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Energy characteristics of subordination chains

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Abstract. We consider subordination chains of simply connected domains with smooth boundaries in the complex plane. Such chains admit Hamiltonian and Lagrangian interpretations through the Löwner–Kufarev evolution equations. The action functional is constructed and its time variation is obtained. It represents the infinitesimal version of the action of the Virasoro–Bott group over the space of analytic univalent functions.

1. Introduction

Many physical processes may be interpreted as expanding dynamical systems of domains in the complex plane $\mathbb{C}$ or in the Riemann sphere $\widehat{\mathbb{C}}$. This leads to the study of time-parameter Löwner subordination chains. In particular, we are interested in Löwner chains of simply connected univalent domains with smooth (C^∞) boundaries. There exists a canonical identification of the space of such domains (under certain conformal normalization) with the infinite dimensional Kähler manifold whose central extension is the Virasoro–Bott group. A Löwner subordination chain $\Omega(t)$ is described by the time-dependent family of conformal maps $z=f(\zeta,t)$ of the unit disk U onto $\Omega(t)$, normalized by $f(\zeta,t)=a_1(t)\zeta+a_2(t)\zeta^2+\dots$, $a_1(t)>0$. After the 1923 seminal Löwner's paper [16] a fundamental contribution to the theory of Löwner chains was made by Pommerenke [19] and [20] who described governing evolution equations in partial and ordinary derivatives, known now as the Löwner–Kufarev equations due to Kufarev's work [14]. A particular case of subordination dynamics is given by the Laplace growth evolution (or Hele–Shaw advancing evolution) consisting of the Dirichlet problem for a harmonic potential where the boundary of the phase domain is unknown *a priori* (a free boundary), and, in fact, is defined by the normality of its motion (see, e.g., [9], [22] and [27]). The aim of our paper is to give Hamiltonian and Lagrangian descriptions of the subordina-

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tion evolution. In particular, we discuss the relations between the Löwner–Kufarev equations in partial and ordinary derivatives, construct the action functional, and obtain its time variation. This variation represents the infinitesimal version of the action of the Virasoro–Bott group over the space of analytic univalent functions.

2. Hamiltonian formulation of the subordination evolution

The parametric method for univalent functions emerged more than 80 years ago in the celebrated paper by Löwner [16] who studied a time-parameter semigroup of conformal one-slit maps of the unit disk U arriving then at an evolution equation named after him. His main achievement was an infinitesimal description of a semi-flow of such maps by the Schwarz kernel that led him to the Löwner equation. This crucial result was generalized, then, in several ways (see [20] and the references therein).

A time-parameter family $\Omega(t)$ of simply connected hyperbolic univalent domains forms a *subordination chain* in the complex plane $\mathbb{C}$, for $0\le t<\tau$ (where τ may be ∞), if $\Omega(t)\subsetneq\Omega(s)$, whenever $t<s$. We suppose that the origin is an interior point of the kernel of $\{\Omega(t)\}_{t=0}^{\tau}$, and the boundaries $\partial\Omega(t)$ are smooth (C^∞). Let us normalize the growth of the evolution of this subordination chain by the conformal radius of $\Omega(t)$ with respect to the origin to be e^t. By the Riemann mapping theorem we construct a subordination chain of mappings $f(\zeta,t)$, $\zeta\in U$, where each function $f(\zeta,t)=e^t\zeta+a_2(t)\zeta^2+\ldots$ is a holomorphic univalent map of U onto $\Omega(t)$ for every fixed t.

Pommerenke's result [19] and [20] says that given a subordination chain of domains $\Omega(t)$ defined for $t\in[0,\tau)$, there exists an analytic regular function

$$p(\zeta,t)=1+p_1(t)\zeta+p_2(t)\zeta^2+\ldots,\quad \zeta\in U,$$

such that $\operatorname{Re} p(\zeta,t)>0$ and

$$\frac{\partial f(\zeta,t)}{\partial t}=\zeta\frac{\partial f(\zeta,t)}{\partial\zeta}p(\zeta,t) \tag{1}$$

for $\zeta\in U$ and for almost all $t\in[0,\tau)$. The equation (1) is called the Löwner–Kufarev equation due to two seminal papers: by Löwner [16] with

$$p(\zeta,t)=\frac{e^{iu(t)}+\zeta}{e^{iu(t)}-\zeta}, \tag{2}$$

where $u(t)$ is a continuous function for $t\in[0,\tau)$, and by Kufarev [14] in the general case, where this equation appeared for the first time.

