Ark. Mat., **44** (2006), 299–308
DOI: 10.1007/s11512-006-0026-5

A new sufficient condition for Hamiltonian graphs

Ronald J. Gould and Kewen Zhao

Abstract. The study of Hamiltonian graphs began with Dirac's classic result in 1952. This was followed by that of Ore in 1960. In 1984 Fan generalized both these results with the following result: If G is a 2-connected graph of order n and $\max\{d(u), d(v)\}\geq n/2$ for each pair of vertices u and v with distance $d(u, v)=2$, then G is Hamiltonian. In 1991 Faudree–Gould–Jacobson–Lesnick proved that if G is a 2-connected graph and $|N(u)\cup N(v)|+\delta(G)\geq n$ for each pair of nonadjacent vertices $u, v\in V(G)$, then G is Hamiltonian. This paper generalizes the above results when G is 3-connected. We show that if G is a 3-connected graph of order n and $\max\{|N(x)\cup N(y)| +d(u), |N(w)\cup N(z)|+d(v)\}\geq n$ for every choice of vertices x, y, u, w, z and v such that $d(x, y)= d(y, u)=d(w, z)=d(z, v)=d(u, v)=2$ and where x, y and u are three distinct vertices and w, z and v are also three distinct vertices (and possibly $|\{x, y\}\cap\{w, z\}|$ is 1 or 2), then G is Hamiltonian.

1. Introduction

Throughout this paper we consider finite simple undirected graphs whose vertex set, edge set, order and minimum degree are denoted by $V(G)$, $E(G)$, $|G|=|V(G)|$ and $\delta(G)$, respectively. If $u\in V(G)$, and C and H are subgraphs of G, then $N(u)=N_G(u)$ denotes the set of neighbours of u in G, $d(u)=|N(u)|$ is the degree of u, $N_C(u)=V(C)\cap N(u)$, $N_C(H)=\bigcup_{u\in V(H)} N_C(u)$, and $G\setminus C$ denotes the subgraph of G induced by $V(G)\setminus V(C)$. We let $d(u, v)$ denote the distance between the vertices u and v.

The study of Hamiltonian graphs has long been an important topic (see [5]). The following conditions are known to suffice for a graph of order $n\geq 3$ to be Hamiltonian:

(i) (Dirac [2]) For each vertex $u\in V(G)$, $d(u)\geq n/2$.

(ii) (Ore [6]) For each pair of nonadjacent vertices $u, v\in V(G)$, $d(u)+d(v)\geq n$.

(iii) (Fan [3]) G is a 2-connected graph and for each pair of vertices $u, v\in V(G)$ at distance 2, $\max\{d(u), d(v)\}\geq n/2$.

(iv) (Faudree–Gould–Jacobson–Lesnick [4]) G is a 2-connected graph and for each pair of nonadjacent vertices $u, v \in V(G)$, $|N(u) \cup N(v)| + \delta(G) \geq n$.

Let $B(G) = \min\{\max\{|N(x) \cup N(y)| + d(u), |N(w) \cup N(z)| + d(v)\}\}$ for every choice of vertices x, y, u, w, z and v such that $d(x, y) = d(y, u) = d(w, z) = d(z, v) = d(u, v) = 2$, where x, y and u are three distinct vertices, and w, z and v are also three distinct vertices (and possibly $|\{x, y\} \cap \{w, z\}|$ is 1 or 2). For other terminology and notation, we refer the reader to [1].

We shall prove the following result.

Theorem 1.1. *Let G be a 3-connected graph of order n such that $B(G) \geq n$, then G is Hamiltonian.*

It is easy to see that if G satisfies one of the conditions (i)–(iv), then $B(G) \geq n$; thus there is a sense in which Theorem 1.1 generalizes the previous four theorems. However, the condition of being 3-connected is needed in Theorem 1.1. It is easy to see that there are non-Hamiltonian 2-connected graphs of order n such that $B(G) \geq n$.

2. The proof of the theorem

If $C: x_1, x_2, ..., x_m, x_1$ is a cycle, we write $[x_i, x_j]$ for the set of vertices $\{x_i, x_{i+1}, ..., x_j\}$, where subscripts are taken modulo m. We will need the following lemma.

