^{-k-n/2}e^{-c_k|x-y|^2/t} \tag{3.1}$$

for all $t>0$ and almost all $x,y\in\mathbf{R}^n$. For the proof of (3.1), see [5], [7], [14] and [17], Theorem 6.17.

We let $\tilde p_t(x,y)=t(d/dt)p_t(x,y)=t(d/ds)p_s(x,y)|_{s=t}$. It follows from the semigroup property that

$$\tilde p_{2t}(\,\cdot\,,y)=2e^{-tA}\tilde p_t(\,\cdot\,,y)\quad\text{for all } t\ge 0 \text{ and } y\in\mathbf{R}^n. \tag{3.2}$$

In particular, for every fixed y, the function $\tilde p_t(\,\cdot\,,y)\in\mathcal{D}(A)\subset\mathcal{D}(Q)$. This shows that the expression $L_k\tilde p_t(\,\cdot\,,y)(x)$ makes sense. In the sequel, we always use the notation $L_k\tilde p_t(x,y)$ to mean $L_k\tilde p_t(\,\cdot\,,y)(x)$.

The following proposition gives a weighted estimate for $L_k\tilde p_t(x,y)$ which will be useful in the proof of Theorem 1.1.

Proposition 3.1. *There exist constants* $\beta>0$ *and* $c>0$ *such that*

$$\int_{\mathbf{R}^n} |V^{1/2}\tilde{p}_t(x,y)|^2 e^{\beta|x-y|^2/t}\,dx+\sum_{k=1}^n \int_{\mathbf{R}^n} |L_k\tilde{p}_t(x,y)|^2 e^{\beta|x-y|^2/t}\,dx \le ct^{-n/2-1}$$

for all $t>0$ *and* $y\in\mathbf{R}^n$.

The proof of Proposition 3.1 uses the following lemma (see Lemma 2.5 in [21]).

Lemma 3.1. *Assume that* (1.1) *holds. Then for each* $k=1,\dots,n$, *there exists a function* $\lambda_k\in L^2_{\mathrm{loc}}$ *such that*

$$L_k = e^{i\lambda_k}\frac{\partial}{\partial x_k}e^{-i\lambda_k}.$$

Moreover, we have

$$\mathcal{D}(L_k)=\left\{u\in L^2 : \frac{\partial}{\partial x_k}u-ia_k u \quad (\textit{as a distribution}) \in L^2\right\}.$$

Proof of Proposition 3.1. Let ψ be a C^∞ function with compact support on $\mathbf{R}^n$ such that $0\le\psi\le1$. Consider

$$I_t(\psi)=\sum_{k=1}^n \int_{\mathbf{R}^n} |L_k\tilde{p}_t(x,y)|^2 e^{\beta|x-y|^2/t}\psi(x)\,dx.$$

Using Lemma 3.2, we have

$$\begin{aligned}
I_t(\psi) &= \sum_{k=1}^n \int_{\mathbf{R}^n} \frac{\partial}{\partial x_k}(e^{-i\lambda_k}\tilde{p}_t(x,y))\overline{\frac{\partial}{\partial x_k}(e^{-i\lambda_k}\tilde{p}_t(x,y))}e^{\beta|x-y|^2/t}\psi(x)\,dx \\
&= \sum_{k=1}^n \int_{\mathbf{R}^n} \frac{\partial}{\partial x_k}(e^{-i\lambda_k}\tilde{p}_t(x,y))\overline{\frac{\partial}{\partial x_k}(e^{-i\lambda_k}\tilde{p}_t(x,y)e^{\beta|x-y|^2/t}\psi(x))}\,dx \\
&\quad -\sum_{k=1}^n \int_{\mathbf{R}^n} \frac{\partial}{\partial x_k}(e^{-i\lambda_k}\tilde{p}_t(x,y))\overline{e^{-i\lambda_k}\tilde{p}_t(x,y)}\frac{\partial}{\partial x_k}(e^{\beta|x-y|^2/t}\psi(x))\,dx \\
&= II_1 - II_2.
\end{aligned}$$