In [28], the case of smooth boundaries $\partial\Omega(t)$, being embedded into the class of quasidisks, has been proved to admit a specific integral form of the function $p(\zeta,t)$ as

$$p(\zeta,t)=1+\frac{\zeta}{2\pi i}\int_{S^1=\partial U}\frac{\nu(\omega,t)}{\omega(\omega-\zeta)}\,d\omega$$

for almost all $t\in[0,\tau]$, where the function $\nu(\omega,t)$ belongs to the Lie algebra Vect S^1 of the vector fields on the unit circle S^1, with the the Poisson–Lie bracket given by

$$[\nu_1,\nu_2]=\nu_1\nu_2'-\nu_2\nu_1',$$

where the derivatives are taken with respect to the angle variable of S^1. Comparing with the Herglotz representation

$$p(\zeta,t)=\zeta\int_{S^1}\frac{\omega+\zeta}{\omega-\zeta}\,d\mu(\omega,t)$$

for the family of Herglotz measures μ normalized by $\int_{S^1}d\mu(\omega,t)=1$, we deduce that $d\mu(\omega,t)=\rho(\omega,t)|d\omega|$, $\omega\in S^1$, and

$$\rho(e^{i\theta},t)\equiv\frac{1}{4\pi}\nu(e^{i\theta},t).$$

From the other side, $\operatorname{Re}p(e^{i\theta},t)=2\pi\rho(e^{i\theta},t)$ and the real-valued function $\rho(e^{i\theta},t)$ is non-negative for almost all $t\in[0,\tau)$.

To arrive at the Hamiltonian interpretation of subordination dynamics, let us rewrite equation (1) in the form

$$\frac{\partial f'(\zeta,t)}{\partial t}=\frac{\partial H(\zeta,f',t)}{\partial\zeta}, \tag{3}$$

where $H(\zeta,f',t)=\zeta f'(\zeta,t)p(\zeta,t)$, and the derivative f' is taken with respect to the complex variable ζ. Interpreting the function $H(\zeta,f',t)$ as a Hamiltonian we must write

$$\frac{\partial\zeta}{\partial t}=-\frac{\partial H(\zeta,f',t)}{\partial f'}=-\zeta p(\zeta,t), \tag{4}$$

formally yet. Equations (3) and (4) constitute the conjugate pair of Hamilton's relations, however this requires some additional clearance to give sense to the equation (4). This equation is just the Löwner–Kufarev equation in ordinary derivatives $\dot\zeta=-\zeta p(\zeta,t)$.

The equation (1) represents a growing evolution of simply connected domains. Let us consider the reverse process. Given an initial domain $\Omega(0)\equiv\Omega_0$ (and therefore, the initial mapping $f(\zeta,0)\equiv f_0(\zeta)$), and a function $p(\zeta,t)$ with positive real part normalized by $p(\zeta,t)=1+p_1\zeta+...$, we solve the equation (1) and ask whether the solution $f(\zeta,t)$ represents a subordination chain of simply connected domains. The initial condition $f(\zeta,0)=f_0(\zeta)$ is not given on the characteristics of the partial differential equation (1), hence the solution exists and is unique. Assuming s as a parameter along the characteristics we have

$$\frac{dt}{ds}=1, \quad \frac{d\zeta}{ds}=-\zeta p(\zeta,t) \quad \text{and} \quad \frac{df}{ds}=0,$$

with the initial conditions $t(0)=0$, $\zeta(0)=z$ and $f(\zeta,0)=f_0(\zeta)$, where z is in U. Obviously, $t=s$. We still need to give sense to this formalism because the domain of ζ is the entire unit disk, however the solutions to the second equation of the characteristic system range within the unit disk but do not fill it. Therefore, introducing another letter w in order to distinguish the function $w(z,t)$ from the variable ζ, we arrive at the Cauchy problem for the Löwner–Kufarev equation in ordinary derivatives for a function $\zeta=w(z,t)$,

$$\frac{dw}{dt}=-wp(w,t), \tag{5}$$

with the initial condition $w(z,0)=z$. The equation (5) is a non-trivial characteristic equation for (1). Unfortunately, this approach requires the extension of $f_0(w^{-1}(\zeta,t))$ into U (w^{-1} denotes the inverse function) because the solution to (1) is the function $f(\zeta,t)$ given as $f_0(w^{-1}(\zeta,t))$, where $\zeta=w(z,s)$ is a solution of the initial value problem for the characteristic equation (5) (or (4)) that maps U into U. Therefore, the solution of the initial value problem for the equation (1) may be non-univalent.