Lemma 2.1. *Suppose that $C: x_1, x_2, ..., x_m, x_1$ is a longest cycle in a graph G and H is a component of $G \backslash C$. Let x_i and x_j be distinct vertices in $N_C(H)$. Then*

(a) $\{x_{i-1}, x_{i+1}, x_{j-1}, x_{j+1}\} \cap N_C(H) = \varnothing$;

(b) $x_{i+1}x_{j+1} \notin E(G)$ and $x_{i-1}x_{j-1} \notin E(G)$;

(c) if $x_t x_{j+1} \in E(G)$ for some vertex $x_t \in [x_{j+2}, x_i]$, then $x_{t-1}x_{i+1} \notin E(G)$ and $x_{t-1} \notin N_C(H)$;

(d) if $x_t x_{j+1} \in E(G)$ for some vertex $x_t \in [x_{i+1}, x_{j-2}]$, then $x_{t+1}x_{i+1} \notin E(G)$;

(e) no vertex of $(G \backslash C) \backslash H$ is adjacent to both x_{i+1} and x_{j+1}.

Proof. If any of (a), (b) and (e) were to fail, then it is easy to show that G would contain a cycle that is longer than C.

If (c) fails, then there exists some vertex $x_t \in [x_{j+1}, x_i]$ satisfying $x_t x_{j+1} \in E(G)$ and $x_{t-1}x_{i+1} \in E(G)$, or x_{t-1} is adjacent to some vertex of H. When $x_{t-1}x_{i+1} \in E(G)$, let $P_1(H) = y_1, y_2, ..., y_r$ denote a path of H with two end vertices y_1 and y_r which are adjacent to x_i and x_j, respectively. Then

$$x_i, P_1(H), x_j, x_{j-1}, ..., x_{i+1}, x_{t-1}, x_{t-2}, ..., x_{j+1}, x_t, x_{t+1}, ..., x_i$$

is a cycle that is longer than C, a contradiction. This contradiction shows that $x_{t-1}x_{i+1}\notin E(G)$. When x_{t-1} is adjacent to some vertex of H, we let $P_2(H)= w_1, w_2, ..., w_r$ denote a path of H with end vertices w_1, w_r which are adjacent to x_j and x_{t-1}, respectively. Then

$$x_j, P_2(H), x_{t-1}, x_{t-2}, ..., x_{j+1}, x_t, x_{t+1}, ..., x_j$$

is a cycle longer than C, a contradiction. Thus, x_{t-1} is not adjacent to any vertex of H.

A similar argument also proves (d) and therefore, the lemma is proven. □

We now proceed to the proof of the theorem.

Proof of Theorem 1.1. Suppose G is as in the theorem, but G is not Hamiltonian. Let $C{:}x_1, x_2, ..., x_m, x_1$ be a longest cycle of G. If $u\in V(G\setminus C)$ we write $N_C^+(u)=\{x_{i+1}{:}x_i\in N_C(u)\}$, where subscripts are taken modulo m.

There are several cases to consider.

Case 1. Suppose there exists a component H of $G\setminus C$ and vertices $x_i, x_j\in N_C(H)$ such that $[x_{i+1}, x_{j-1}]\cap N_C(H)=\varnothing$ and $d(x_{i+1}, x_{j+1})=2$.

Choose vertices $u, v\in V(H)$ such that $ux_i, vx_j\in E(G)$ (note that possibly $u=v$). Since C is a longest cycle, by the lemma we have that every vertex of $N_{G\setminus C}(u)\cup N_C^+(u)\cup\{u\}$ is nonadjacent to x_{i+1} and x_{j+1}. Hence

$$|N(x_{j+1})\cup N(x_{i+1})|\le|V(G)|-|N_{G\setminus C}(u)\cup N_C^+(u)\cup u|=n-d(u)-1. \tag{1}$$

Similarly, we have

$$|N(x_{i+1})\cup N(x_{j+1})|\le n-d(v)-1. \tag{2}$$

We now consider two subcases.

Subcase 1.1. Suppose that either $x_{j-1}x_{j+1}\notin E(G)$ or $x_{i-1}x_{i+1}\notin E(G)$.

Suppose $x_{j-1}x_{j+1}\notin E(G)$. By Lemma 2.1 (a) and (e), no vertex of $N_{G\setminus C}(x_{j+1})$ is adjacent to u or x_{i+1}. By Lemma 2.1 (c), if $x_t\in[x_{j+2}, x_i]\cap N(x_{j+1})$, then x_{t-1} is nonadjacent to u and x_{i+1}. By Lemma 2.1 (d), the hypothesis of Case 1, and the fact that $x_{j-1}\notin N(x_{j+1})$, if $x_t\in[x_{i+1}, x_{j-1}]\cap N(x_{j+1})$, then x_{t+1} is nonadjacent to u and x_{i+1}. It follows that

$$|N(u)\cup N(x_{i+1})|\le|V(G)|-|N_G(x_{j+1})\setminus\{x_j\}|-|\{u, x_{i+1}\}|\le n-d(x_{j+1})-1. \tag{3}$$

However, $d(x_{i+1}, x_{j+1})=d(x_{j+1}, v)=2$ and $d(u, x_{i+1})=d(x_{i+1}, x_{j+1})=d(v, x_{j+1})=2$, so (2) and (3) contradict the hypothesis that $B(G)\ge n$.