Using (3.2) and the fact that ψ has compact support, we have

$$\tilde{p}_t(\,\cdot\,,y)e^{\beta|x-y|^2/t}\psi(\,\cdot\,)\in\mathcal{D}(Q)\subset\mathcal{D}(L_k).$$

We can then write the first term II_1 as

$$II_1=\sum_{k=1}^n \int_{\mathbf{R}^n} L_k\tilde{p}_t(x,y)\overline{L_k(\tilde{p}_t(\,\cdot\,,y)e^{\beta|\cdot-y|^2/t}\psi)(x)}\,dx$$

Since $0\le V$ and $0\le\psi$, we obtain

$$\begin{aligned} II_1 &\le \sum_{k=1}^{n}\int_{\mathbf{R}^n} L_k\tilde{p}_t(x,y)\overline{L_k(\tilde{p}_t(\,\cdot\,,y)e^{\beta|\cdot-y|^2/t}\psi)(x)}\,dx \\ &\quad+\int_{\mathbf{R}^n} V\tilde{p}_t(x,y)\overline{\tilde{p}_t(x,y)e^{\beta|x-y|^2/t}\psi(x)}\,dx \\ &= Q(\tilde{p}_t(\,\cdot\,,y),\ \tilde{p}_t(\,\cdot\,,y)e^{\beta|\cdot-y|^2/t}\psi) \\ &= \int_{\mathbf{R}^n} A\tilde{p}_t(x,y)\overline{\tilde{p}_t(x,y)}e^{\beta|x-y|^2/t}\psi(x)\,dx. \end{aligned}$$

From the semigroup property, it follows easily that $A\tilde{p}_t(x,y)=-t(d^2/dt^2)p_t(x,y)$. We then apply (3.1) to obtain

$$II_1 \le c\int_{\mathbf{R}^n}\frac{1}{t^{n/2+1}}e^{-c_2|x-y|^2/t}\frac{1}{t^{n/2}}e^{-c_1|x-y|^2/t}e^{\beta|x-y|^2/t}\psi(x)\,dx.$$

Hence for any constant $\beta<c_1$, there exists a constant $c>0$ independent of ψ such that

$$II_1 \le \frac{c}{t^{n/2+1}} \tag{3.3}$$

since $0\le\psi\le1$.

Next, we rewrite II_2 as follows:

$$\begin{aligned} II_2 &= \sum_{k=1}^{n}\int_{\mathbf{R}^n} e^{i\lambda_k}\frac{\partial}{\partial x_k}\Big(e^{-i\lambda_k}\tilde{p}_t(x,y)\Big)\overline{\tilde{p}_t(x,y)}e^{\beta|x-y|^2/t} \\ &\quad\times\left[\frac{\partial}{\partial x_k}\psi(x)+\frac{2\beta(x_k-y_k)}{t}\psi(x)\right]dx. \end{aligned}$$

This gives

$$\begin{aligned} II_2 &\le \sum_{k=1}^{n}\frac{c}{\sqrt{t}}\int_{\mathbf{R}^n}|L_k\tilde{p}_t(x,y)|\,|\tilde{p}_t(x,y)|e^{2\beta|x-y|^2/t}\psi(x)\,dx \\ &\quad+\sum_{k=1}^{n}\int_{\mathbf{R}^n}|L_k\tilde{p}_t(x,y)|\,|\tilde{p}_t(x,y)e^{\beta|x-y|^2/t}\left|\frac{\partial}{\partial x_k}\psi(x)\right|dx \\ &= J_1(\psi)+J_2(\psi). \end{aligned}$$