Let A stand for the usual class of all univalent holomorphic functions $f(z)=z+a_2z^2+...$ in the unit disk. Solutions to the equation (5) are regular univalent functions $w(z,t)=e^{-t}z+a_2(t)z^2+...$ in the unit disk that map U into itself. Conversely, every function from the class A can be represented by the limit

$$f(z)=\lim_{t\to\infty} e^t w(z,t), \tag{6}$$

where there exists a function $p(z,t)$ with positive real part for almost all $t\geq 0$, such that $w(z,t)$ is a solution to the equation (4) (see [20, pp. 159–163]). Each function $p(z,t)$ generates a unique function from the class A. The reciprocal statement is not true. In general, a function $f\in A$ can be determined by different functions p.

From [20, p. 163] it follows that we can guarantee the univalence of the solutions to the Löwner–Kufarev equation in partial derivatives (1) assuming the initial condition $f_0(\zeta)$ given by the limit (6) with the function $p(\cdot,t)$ chosen to be the same in the equations (1) and (5). Originally, these arguments have been made by

Prokhorov and the author in [21]. We remark also that an analogous Hamiltonian H can also be found in [4].

The Hamiltonian H is linear with respect to the variable f', therefore, the Hamiltonian dynamics which is generated by H is trivial and the velocity is constant. In [21] we studied another Hamiltonian system for a finite number of the coefficients of the function $w(z,t)$ generated by the equation (5). It turns out that the Hamiltonian system generated by the coefficients is Liouville partially integrable and the first integrals were obtained and were proved to form a finite-dimensional grading algebra generated by truncated Kirillov operators [1], [10] and [11] for a representation of the Virasoro algebra. However, the Hamiltonian was again linear with respect to the conjugate system, and we have both systems accelerationless. In order to describe a non-trivial motion we proceed with the Lagrangian formulation.

3. Lagrangian formulation of the subordination evolution

Let us consider a subordination chain $\{\Omega(t)\}_{t=0}^{\tau}$, $0\in\Omega(t)$, and the time-parameter family of the real-valued Green functions $G(z,t)$ of $\Omega(t)$ with the logarithmic singularity at 0. If $z=f(\zeta,t)$ is the Riemann map from the unit disk U onto $\Omega(t)$, $f'(0,t)=e^t$, then $G(z,t)=-\log|f^{-1}(z,t)|$. The unit normal vector $\mathbf{n}$ to $\partial\Omega(t)$ in the outward direction can be written as

$$\mathbf{n}=\frac{\zeta f'(\zeta,t)}{|f'(\zeta,t)|},\quad |\zeta|=1.$$

Therefore, the normal velocity v_n of the boundary $\partial\Omega(t)$ at the point $f(e^{i\theta},t)$ may be expressed as

$$v_n=\operatorname{Re}\dot{f}\frac{\overline{e^{i\theta}f'}}{|f'|}=|f'|\operatorname{Re}\frac{\dot{f}}{e^{i\theta}f'}=|f'|\rho(e^{i\theta},t),$$

where the function $\rho(e^{i\theta},t)$ was defined in the preceding section. Thus, $\rho(e^{i\theta},t)=v_n|\nabla G|$.

The easiest Lagrangian is given by the Dirichlet integral

$$\iint_D |\nabla G|^2\, d\sigma_z,$$

where $d\sigma_z=\left|\frac{1}{2}dz\wedge d\bar{z}\right|$, locally for any measurable set $D\subset\Omega(t)\setminus\{0\}$. However, this functional cannot be defined globally in $\Omega(t)\setminus\{0\}$ because of the parabolic singularity at the origin. To overcome this obstacle we define the energy represented by this Lagrangian in the following way. Let $\Omega_\varepsilon(t)=\Omega(t)\setminus\{z:|z|\le\varepsilon\}$ for a sufficiently small ε, $U_\varepsilon=\{\zeta:\varepsilon<|\zeta|<1\}$. Then the finite limit

$$\mathcal{E}=\lim_{\varepsilon\to 0}\left(\iint_{\Omega_\varepsilon(t)}|\nabla G|^2\,d\sigma_z+2\pi\log\varepsilon\right) \tag{7}$$

exists. Applying the conformal map $z=f(\zeta,t)$ and changing variables we arrive at the classical representation of the energy

$$\mathcal{E}=\mathcal{E}[f]=2\pi\log|f'(0,t)|=2\pi t,$$

as the capacity (or the conformal radius in this case) of $\partial\Omega(t)$. In other words, $\mathcal{E}$ represents the classical action for the Lagrangian defined by the Dirichlet integral. This interpretation allows us to get a less trivial Lagrangian description of subordination dynamics, that in particular, emerges in the Liouville part of the classical field theory [18] and [24].