Now suppose $x_{i-1}x_{i+1}\notin E(G)$.

(a) First suppose $|N_C(u)|\geq 2$.

Under this hypothesis, there exists some vertex $x_h\in N_C(u)\setminus\{x_i\}$ such that $[x_{h+1},x_{i-1}]\cap N_C(u)=\varnothing$. Combining this with the proof of inequality (3) (with i and j replaced by h and i, respectively) gives

$$(4)\quad |N(x_{h+1})\cup N(u)|\leq|V(G)|-|N_G(x_{i+1})\setminus\{x_i\}|-|\{x_{h+1},u\}|\leq n-d(x_{i+1})-1.$$

However, $d(x_{j+1},x_{i+1})=d(x_{i+1},u)=2$ and $d(x_{h+1},u)=d(u,x_{i+1})=d(u,x_{i+1})=2$, so (1) and (4) contradict the hypothesis that $B(G)\geq n$.

(b) Now suppose $|N_C(u)|=|\{x_i\}|=1$.

In this case we only consider $x_{j-1}x_{j+1}\in E(G)$ for if $x_{j-1}x_{j+1}\notin E(G)$, then we are back in the first case of Subcase 1.1. By our assumption x_j is not adjacent to u. Since C is a longest cycle, x_j is not adjacent to x_{i+1}, for otherwise, if we let $P(H)$ denote a path of H with two end vertices which are adjacent to x_i and x_j, respectively, we see that $x_i,P(H),x_j,x_{i+1},x_{i+2},...,x_{j-1},x_{j+1},x_{j+2},...,x_i$ is a cycle that is longer than C, a contradiction. Together with the arguments that gave rise to (3), we have

$$(5)\quad |N(u)\cup N(x_{i+1})|\leq|V(G)|-|N_G(x_{j+1})\setminus\{x_j\}|-|\{u,x_{i+1}\}|\leq n-d(x_{j+1})-1.$$

However, $d(x_{i+1},x_{j+1})=d(x_{j+1},v)=d(u,x_{i+1})=d(x_{i+1},x_{j+1})=d(v,x_{j+1})=2$, so (2) and (5) contradict the hypothesis that $B(G)\geq n$.

Subcase 1.2. Suppose $x_{i-1}x_{i+1}\in E(G)$ and $x_{j-1}x_{j+1}\in E(G)$ for every pair of vertices x_i and x_j satisfying the hypothesis of Case 1.

Since G is 3-connected, there exist $x_k,x_i,x_j\in N_C(H)$ such that $[x_{k+1},x_{i-1}]\cap N_C(H)=\varnothing$ and $[x_{i+1},x_{j-1}]\cap N_C(H)=\varnothing$, and we still have $d(x_{i+1},x_{j+1})=2$.

Subcase 1.2a. Suppose $d(x_{k+1},x_{i+1})=2$.

In this case, $x_{k-1}x_{k+1}\notin E(G)$ by the assumptions of Subcase 1.2 and Subcase 1.2a, so by an argument similar to that of Subcase 1.1, we obtain a contradiction. Thus, we may assume that $x_{k-1}x_{k+1}\in E(G)$.

If $x_{k-1}x_{k+1}\in E(G)$, it is easy to see that x_{k+2} is not adjacent to x_{j+1}. Otherwise, if $x_{k+2}x_{j+1}\in E(G)$, let P be a path in H with end vertices adjacent to x_k and x_j, respectively. Then the cycle $x_k,P,x_j,x_{j-1},...,x_{k+2},x_{j+1},x_{j+2},...,x_{k-1},x_{k+1},x_k$ is longer than C, a contradiction.

Clearly, $x_{k+1}u,x_{k+1}x_{i+1}\notin E(G)$. In this case $x_{j-1}x_{j+1}\in E(G)$ and it is also possible that $x_ju\in E(G)$. Together with the arguments that gave rise to (3), we have

$$(6)\quad \begin{aligned}|N(u)\cup N(x_{i+1})|&\leq|V(G)|-|N_G(x_{j+1})\setminus\{x_{j-1},x_j\}|-|\{u,x_{i+1},x_{k+1}\}|\\&\leq n-d(x_{j+1})-1.\end{aligned}$$

Now, (2) and (6) contradict the hypothesis that $B(G)\geq n$.