To estimate the first term $\mathrm{J}_1(\psi)$, we use (3.1) and Cauchy–Schwarz' inequality to obtain

$$\begin{aligned} &J_1(\psi) \\ &\le \frac{c}{\sqrt{t}}\sum_{k=1}^{n}\Big(\int_{\mathbf{R}^n} t^{-n}e^{(3\beta-2c_1)|x-y|^2/t}\,dx\Big)^{1/2}\Big(\int_{\mathbf{R}^n}|L_k\tilde{p}_t(x,y)|^2e^{\beta|x-y|^2/t}\psi(x)\,dx\Big)^{1/2}. \end{aligned}$$

Hence, if β is small enough (for example, $\beta<2c_1/3$), there exists a constant $c>0$, independent of ψ, such that

$$J_1(\psi) \le \frac{c}{\sqrt{t^{n/2+1}}}\sqrt{I_t(\psi)}.$$

Using this estimate and (3.3), we obtain

$$I_t(\psi) \le c(t^{-n/2-1}+J_2(\psi)), \tag{3.4}$$

where $c>0$ is a positive constant independent of ψ.

We now apply (3.4) with $\psi_j(x)=\psi(x/j)$, where ψ is a function such that $\psi(x)=1$ for all x with $|x|\le 1$. Simple computations show that the sequence $\mathrm{J}_2(\psi_j)$ converges to 0 as $j\to\infty$. An application of Fatou's lemma and (3.4) (with ψ_j in place of ψ) gives

$$\int_{\mathbf{R}^n} |L_k\tilde{p}_t(x,y)|^2 e^{\beta|x-y|^2/t}\psi(x)\,dx \le ct^{-n/2-1}.$$

The term

$$\int_{\mathbf{R}^n} |V^{1/2}\tilde{p}_t(x,y)|^2 e^{\beta|x-y|^2/t}\psi(x)\,dx$$

can be written as

$$Q(\tilde{p}_t(\,\cdot\,,y),\tilde{p}_t(\,\cdot\,,y)e^{\beta|\cdot-y|^2/t}\psi)-II_1.$$

Both terms have already been estimated. This finishes the proof. □

Proof of Theorem 1.1. Firstly, the operators $L_kA^{-1/2}$ and $V^{1/2}A^{-1/2}$ are both bounded on L^2 for all $k=1,2,...,n$. Indeed, for $f\in L^2$ we have

$$\begin{aligned}\sum_{k=1}^n\int_{\mathbf{R}^n}|L_kA^{-1/2}f(x)|^2\,dx+\int_{\mathbf{R}^n}V(x)|A^{-1/2}f(x)|^2\,dx &= Q(A^{-1/2}f,A^{-1/2}f)\\ &= \|f\|_{L^2}^2.\end{aligned} \tag{3.5}$$

The latter equality follows from the fact that for any symmetric form Q associated with an operator A, one has $\mathcal{D}(Q)=\mathcal{D}(A^{1/2})$ and $Q(u,v)=(A^{1/2}u,A^{1/2}v)$ (see, for example, Chapter VI of [15] or Chapter 7 of [17]).

We now prove that the operators $L_kA^{-1/2}$ are bounded from H_A^1 into L^1. By Proposition 2.2, it suffices to verify that for any A-molecule $\alpha(x)$, there exists a positive constant c independent of α such that

$$\|L_kA^{-1/2}(\alpha)\|_{L^1} \le c. \tag{3.6}$$

Assume that $\alpha(x)=\pi_A(a)(x)$ is an A-molecule as in (2.4), where $a=a(t,x)$ is a standard T_2^1-atom supported in $\widehat{B}$ (for some ball $B=B(x_B,r_B)\subset\mathbf{R}^n$). One writes

$$\|L_kA^{-1/2}(\alpha)\|_{L^1}=\int_{2B}|L_kA^{-1/2}(\alpha)(x)|\,dx+\int_{(2B)^c}|L_kA^{-1/2}(\alpha)(x)|\,dx=I+II.$$