The classical field theory studies the extremum of the action functional, and its critical value is called the classical action. The critical point ϕ^* satisfies Hamilton's principle (or the principle of the least action), i.e., $\delta\mathcal{S}[\phi^*]=0$, which is the Euler–Lagrange equation for the variational problem defined by the action functional $\mathcal{S}[\phi]$. For the action given by the Dirichlet integral the classical action is achieved for the harmonic ϕ^* and the principle of the least action leads to the Laplacian equation $\Delta\phi=0$ with relevant boundary conditions as above. The Liouville action plays a key role in two-dimensional gravity and leads to the Liouville equation, a solution of which is the Poincaré metric of constant negative curvature. This conformal metric is of a particular interest, because no flat metric satisfies the Einstein field equation. The conformal symmetry of classical field theory is generated by its energy-momentum tensor T whose mode expansion is expressed in terms of the operators satisfying the commutation relations of the Virasoro algebra. The $(2,0)$-component of the energy-momentum tensor in the Liouville theory is given by the expression $T_\varphi=\varphi_{zz}-\frac{1}{2}\varphi_z^2$ that leads to the classical Schwarz result $T_\varphi=S_f(\zeta)$ with the Schwarzian derivative S_f, where f is the ratio of two linearly independent solutions to the Fuchsian equation $w''+\frac{1}{2}T_\varphi w=0$.

In the case of subordination chains our starting point will be the Riemannian metric $ds^2=e^{\varphi(z)}|dz|^2$. In the case of the Liouville theory, the real-valued potential φ satisfies the Liouville equation $\varphi_{z\bar{z}}=\frac{1}{2}e^{\varphi}$ (generally, with certain prescribed asymptotics which guarantee the uniqueness). Geometrically, this means that the conformal metric ds^2 has constant negative curvature -1 on the underlying Riemann surface corresponding to the prescribed singularities. Let us consider the complex Green function $W(z,t)$ whose real part is $G(z,t)=\log|f^{-1}(z,t)|$, $z\in\Omega(t)\setminus 0$, as before. We have the representation $W(z,t)=-\log z+w_0(z,t)$, where $w_0(z,t)$ is an analytic function in $\Omega(t)$. Because of the conformal invariance of the Green function we have the superposition

$$(W\circ f)(\zeta,t) = -\log\zeta,$$

and the conformally invariant complex velocity field is just

$$W'(z,t) = -((f^{-1})'/f^{-1})(z,t),$$

where $\zeta = f^{-1}(z,t)$ is the inverse to our parametric function f and prime denotes the complex derivative. Rewriting this relation we get

$$(W'(z,t)\,dz)^2 = \frac{d\zeta^2}{\zeta^2}. \tag{8}$$

The velocity field is the conjugation of $-W'$. In other words the velocity field is directed along the trajectories of the quadratic differential in the left-hand side of (8) for each fixed moment t. The equality (8) implies that the boundary $\partial\Omega(t)$ is the orthogonal trajectory of the differential $(W'(z,t)\,dz)^2$, which has a double pole at the origin. The dependence on t yields that the trajectory structure of this differential changes in time, and in general, the stream lines are not inherited in time. These lines are geodesic in the conformal metric $|W'(z,t)||dz|$ generated by this differential. Let us use the conformal logarithmic metric generated by (8),

$$ds^2 = \frac{|(f^{-1})'|^2}{|f^{-1}|^2}|dz|^2 = |W'|^2|dz|^2,$$

which is intrinsically flat. Unlike the Poincaré metric, the hyperbolic boundary is not singular for the logarithmic metric whereas the origin is. But it is a parabolic singularity which can be easily regularized similarly to the Dirichlet integral (7). The density of this metric satisfies the usual Laplacian equation $\varphi_{z\bar z}=0$ in $\Omega(t)\setminus\{0\}$, where $\varphi(z)=\log(|(f^{-1})'|^2/|f^{-1}|^2)$. The function φ possesses the asymptotics

$$\varphi\sim\log\frac{1}{|z|^2}, \text{ and } |\varphi_z|\sim\frac{1}{|z|^2}, \quad \text{as } z\to 0,$$

therefore, the finite limit

$$\mathcal{S} = \mathcal{S}[\varphi] = \lim_{\varepsilon\to 0}\left(\iint_{\Omega_\varepsilon(t)} |\varphi_z|^2\,d\sigma_z + 2\pi\log\varepsilon\right) \tag{9}$$

exists and is called the logarithmic action.