Subcase 1.2b. Suppose $d(x_{k+1},x_{i+1})\geq 3$.

(I). Suppose there exists some vertex x in H such that $xx_k, xx_i\in E(G)$.

Without loss of generality assume $ux_k, ux_i\in E(G)$. In this case, since $[x_{k+1}, x_{i-1}]\cap N_C(H)=\varnothing$ and $d(x_{k+1},x_{i+1})\geq 3$, when $x_t\in[x_{k+1},x_{i-1}]\cap N(x_{i+1})$, then x_t is nonadjacent to u and x_{k+1}. Combining this with the first part of the proof of inequality (3) (with i and j replaced by k and i, respectively) gives

$$|N(x_{k+1})\cup N(u)|\leq|V(G)|-|N_G(x_{i+1})\setminus\{x_i\}|-|\{x_{k+1},u\}|\leq n-d(x_{i+1})-1. \tag{7}$$

However, $d(x_{j+1},x_{i+1})=d(x_{i+1},u)=d(x_{k+1},u)=d(u,x_{i+1})=d(u,x_{i+1})=2$, so (1) and (7) contradict the hypothesis that $B(G)\geq n$.

(II). Suppose $xx_k\notin E(G)$ for every vertex x of H for which $xx_i\in E(G)$.

Recall that $ux_i\in E(G)$. Further, $ux_k\notin E(G)$ by our assumptions. Since $d(x_{k+1},x_{i+1})\geq 3$ and since C is a longest cycle of G, clearly, $x_{k+1}x_{j+1}\notin E(G)$ and $x_{i+1}x_k\notin E(G)$. Together with the arguments that gave rise to (6), we have

$$\begin{aligned}|N(u)\cup N(x_{i+1})|&\leq|V(G)|-|N_G(x_{j+1})\setminus\{x_{j-1},x_j\}|-|\{u,x_{i+1},x_k\}|\\&\leq n-d(x_{j+1})-1.\end{aligned}\tag{8}$$

However, $d(x_{i+1},x_{j+1})=d(x_{j+1},v)=d(u,x_{i+1})=d(x_{i+1},x_{j+1})=d(v,x_{j+1})=2$, so (2) and (8) contradict the hypothesis that $B(G)\geq n$.

Case 2. Suppose $d(x_{i+1},x_{j+1})\geq 3$ for each pair of vertices $x_{i+1},x_{j+1}\in N_C^+(H)$ which satisfies $[x_{i+1},x_{j-1}]\cap N_C(H)=\varnothing$ for every component H of $G\setminus C$.

Subcase 2.1. Suppose there exists a vertex u in $G\setminus C$ such that $|N_C(u)|\geq 2$.

Subcase 2.1a. There exists a vertex $x_i\in N_C(u)$ such that $x_{i+1}x_{i-1}\notin E(G)$.

Recall $|N_C(u)|\geq 2$. Let $x_h\in N_C(u)\setminus\{x_i\}$ be such that $[x_{h+1},x_{i-1}]\cap N_C(u)=\varnothing$. Since C is a longest cycle of G, by arguments similar to those in the proof of Lemma 2.1, we have the following facts.

(A) If $x_kx_{i+1}\in E(G)$ for some vertex $x_k\in[x_{i+1},x_h]$, then $ux_{k-1}, x_{h+1}x_{k-1}\notin E(G)$;

(B) If $x_kx_{i+1}\in E(G)$ for some vertex $x_k\in[x_{h+1},x_{i-1}]$, considering the fact that $x_{i+1}x_{i-1}\notin E(G)$, we have that $ux_{k+1}, x_{h+1}x_{k+1}\notin E(G)$;

(C) If $xx_{i+1}\in E(G)$ for some vertex $x\in V(G\setminus C)$, then $x_{h+1}x, ux\notin E(G)$, or a cycle longer than C results.

By (A), (B) and (C), we have

$$(9)\quad |N(x_{h+1})\cup N(u)|\le |V(G)|-|N(x_{i+1})\setminus\{x_i\}|-|\{x_{h+1},u\}|\le n-d(x_{i+1})-1.$$

Next, let $x_{k+1}\in N_C^+(u)$ which satisfies $[x_{i+1},x_{k-1}]\cap N_C(u)=\varnothing$. (If $|N_C(u)|=2$, clearly $x_{k+1}=x_{h+1}$.) By considering the reverse orientation on the cycle C, and by arguments similar to that for (9), we have

$$(10)\quad |N(x_{k-1})\cup N(u)|\le |V(G)|-|N(x_{i-1})\setminus\{x_i\}|-|\{x_{k-1},u\}|\le n-d(x_{i-1})-1.$$

However, $d(x_{h+1},u)=d(u,x_{i+1})=d(x_{k-1},u)=d(u,x_{i-1})=d(x_{i+1},x_{i-1})=2$, so (9) and (10) contradict the hypothesis that $B(G)\ge n$.