It follows from (a) of Lemma 4.3 in [11] that $\|\pi_A(a)\|_{L^2}\le c\|a\|_{T_2^2}\le c|B|^{-1/2}$. Using Hölder's inequality and (3.5), we obtain

$$I\le c|B|^{1/2}\|L_kA^{-1/2}(\pi_A(a))\|_{L^2}\le c|B|^{1/2}\|\pi_A(a)\|_{L^2}\le c|B|^{1/2}|B|^{-1/2}\le c.$$

We now estimate the term II. Substituting

$$\alpha(x)=\pi_A(a)(x)=\frac{36}{5}\int_0^\infty tAe^{-tA}(\mathcal{I}-e^{-tA})(a(\sqrt{t},\cdot\,))(x)\,\frac{dt}{t}$$

into the formula

$$A^{-1/2}(\alpha)=\frac{1}{2\sqrt{\pi}}\int_0^\infty e^{-sA}(\alpha)\frac{ds}{\sqrt{s}},$$

we have

$$\begin{aligned}
&L_kA^{-1/2}(\alpha)(x)\\
&\quad=\frac{18}{5\sqrt{\pi}}\int_0^\infty\!\!\int_0^\infty t\Big(L_kAe^{-(s+t)A}-L_kAe^{-(s+2t)A}\Big)(a(\sqrt{t},\cdot\,))(x)\,\frac{dt}{t}\,\frac{ds}{\sqrt{s}}\\
&\quad=\frac{18}{5\sqrt{\pi}}\int_0^\infty\!\!\int_0^\infty\bigg(\frac{\chi_{\{s>t\}}}{s\sqrt{s-t}}-\frac{\chi_{\{s>2t\}}}{s\sqrt{s-2t}}\bigg)L_k\bigg(s\frac{d}{ds}e^{-sA}\bigg)(a(\sqrt{t},\cdot\,))(x)\,dt\,ds\\
&\quad=\frac{18}{5\sqrt{\pi}}\int_0^\infty\!\!\int_0^{r_B^2}\!\!\int_B\bigg(\frac{\chi_{\{s>t\}}}{s\sqrt{s-t}}-\frac{\chi_{\{s>2t\}}}{s\sqrt{s-2t}}\bigg)L_k\tilde{p}_s(x,y)a(\sqrt{t},y)\,dy\,dt\,ds.
\end{aligned}$$

Thus the key to proving Theorem 1.1 is to estimate $\int_{(2B)^c}|L_k\tilde{p}_s(x,y)|\,dx$. Let $\beta>0$ be the constant as in Proposition 3.1 and let $\gamma=\beta/4$. We have

$$\begin{aligned}
&\int_{(2B)^c}|L_k\tilde{p}_s(x,y)|\,dx\\
&\qquad\le\int_{|x-y|\ge r_B}|L_k\tilde{p}_s(x,y)|e^{\beta|x-y|^2/2s}e^{-\beta|x-y|^2/2s}\,dx\\
&\qquad\le\bigg(\int_{\mathbf{R}^n}|L_k\tilde{p}_s(x,y)|^2e^{\beta|x-y|^2/s}dx\bigg)^{1/2}\bigg(\int_{|x-y|\ge r_B}e^{-\beta|x-y|^2/s}\,dx\bigg)^{1/2}\\
&\qquad\le cs^{-n/4-1/2}e^{-\gamma r_B^2/s}\bigg(\int_{\mathbf{R}^n}e^{-\beta|x-y|^2/2s}\,dx\bigg)^{1/2}
\end{aligned}$$

$$\le cs^{-1/2}e^{-\gamma r_B^2/s}$$
$$\le \frac{c}{\sqrt{s}}\frac{1}{(1+r_B^2/s)^{\varepsilon}}$$