The construction of the Riemannian metric ds^2 implies the rotation invariance instead of the Poincaré group in the classical field theory. Like in the Liouville theory we have no reason to discard the conformal factor (Weyl rescaling). Another important fact is that the infinitesimal structure at the boundary is invariant under the mapping f, i.e., the stream lines within the unit disk are radial, orthogonal to the unit circle and remain orthogonal under the conformal map f. In order to complete the boundary conditions we assign the boundary values $\varphi(e^{i\theta})=v_n$ to the potential φ, where the normal velocity v_n is taken at the boundary point $f(e^{i\theta},t)$.

The logarithmic action (9) represents the reduced area of $\Omega(t)$ in the Riemannian metric ds^2.

The potential $\varphi(z)$ possesses the following geometric meaning. It gives rise to a vector field $\nabla\varphi$, whose projection onto each normal outward vector to the level line of the Green function G is the curvature of this line, $\partial\varphi/\partial n=\varkappa$.

Lemma 1. *The Euler–Lagrange equation for the variational problem for the logarithmic action* $\mathcal{S}[\phi]$ *is the Laplacian equation* $\Delta\phi=-4\pi\delta_0(z)$, $z\in\Omega(t)$, *where* $\delta_0(z)$ *is the Dirac distribution supported at the origin, where* ϕ *is taken from the class of twice differentiable functions in* $\Omega(t)\setminus\{0\}$ *with the asymptotics* $\phi\sim-\log|z|^2$, *as* $z\to 0$.

Proof. Let us consider first the integral

$$\mathcal{S}_\varepsilon[\phi]=\iint_{\Omega_\varepsilon(t)}|\phi_z|^2\,d\sigma_z=\iint_{\mathbb{C}}\chi_{\Omega_\varepsilon(t)}|\phi_z|^2\,d\sigma_z,$$

where $\chi_{\Omega_\varepsilon(t)}$ is the characteristic function of $\Omega_\varepsilon(t)$. Then, due to Green's theorem,

$$\begin{aligned}(10)\qquad \lim_{h\to 0}\frac{\mathcal{S}_\varepsilon[\phi+hu]-\mathcal{S}_\varepsilon[\phi]}{h}&=2\iint_{\mathbb{C}}\chi_{\Omega_\varepsilon(t)}\operatorname{Re}\phi_z\overline{u}_z\,d\sigma_z\\ &=-\frac{1}{2}\iint_{\Omega_\varepsilon(t)}u\Delta\phi\;d\sigma_z+\frac{1}{2}\int_{\partial\Omega_\varepsilon(t)}u\frac{\partial\phi}{\partial n}\;ds,\end{aligned}$$

in distributional sense for every $C^\infty(\mathbb{C})$ test function u supported in $\Omega(t)$. On the other hand, we have $\partial\phi/\partial n\sim-2/\varepsilon$, as $\varepsilon\to 0$, and $u=0$ on $\partial\Omega(t)$. Therefore, the expression (10) tends to

$$-\frac{1}{2}\iint_{\Omega(t)}u\Delta\phi\,d\sigma_z-2\pi u(0),$$

as $\varepsilon\to 0$, and the latter must vanish, which is equivalent to the Laplacian equation mentioned in the statement of the lemma. Obviously, the logarithmic term in the definition of $\mathcal{S}[\phi]$ does not contribute to the variation. □

The above Dirichlet problem is related to an interesting property of plane domains discussed in [23]. If $\omega(z)$ is harmonic in a domain Ω and satisfies, on its boundary, the relation

$$\frac{\partial^2\omega}{\partial s^2}=\varkappa\frac{\partial\omega}{\partial n},$$

then

$$-i\frac{\partial^2\omega}{\partial z^2}\,dz^2$$

is a quadratic differential on Ω.