Subcase 2.1b. Suppose $x_{i+1}x_{i-1}\in E(G)$ for any $x_{i+1}\in N_C^+(u)$.

By Subcase 2.1, $|N_C(u)|\ge 2$. Let $x_h,x_i\in N_C(u)$ be such that $[x_{h+1},x_{i-1}]\cap N_C(u)=\varnothing$. Let $x_r\in N_C(H)$ be such that $[x_{r+1},x_{i-1}]\cap N_C(H)=\varnothing$. Then clearly $x_r\in[x_h,x_{i-2}]$. Let $x_k\in[x_{h+1},x_{i-1}]$ satisfy $x_kx_{i+1}\in E(G)$ with k as small as possible.

(D) Suppose $x_h=x_r$.

By the hypothesis of Case 2, we have that $d(x_{h+1},x_{i+1})=d(x_{r+1},x_{i+1})\ge 3$. Then $x_kx_{h+1},x_ku\notin E(G)$ and since $ux_{h+1}\notin E(G)$ together with the arguments of the lemma, we have

$$|N(x_{h+1})\cup N(u)|\le |V(G)|-|N(x_{i+1})\setminus\{x_{i-1},x_i\}|-|\{x_{h+1},u,x_k\}|\le n-d(x_{i+1})-1.$$

(E) Suppose $x_h\ne x_r$.

By the hypothesis of Case 2, $d(x_{r+1},x_{i+1})\ge 3$. Then clearly $x_rx_{i+1},x_{r+1}x_{i+1}\notin E(G)$ and since C is a longest cycle, $x_{r+1}x_{h+1},x_{r+1}u\notin E(G)$. Together with the arguments of the lemma, we have that

$$\begin{aligned}|N(x_{h+1})\cup N(u)|&\le |V(G)|-|N(x_{i+1})\setminus\{x_{i-1},x_i\}|-|\{x_{h+1},u,x_{r+1}\}|\\&\le n-d(x_{i+1})-1.\end{aligned}$$

In (D) and (E), we see that whether $x_h=x_r$ or $x_h\ne x_r$, we have

$$(11)\qquad |N(x_{h+1})\cup N(u)|\le n-d(x_{i+1})-1.$$

Next, suppose $x_k\in[x_{i+1},x_{j-1}]$ is such that $x_kx_{i-1}\in E(G)$ and $x_{k+1}x_{i-1}\notin E(G)$. Since C is a longest cycle of G and by Subcase 2.1b, $x_{i+1}x_{i-1}\in E(G)$ for any $x_{i+1}\in N_C^+(u)$, every vertex of $(N_{G\setminus C}(u)\cup N_C(u)\cup\{u\})\setminus\{x_i,x_j\}$ is nonadjacent to x_{k+1} and x_{i-1}. Now note that if $x_l\in N_C(u)$, then $x_{i-1}x_l\notin E(G)$ for otherwise

$$u,x_l,x_{i-1},x_{i-2},...,x_{l+1},x_{l-1},...,x_{i+1},x_i,u$$

is a cycle longer than C, a contradiction. Similarly, $x_{k+1}x_l \notin E(G)$, for otherwise

$$u, x_l, x_{k+1}, x_{k+2}, ..., x_{i-1}, x_k, x_{k-1}, ..., x_{l+1}, x_{l-1}, ..., x_{i+1}, x_i, u$$

is a cycle longer than C. Hence, we have

$$\begin{aligned} |N(x_{k+1}) \cup N(x_{i-1})| &\le |V(G)| - |(N_{G\setminus C}(u) \cup N_C(u) \cup \{u\}) \setminus \{x_i, x_j\}| \\ &\quad - |\{x_{k+1}, x_{i-1}\}| \\ &\le n - d(u) - 1. \end{aligned} \tag{12}$$

However, $d(x_{h+1}, u) = d(u, x_{i+1}) = d(x_{k+1}, x_{i-1}) = d(x_{i-1}, u) = d(x_{i+1}, u) = 2$, so (11) and (12) contradict the hypothesis that $B(G) \ge n$.