for any $\varepsilon>0$. This gives

$$\begin{aligned} II &\le c\int_0^\infty\int_0^{r_B^2}\int_B \frac{1}{s^{3/2}(1+r_B^2/s)^{\varepsilon/2}}\left|\frac{\chi_{\{s>t\}}}{\sqrt{s-t}}-\frac{\chi_{\{s>2t\}}}{\sqrt{s-2t}}\right||a(\sqrt{t},y)|\,dy\,dt\,ds \\ &\le c\int_0^{r_B^2}\int_B\int_t^{2t}\frac{1}{s^{3/2}(1+r_B^2/s)^{\varepsilon/2}}\frac{1}{\sqrt{s-t}}|a(\sqrt{t},y)|\,ds\,dy\,dt \\ &\quad +c\int_0^{r_B^2}\int_B\int_{2t}^\infty\frac{1}{s^{3/2}(1+r_B^2/s)^{\varepsilon/2}}\left|\frac{1}{\sqrt{s-t}}-\frac{1}{\sqrt{s-2t}}\right||a(\sqrt{t},y)|\,ds\,dy\,dt \\ &= II_1+II_2, \quad \text{respectively.} \end{aligned}$$

It follows from the estimate

$$\int_t^{2t}\frac{1}{s^{3/2}(1+r_B^2/s)^{\varepsilon/2}}\frac{1}{\sqrt{s-t}}\,ds\le cr_B^{-\varepsilon}\ t^{\varepsilon/2-1}$$

and Hölder's inequality that

$$\begin{aligned} II_1 &\le cr_B^{-\varepsilon}\int_0^{t_B}\int_B t^{\varepsilon/2-1}|a(\sqrt{t},y)|\,dy\,dt \\ &\le cr_B^{-\varepsilon}\left(\int_0^{r_B^2}\int_B t^{\varepsilon-1}\,dy\,dt\right)^{1/2}\left(\int_{\widehat{B}}|a(t,x)|^2\frac{d\mu(x)\,dt}{t}\right)^{1/2}\le c. \end{aligned}$$

We now estimate the term II_2. We have $|1/\sqrt{s-t}-1/\sqrt{s-2t}|\le c(t/s\sqrt{s-2t})$ for $s>2t$. Therefore,

$$\int_{2t}^\infty\frac{1}{s^{3/2}(1+r_B^2/s)^{\varepsilon/2}}\left|\frac{1}{\sqrt{s-t}}-\frac{1}{\sqrt{s-2t}}\right|ds\le cr_B^{-\varepsilon}t\int_{2t}^\infty\frac{s^{\varepsilon/2}}{s^{5/2}\sqrt{s-2t}}\,ds\le cr_B^{-\varepsilon}t^{\varepsilon/2-1}.$$

This gives

$$II_2\le cr_B^{-\varepsilon}\int_0^{t_B}\int_B t^{\varepsilon/2-1}|a(\sqrt{t},y)|\,dy\,dt\le c.$$

By combining the estimates of I, II_1 and II_2, we obtain the desired estimate (3.6).

All the above arguments work in the same way if we replace $L_kA^{-1/2}$ by $V^{1/2}A^{-1/2}$. We obtain (3.6) for $V^{1/2}A^{-1/2}$. The proof is complete. □

4. A characterization of the Hardy space H_A^1

In this section, we continue with the assumption that the magnetic Schrödinger operator $A=-(\nabla-i\vec{a})^2+V$ satisfies conditions (1.1) and (1.2). We define a Hardy space $\widetilde{H}^1_A$ by means of a maximal function associated with the semigroup $\{e^{-tA}\}_{t>0}$ as

$$\widetilde{H}^1_A = \Big\{ f \in L^1 : \sup_{t>0} |e^{-tA} f(x)| \in L^1 \Big\}. \tag{4.1}$$

We first have the following proposition.

Proposition 4.1. *Let* $A=-(\nabla - i\vec{a})^2+V$ *satisfy conditions* (1.1) *and* (1.2). *Then,* $H^1_A \subset \widetilde{H}^1_A$, *i.e., the Hardy space* H^1_A *is a subspace of* $\widetilde{H}^1_A$.