A straightforward calculation gives

$$\varphi_z = -\frac{1}{f'}\left(\frac{f''}{f'}+\frac{1}{\zeta}\right)\circ f^{-1}(z,t).$$

Hence, the action $\mathcal{S}$ can be expressed in terms of the parametric function f as

$$\mathcal{S}=\mathcal{S}[f]=\lim_{\varepsilon\to 0}\left(\iint_{U_\varepsilon}\left|\frac{f''}{f'}+\frac{1}{\zeta}\right|^2 d\sigma_\zeta+2\pi\log\varepsilon\right)+2\pi\log|f'(0,t)|, \tag{11}$$

or adding the logarithmic term into the integral we obtain

$$\mathcal{S}[f]=\iint_U\left(\left|\frac{f''}{f'}+\frac{1}{\zeta}\right|^2-\frac{1}{|\zeta|^2}\right)d\sigma_\zeta+2\pi\log|f'(0,t)|. \tag{12}$$

Within the quantum theory of Riemann surfaces the Liouville action is a Kähler potential of the Weil–Petersson metric on the space of deformations (the Teichmüller space), see [29] and [30]. We use a flat metric instead. Nevertheless, as we show further on, there are several common features between smooth subordination evolution and the Liouville theory. In particular, we shall derive the variation of the logarithmic action $\mathcal{S}$ and give connections with a representation of the Virasoro algebra and the Kähler geometry on the infinite dimensional manifold $\operatorname{Diff} S^1/S^1$.

4. Variation of the logarithmic action and the Kähler geometry on $\operatorname{Diff} S^1/S^1$

We denote the Lie group of C^∞ sense-preserving diffeomorphisms of the unit circle $S^1=\partial U$ by $\operatorname{Diff} S^1$. Each element of $\operatorname{Diff} S^1$ is represented as $z=e^{i\phi(\theta)}$ with an increasing, C^∞ real-valued function $\phi(\theta)$, such that $\phi(\theta+2\pi)=\phi(\theta)+2\pi$. The Lie algebra for $\operatorname{Diff} S^1$ is identified with the Lie algebra $\operatorname{Vect} S^1$ of smooth (C^∞) tangent vector fields to S^1 with the Poisson–Lie bracket given by

$$[\nu_1,\nu_2]=\nu_1\nu_2'-\nu_2\nu_1'.$$

There is no general theory of such infinite-dimensional Lie groups, as the one under consideration. The interest in this particular case comes first of all from the string theory where the Virasoro algebra appears as the central extension of $\operatorname{Vect} S^1$ and gives the mode expansion for the energy-momentum tensor. The central extension of Diff S^1 is called the Virasoro–Bott group. The entire necessary background for the construction of the theory of unitary representations of Diff S^1 is found in the study of Kirillov's homogeneous Kählerian manifold $\mathcal{M}=\text{Diff}\, S^1/S^1$. The group Diff S^1 acts as a group of left translations on the manifold $\mathcal{M}$ with the group S^1 as a stabilizer. The Kählerian geometry of $\mathcal{M}$ has been described by Kirillov and Yuriev in [12]. The manifold $\mathcal{M}$ admits several representations, in particular, in the space of smooth probability measures, a symplectic realization in the space of quadratic differentials, and an analytic representation used in this paper. Let A stand for the class of all analytic regular univalent functions f in U normalized by $f(0)=0$ and $f'(0)=1$. The analytic representation of $\mathcal{M}$ is based on the class $\tilde{A}$ of functions from A which being extended onto the closure $\overline{U}$ of U are supposed to be smooth on S^1. The class $\tilde{A}$ is dense in A in the local uniform topology of U. There exists a canonical identification of $\tilde{A}$ with $\mathcal{M}$ by the conformal welding. As a consequence, $\tilde{A}$ is a homogeneous space under the left action of Diff S^1, see [1, Theorem 1.4.1], [10], [11], [12] and [13]. As was mentioned in Section 2, see also [28], a smooth subordination evolution is governed by the Löwner–Kufarev equation (1) with a function $p(\zeta,t)$ which may be represented by the Cauchy–Schwarz formula with the boundary values $\rho(e^{i\theta},t)$, such that $4\pi\rho\in\text{Vect}\, S^1$.