Subcase 2.2. Suppose $|N_C(u)| \le 1$ for every vertex u of $G \setminus C$.

Let H be a component of $G \setminus C$. Since G is 3-connected, there exist three distinct vertices $x_k, x_i, x_j \in N_C(H)$ such that $[x_{k+1}, x_{j-1}] \cap N_C(H) \setminus \{x_i\} = \varnothing$. By Subcase 2.2, $|N_C(x)| \le 1$ for every vertex x of $G \setminus C$, so there exists a vertex $u \in V(H)$ adjacent to x_k and $v \in V(H) \setminus \{u\}$ adjacent to x_j, and $w \in V(H) \setminus \{u, v\}$ adjacent to x_i.

Claim. Under the conditions of Case 2 and Subcase 2.2, if some vertex u of H is adjacent to some vertex of C, then u is adjacent to every vertex of H.

Proof of claim. If the claim fails, then there exists a vertex y in H such that $d(u, y) = 2$. Without loss of generality, let u be adjacent to x_k in C. Since C is a longest cycle of G, then x_{k+1} is nonadjacent to any vertex of H. Thus, every vertex of $N_{G\setminus C}(x_{k+1})$ is nonadjacent to u and y, and every vertex of $N_C(x_{k+1}) \setminus \{N_C(u), N_C(y)\}$ is nonadjacent to u and y. Together with the fact that $|N_C(u)| \le 1$ and $|N_C(y)| \le 1$, we have

$$\begin{aligned} |N(y) \cup N(u)| &\le |V(G)| - |(N_{G\setminus C}(x_{k+1}) \cup N_C(x_{k+1})) \setminus \{N_C(u), N_C(y)\}| \\ &\quad - |\{y, u, x_{k+1}\}| \\ &\le n - d(x_{k+1}) - 1. \end{aligned} \tag{13}$$

Next, let $x_{h+1} \in N_C^+(H)$ with $[x_{h+1}, x_{k-1}] \cap N_C(H) = \varnothing$. Since $x_{k+1}x_{k-1} \in E(G)$, there exists $x_q \in [x_{h+1}, x_{k-1}]$ with $x_{k+1}x_{q+1} \in E(G)$ and $x_{k+1}x_q \notin E(G)$, that is, $d(x_{k+1}, x_q) = 2$. Clearly every vertex of H is nonadjacent to x_q and x_{k+1}, and $|N_C(u)| = 1$. Thus, we have

$$|N(x_q) \cup N(x_{k+1})| \le |V(G)| - |N(u) \setminus N_C(u)| - |\{x_q, x_{k+1}\}| \le n - d(u) - 1. \tag{14}$$

However, $d(y, u) = d(u, x_{k+1}) = d(x_q, x_{k+1}) = d(x_{k+1}, u) = d(x_{k+1}, u) = 2$, so (13) and (14) contradict the hypothesis that $B(G) \ge n$. This contradiction shows that the claim holds.

Next we consider the following case.

Subcase 2.2a. Suppose $x_{i-1}x_{i+1}\notin E(G)$ for every $x_i\in N_C(H)$.

Let $x_k, x_i, x_j\in N_C(H)$ be such that $[x_{k+1}, x_{i-1}]\cap N_C(H)=\varnothing$ and $[x_{i+1}, x_{j-1}]\cap N_C(H)=\varnothing$. Then Case 2 implies that $d(x_{i+1}, x_{j+1})\geq 3$ and by a symmetric argument we may assume that $d(x_{i-1}, x_{j-1})\geq 3$ for each pair of vertices $x_i, x_j\in N_C(H)$ which satisfy $[x_{i+1}, x_{j-1}]\cap N_C(H)=\varnothing$. Thus, $N(x_{k-1})\cap N(x_{i-1})=\varnothing$ and $N(x_{i-1})\cap N(x_{j-1})=\varnothing$. Since C is a longest cycle of G, every vertex of H is nonadjacent to x_{k-1} and x_{i-1}, hence we have

$$d(x_{k-1})+d(x_{i-1})=|N(x_{k-1})\cup N(x_{i-1})|\leq |V(G\backslash H)\backslash\{x_{k-1}, x_{i-1}\}|, \tag{15}$$

and

$$d(x_{i-1})+d(x_{j-1})=|N(x_{i-1})\cup N(x_{j-1})|\leq |V(G\backslash H)\backslash\{x_{i-1}, x_{j-1}\}|. \tag{16}$$

We now claim that

$$d(x_{j-1})+d(x_{k-1})=|N(x_{j-1})\cup N(x_{k-1})|\leq |V(G\backslash H)\backslash\{x_{j-1}, x_{k-1}\}|. \tag{17}$$

First note that clearly, $|N(x_{j-1})\cup N(x_{k-1})|\leq |V(G\backslash H)\backslash\{x_{j-1}, x_{k-1}\}|$. Thus, to prove (17), we only need show that $d(x_{j-1})+d(x_{k-1})=|N(x_{j-1})\cup N(x_{k-1})|$. That is, we only need to prove that $d(x_{k-1}, x_{j-1})\neq 2$.