Proof. Define

$$\mathcal{P}^* f(x) = \sup_{t>0} |e^{-tA} f(x)|.$$

Because of the decay of the kernel $p_t(x,y)$ in (2.1), one has $\mathcal{P}^* f(x) \le c\mathcal{M}f(x)$, where $\mathcal{M}$ is the Hardy–Littlewood maximal function, given by

$$\mathcal{M}f(x) = \sup_{B\ni x} \frac{1}{|B|} \int_B |f(y)|\, dy.$$

In order to prove Proposition 4.1, by Proposition 2.2 it suffices to verify that for any A-molecule $\alpha(x)$, there exists a positive constant c independent of α such that

$$\|\mathcal{P}^*(\alpha)\|_{L^1} \le c. \tag{4.2}$$

Assume that $\alpha(x) = \pi_A(a)(x)$ is an A-molecule as in (2.4), where $a=a(t,x)$ is a standard T^1_2-atom supported in $\widehat{B}$ (for some ball $B=B(x_B, r_B) \subset \mathbf{R}^n$). One writes

$$\|\mathcal{P}^*(\alpha)\|_{L^1} = \int_{2B} |\mathcal{P}^*(\alpha)(x)|\, dx + \int_{(2B)^c} |\mathcal{P}^*(\alpha)(x)|\, dx = I + II.$$

It follows from (a) of Lemma 4.3 in [11] that $\|\pi_A(a)\|_{L^2} \le c\|a\|_{T^2_2} \le c|B|^{-1/2}$. Using Hölder's inequality and (3.5), we obtain

$$I \le c|B|^{1/2} \|\mathcal{M}(\pi_A(a))\|_{L^2} \le c|B|^{1/2} \|\pi_A(a)\|_{L^2} \le c|B|^{1/2}|B|^{-1/2} \le c.$$

We now estimate the term *II*. Substituting

$$\alpha(x) = \pi_A(a)(x) = \frac{36}{5} \int_0^\infty tAe^{-tA}(\mathcal{I} - e^{-tA})(a(\sqrt{t}, \cdot))(x)\, \frac{dt}{t}$$

into the formula

$$\mathcal{P}^* f(x) = \sup_{s>0} |e^{-sA} f(x)|,$$

we have

$$\begin{aligned}\mathcal{P}^*(\alpha)(x) &= \frac{36}{5\sqrt{\pi}}\sup_{s>0}\bigg|\int_0^{r_B^2} tA(e^{-(s+t)A}-e^{-(s+2t)A})(a(\sqrt{t},\cdot))(x)\,\frac{dt}{t}\bigg| \\ &\le c\sup_{s>0}\bigg|\int_0^{r_B^2} t(s+t)^{-1}((s+t)A\ e^{-(s+t)A})(a(\sqrt{t},\cdot))(x)\,\frac{dt}{t}\bigg| \\ &\quad + c\sup_{s>0}\bigg|\int_0^{r_B^2} t(s+2t)^{-1}((s+2t)A\ e^{-(s+2t)A})(a(\sqrt{t},\cdot))(x)\,\frac{dt}{t}\bigg| \\ &= II_1 + II_2.\end{aligned}$$

Note that for any $x\notin 2B$ and $y\in B$, we have that $|x-y|\ge|x-x_B|/2$. This, together with (3.1), give

$$\begin{aligned}II_1 &\le c\sup_{s>0}\int_0^{r_B^2}\int_B t(s+t)^{-1}(\sqrt{t+s})^{\varepsilon}|a(\sqrt{t},y)|\,\frac{dt\,dy}{t}|x-x_B|^{-n-\varepsilon} \\ &\le c\bigg(\int_0^{r_B^2}\int_B t^{\varepsilon-1}\,dy\,dt\bigg)^{1/2}\|a\|_{T_2^2}|x-x_B|^{-n-\varepsilon} \\ &\le cr_B^{\varepsilon}|x-x_B|^{-n-\varepsilon}\end{aligned}$$

for any $\varepsilon>0$.