Theorem 1. *Let $z=f(\zeta,t)$ be the parametric function for the subordination evolution $\Omega(t)$, $t\in[0,\tau)$, $f(\zeta,t)=e^t\zeta+\dots$. Let $\mathcal{S}[f]$ stand for the logarithmic action. Then,*

$$\frac{d}{dt}\mathcal{S}[f]=\int_0^{2\pi}\left[\text{Re}\left(1+\frac{e^{i\theta}f''}{f'}\right)\right]^2\nu(e^{i\theta},t)\,d\theta+\int_0^{2\pi}\text{Re}(e^{2i\theta}S_f)\nu(e^{i\theta},t)\,d\theta-2\pi,$$

where $\nu\in\text{Vect}\, S^1$, $\nu>0$ and $\int_0^{2\pi}\nu(e^{i\theta},t)\,d\theta=4\pi$.

Proof. We start rewriting the expression for $\mathcal{S}[f]$ as

$$\mathcal{S}[f]=\iint_U\left(\left|\frac{f''(\zeta,t)}{f'(\zeta,t)}\right|^2+2\,\text{Re}\,\frac{f''(\zeta,t)}{\bar{\zeta}f'(\zeta,t)}\right)d\sigma_\zeta+2\pi t,$$

and therefore,

$$\frac{d}{dt}\mathcal{S}[f]=2\,\text{Re}\iint_U\overline{\left(\frac{f''}{f'}+\frac{1}{\zeta}\right)}\left(\frac{\dot{f}''}{f'}-\frac{f''\dot{f}'}{(f')^2}\right)d\sigma_\zeta+2\pi.$$

Now applying the Löwner–Kufarev representation $\dot{f}=\zeta f' p(\zeta,t)$, we get

$$\frac{d}{dt}\mathcal{S}[f]=2\,\text{Re}\iint_U\overline{\left(\frac{f''}{f'}+\frac{1}{\zeta}\right)}\left(\left(1+\zeta\frac{f''}{f'}\right)p(\zeta,t)+\zeta p'(\zeta,t)\right)'d\sigma_\zeta+2\pi.$$

In order to apply Green's theorem, we remove the singularity at the origin by splitting the integral into two terms as

$$\frac{d}{dt}\mathcal{S}[f] = 2\,\mathrm{Re}\Big(\iint_{U_\varepsilon} \dots + \iint_{|\zeta|<\varepsilon} \dots\Big) + 2\pi,$$

where the second term

$$\iint_{|\zeta|<\varepsilon} \dots = \int_0^\varepsilon\int_0^{2\pi} \left(\overline{\frac{f''}{f'}} + \frac{e^{i\theta}}{r}\right)(\text{holomorphic function}) r\,d\theta\,dr \to 0,$$

as $\varepsilon\to 0$. Applying Green's theorem to the first term and taking into account that $p(0,t)=1$, we obtain

$$\iint_{U_\varepsilon} \dots = \left(-\frac{1}{2i}\int_{S^1} \dots\, d\bar{\zeta} + \frac{1}{2i}\int_{|\zeta|=\varepsilon} \dots\, d\bar{\zeta}\right) \longrightarrow \left(-\frac{1}{2i}\int_{S^1} \dots\, d\bar{\zeta} - \pi\right),$$

as $\varepsilon\to 0$. Thus, we have

$$\frac{d}{dt}\mathcal{S}[f] = \mathrm{Re}\int_0^{2\pi} \overline{\left(1+e^{i\theta}\frac{f''}{f'}\right)}\left(\left(1+e^{i\theta}\frac{f''}{f'}\right)p(e^{i\theta},t) + e^{i\theta}p'(e^{i\theta},t)\right)d\theta,$$

or

$$\begin{aligned}\frac{d}{dt}\mathcal{S}[f] = 2\pi\int_0^{2\pi}\left|1+e^{i\theta}\frac{f''}{f'}\right|^2 \rho(e^{i\theta},t)\,d\theta + \int_0^{2\pi}\mathrm{Re}\left(1+e^{i\theta}\frac{f''}{f'}\right)\mathrm{Re}\,e^{i\theta}p'(e^{i\theta},t)\,d\theta \\ + \int_0^{2\pi}\mathrm{Im}\left(1+e^{i\theta}\frac{f''}{f'}\right)\mathrm{Im}\,e^{i\theta}p'(e^{i\theta},t)\,d\theta.\end{aligned}$$

These equalities are thought of as limiting values making use of the smoothness of f on the boundary. Let us denote by J_1, J_2 and J_3, the first, the second and the third term, respectively, in the latter expression. We have

$$J_3 = \int_0^{2\pi}\mathrm{Im}\left(1+e^{i\theta}\frac{f''}{f'}\right)\mathrm{Im}\int_0^{2\pi}\frac{2e^{i\theta}e^{i\alpha}}{(e^{i\alpha}-e^{i\theta})^2}\rho(e^{i\alpha},t)\,d\alpha\,d\theta.$$