Thus, suppose $d(x_{k-1}, x_{j-1})=2$ and note that $x_{j-1}\in N_C^-(v)$. By arguments similar to those in the proof of Case 1, we have the following inequality:

$$\begin{aligned} |N(x_{k-1})\cup N(x_{j-1})| &\leq |V(G)|-|N_{G\backslash C}(u)\cup N_C^-(u)\cup\{u\}|-|\{x_{k-1}, x_{j-1}\}| \\ &\leq n-d(u)-1. \end{aligned} \tag{i}$$

Then let $v\in V(H)$ and $x_k\in N_C(v)$. Since C is a longest cycle of G and $x_{j-1}x_{j+1}\notin E(G)$, by arguments similar to those in the lemma we have the following inequality:

$$|N(v)\cup N(x_{k-1})|\leq n-|N(x_{j-1})\backslash\{x_j\}|-|\{v, x_{k-1}\}|\leq n-d(x_{j-1})-1. \tag{ii}$$

However, $d(x_{k-1}, x_{j-1})=d(x_{j-1}, u)=d(v, x_{k-1})=d(x_{k-1}, x_{j-1})=d(u, x_{j-1})=2$. Thus, (i) and (ii) contradict the hypothesis that $B(G)\geq n$. This contradiction shows that (17) holds.

Let $d(x_{r-1})=\max\{d(x_{k-1}), d(x_{i-1}), d(x_{j-1})\}$, where $r\in\{k, i, j\}$, and let $\{t, h\}=\{k, i, j\}\backslash\{r\}$. By inequalities (15), (16) and (17), we have that $d(x_{t-1})\leq(|V(G\backslash H)|-2)/2$ and $d(x_{h-1})\leq(|V(G\backslash H)|-2)/2$.

On the other hand, by arguments similar to those for (15), (16) and (17), we have

$$d(x_{k+1})+d(x_{i+1})=|N(x_{k+1})\cup N(x_{i+1})|\leq |V(G\backslash H)\backslash\{x_{k+1}, x_{i+1}\}|, \tag{18}$$

$$d(x_{i+1})+d(x_{j+1})=|N(x_{i+1})\cup N(x_{j+1})|\leq |V(G\backslash H)\backslash\{x_{i+1}, x_{j+1}\}|, \tag{19}$$

and

(20) $$d(x_{j+1})+d(x_{k+1})=|N(x_{j+1})\cup N(x_{k+1})|\leq |V(G\backslash H)\backslash\{x_{j+1},x_{k+1}\}|.$$

Let $d(x_{q+1})=\max\{d(x_{k+1}),d(x_{i+1}),d(x_{j+1})\}$, where $q\in\{k,i,j\}$ and let $\{p,m\}=\{k,i,j\}\backslash\{q\}$. By inequalities (18), (19) and (20), we have

$$d(x_{p+1})\leq (|V(G\backslash H)|-2)/2 \quad\text{and}\quad d(x_{m+1})\leq (|V(G\backslash H)|-2)/2.$$

Clearly $\{t,h\}\cap\{p,m\}\neq\varnothing$.

Without loss of generality assume that $k\in\{t,h\}\cap\{p,m\}$. Thus, $d(x_{k-1})\leq (|V(G\backslash H)|-2)/2$ and $d(x_{k+1})\leq(|V(G\backslash H)|-2)/2$. By Subcase 2.2 we have $d(u)\leq |V(H)\backslash\{u\}|+|N_C(u)|\leq|V(H)|$ for every vertex u of H, which implies that $d(x_{k-1})+d(x_{k+1})+d(u)\leq n-1$. Then we have the following inequalities:

(21) $$|N(x_{k+1})\cup N(u)|+d(x_{k-1})\leq d(x_{k-1})+d(x_{k+1})+d(u)\leq n-1,$$

and

(22) $$|N(x_{k-1})\cup N(u)|+d(x_{k+1})\leq d(x_{k-1})+d(x_{k+1})+d(u)\leq n-1.$$

Now, (21) and (22) contradict the hypothesis that $B(G)\geq n$, completing this subcase.