Similarly, we also have $II_2\le cr_B^{\varepsilon}|x-x_B|^{-n-\varepsilon}$ for any $\varepsilon>0$. Therefore,

$$\int_{(2B)^c}|\mathcal{P}^*(\alpha)(x)|\,dx\le cr_B^{\varepsilon}\int_{(2B)^c}|x-x_B|^{-n-\varepsilon}\,dx\le c.$$

We obtain (4.2) from the estimates of I and II. Thus we have $H_A^1\subset\widetilde{H}_A^1$. The proof of Proposition 4.1 is complete. □

Next, a natural question is to ask whether the space H_A^1 is the same as $\widetilde{H}_A^1$. The question is still open in general. However, we will give an affirmative answer in the case of $A=-\Delta+V$ with the assumption that V is a fixed non-negative function on $\mathbf{R}^n$, $n\ge3$, belonging to the reverse Hölder class $RH_s(\mathbf{R}^n)$ for some $s>n/2$; that is, there exists a constant $c=c(s,V)>0$ such that the reverse Hölder inequality

$$\bigg(\frac{1}{|B|}\int_B V^s\,dx\bigg)^{1/s}\le c\bigg(\frac{1}{|B|}\int_B V\,dx\bigg) \tag{4.3}$$

holds for every ball B in $\mathbf{R}^n$.

We now show that the spaces $\widetilde{H}_A^1$ and H_A^1 coincide, and that they contain the classical Hardy space H^1 as a proper subspace.

Theorem 4.1. *Let* $A=-\Delta+V(x)$*, where* V *satisfies condition* (4.3) *and is not identically zero. Then, we have*

$$H^1 \subsetneq \widetilde{H}^1_A \equiv H^1_A. \tag{4.4}$$

As a consequence, the Riesz transform $\nabla A^{-1/2}$ *is bounded from* H^1_A *into* L^1*, and from* H^1 *into* L^1*. More specifically, there exist positive constants* c_1 *and* c_2 *such that*

$$\|\nabla A^{-1/2}u\|_{L^1} \le c_1\|u\|_{H^1_A} \le c_2\|u\|_{H^1}.$$

Proof. By Proposition 4.1, we have $H^1_A\subset\widetilde{H}^1_A$. By Lemma 6 of [12], $\widetilde{H}^1_A\subset H^1_A$. Therefore, $\widetilde{H}^1_A\equiv H^1_A$.

Theorem 1.5 of [13] implies that the classical Hardy space H^1 is a subspace of $\widetilde{H}^1_A$. In order to see that H^1 is a proper subspace of $\widetilde{H}^1_A$, we note that if they coincide then their duals BMO and BMO_A coincide. Therefore, by Proposition 6.7 of [11], the semigroup e^{-tA} satisfies the conservation property $e^{-tA}1=1$ for all $t\ge 0$. This latter property does not hold if V is not identically zero.

It follows then from Theorem 1.1 that the Riesz transform $\nabla A^{-1/2}$ is bounded from H^1_A into L^1, and from H^1 into L^1. This proves Theorem 4.2. □

We observe that [13] gives an atomic decomposition for functions in $\widetilde{H}^1_A$ which shows that the atoms for the $\widetilde{H}^1_A$ space satisfy the same size conditions as the classical H^1 atoms, but the mean-value zero condition for $\widetilde{H}^1_A$ atoms is required only for those supported on small balls. This, together with Theorem 1.5 of [13], imply that the classical Hardy space H^1 is a proper subspace of $\widetilde{H}^1_A$, and thus $H^1\subsetneq\widetilde{H}^1_A\equiv H^1_A$.

Notes. Let $A=-(\nabla-i\vec{a})^2+V$ be a magnetic Schrödinger operator with assumptions (1.1) and (1.2).

(α) Weak type $(1,1)$ of the Riesz transforms $L_kA^{-1/2}$ associated with A can be obtained by modifying the proof in [4]. See also [20].

(β) To the best of our knowledge, the question of boundedness of Riesz transform $L_kA^{-1/2}$ from the classical H^1 space into L^1 is still open.

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