Subcase 2.3. Suppose there is a vertex $x_i\in N_C(H)$ such that $x_{i-1}x_{i+1}\in E(G)$.

In this case let $x_h\in[x_k,x_{i-1}]$ satisfy $x_hx_{k-1}\in E(G)$ and $x_{h+1}x_{k-1}\notin E(G)$. Let $u\in V(H)$ and $x_k\in N(u)$. Since C is a longest cycle and $[x_{k+1},x_{i-1}]\cap N_C(H)=\varnothing$, every vertex of $N(u)\backslash N_C(u)$ is nonadjacent to x_{h+1} and x_{k-1}, hence we have

(23) $$|N(x_{h+1})\cup N(x_{k-1})|\leq |V(G)|-|N(u)\backslash N_C(u)|-|\{x_{h+1},x_{k-1}\}|\leq n-d(u)-1.$$

On the other hand, we have the following cases:

(F) Suppose $x_t\in[x_k,x_{i-1}]$ satisfy $x_tx_{k-1}\in E(G)$, then $x_{t+1}x_i, x_{t+1}u\notin E(G)$. Otherwise, if $x_{t+1}x_i\in E(G)$, then let $P_1(H)$ denote a path of H with two end vertices that are adjacent to x_k and x_i, respectively. Then

$$x_i,x_{t+1},x_{t+2},...,x_{i-1},x_{i+1},x_{i+2},...,x_{k-1},x_t,x_{t-1},...,x_k,P_1(H),x_i$$

is a cycle longer than C, a contradiction. By Subcase 2.2, $|N_C(u)|\leq 1$ for every vertex u of $G\backslash C$, so $x_{t+1}u\notin E(G)$.

(G) Suppose $x_t\in[x_i,x_{k-2}]$ satisfy $x_tx_{k-1}\in E(G)$, then $x_{t+1}x_i, x_{t+1}u\notin E(G)$.

Otherwise, if $x_{t+1}x_i\in E(G)$, then let $P_1(H)$ denote a path of H with two end vertices that are adjacent to x_k and x_i, respectively. Then

$$x_i,x_{t+1},x_{t+2},...,x_{k-1},x_t,x_{t-1},...,x_{i+1},x_{i-1},x_{i-2},...,x_k,P_1(H),x_i$$

is a cycle longer than C, a contradiction.

(H) Suppose $x \in N_{G\backslash C}(x_{k-1})$.

By Subcase 2.2, $|N_C(w)| \leq 1$ for every vertex w of $G\backslash C$. Thus, we have $xx_i \notin E(G)$. Since C is a longest cycle, we have $xu \notin E(G)$. By the claim, we have that $d(x_i,u)=2$. Hence,

$$(24)\quad |N(x_i) \cup N(u)| \leq |V(G)| - |N_{G\backslash C}(x_{k-1}) \cup N_C(x_{k-1})| - |\{u\}| \leq n - d(x_{k-1}) - 1.$$

However, $d(x_{h+1},x_{k-1})=d(x_{k-1},u)=2$ and $d(x_i,u)=d(u,x_{k-1})=d(u,x_{k-1})=2$, so (23) and (24) contradict the hypothesis that $B(G) \geq n$. This completes the proof. □

Acknowledgement. Many thanks to the referee for his/her thoughtful comments which have considerably improved the quality of the paper.

References

1. BONDY, J. A. and MURTY, U. S. R., *Graph Theory with Applications*, Macmillan London, New York, 1976.
2. DIRAC, G. A., Some theorems on abstract graphs, *Proc. London Math. Soc.* **2** (1952), 69–81.
3. FAN, G. H., New sufficient condition for cycles in graphs, *J. Combin. Theory Ser. B* **37** (1984), 221–227.
4. FAUDREE, R. J., GOULD, R. J., JACOBSON, M. S. and LESNIAK, L., Neighborhood unions and highly Hamiltonian graphs, *Ars Combin.* **31** (1991), 139–148.
5. GOULD, R. J., Advances on the Hamiltonian problem: a survey, *Graphs Combin.* **19** (2003), 7–52.
6. ORE, O., A Note on Hamilton circuits, *Amer. Math. Monthly* **67** (1960), 55.

Ronald J. Gould
Department of Mathematics
and Computer Science
Emory University
Atlanta, GA 30322
U.S.A.
rg@mathcs.emory.edu

Kewen Zhao
Department of Mathematics
Qiongzhou University
Sanya, Hainan 572022
P.R. China
kewen@zhao.net.cn

Received August 22, 2005
in revised form April 20, 2006
published online November 29, 2006