Obviously,

$$\frac{\partial}{\partial\alpha}\left(\frac{e^{i\alpha}+\zeta}{e^{i\alpha}-\zeta}\right) = \frac{-2\zeta i e^{i\alpha}}{(e^{i\alpha}-\zeta)^2}.$$

Integrating by parts and applying the Cauchy–Schwarz formula we obtain

$$J_3 = 2\pi\,\mathrm{Re}\int_0^{2\pi}\left(e^{i\theta}\frac{f''}{f'} + e^{2i\theta}\left(\frac{f'''}{f'} - \left(\frac{f''}{f'}\right)^2\right)\right)\rho(e^{i\theta},t)\,d\theta.$$

Using the representation of the function $p(\zeta,t)$ the integral J_2 admits the form

$$J_2=\int_0^{2\pi}\mathrm{Re}\left(1+e^{i\theta}\frac{f''}{f'}\right)\mathrm{Re}\int_0^{2\pi}\frac{2e^{i\theta}e^{i\alpha}}{(e^{i\alpha}-e^{i\theta})^2}\rho(e^{i\alpha},t)\,d\alpha\,d\theta.$$

Changing the order of integration implies that

$$\begin{aligned}J_2&=\int_0^{2\pi}\mathrm{Re}\int_0^{2\pi}\mathrm{Re}\left(1+e^{i\theta}\frac{f''}{f'}\right)\frac{2e^{i\theta}e^{i\alpha}}{(e^{i\alpha}-e^{i\theta})^2}\rho(e^{i\alpha},t)\,d\alpha\,d\theta\\&=\mathrm{Re}\int_0^{2\pi}\rho(e^{i\alpha},t)\left(\int_0^{2\pi}\mathrm{Re}\left(1+e^{i\theta}\frac{f''}{f'}\right)\frac{2e^{i\theta}e^{i\alpha}}{(e^{i\alpha}-e^{i\theta})^2}\,d\theta\right)d\alpha.\end{aligned}$$

Integrating by parts we obtain that

$$J_2=\int_0^{2\pi}\rho(e^{i\alpha},t)\left(\mathrm{Re}(-i)\int_0^{2\pi}\frac{\partial}{\partial\theta}\mathrm{Re}\left(1+e^{i\theta}\frac{f''}{f'}\right)\frac{e^{i\theta}+e^{i\alpha}}{e^{i\alpha}-e^{i\theta}}\,d\theta\right)d\alpha.$$

The inner integral represents an analytic function by the Cauchy formula (modulo an imaginary constant). Taking into account the normalization at the origin we get

$$J_2=2\pi\,\mathrm{Re}\int_0^{2\pi}\left(e^{i\alpha}\frac{f''}{f'}+e^{2i\alpha}\left(\frac{f'''}{f'}-\left(\frac{f''}{f'}\right)^2\right)\right)\rho(e^{i\alpha},t)\,d\alpha=J_3.$$

Summing up $J_1+J_2+J_3$, and taking $\nu=4\pi\rho$ into account, concludes the proof. □

In the particular case of the Laplacian growth evolution this theorem has been proved in [8, Theorem 7.4.10]. The normal velocity of the boundary is equal to the gradient of the Green function and $\nu(e^{i\theta},t)=2/|f'(e^{i\theta},t)|^2$.

In two-dimensional conformal field theories [6], the algebra of the energy-momentum tensor is deformed by a central extension due to the conformal anomaly, and becomes the *Virasoro algebra*. The Virasoro algebra is spanned by the elements $e_k=\zeta^{1+k}\partial$, $\zeta=e^{i\theta}$, $k\in\mathbb{Z}$, and c, where c is a real number, called the *central charge*. The commutator is defined by

$$[e_m,e_n]_{\mathrm{Vir}}=(n-m)e_{m+n}+\frac{c}{12}m(m^2-1)\delta_{n,-m},\quad [c,e_k]=0.$$

The Virasoro algebra Vir can be realized as a central extension $\mathrm{Vect}\,S^1\oplus\mathbb{R}$ of $\mathrm{Vect}\,S^1$ by defining

$$[\phi\partial+ca,\psi\partial+cb]_{\mathrm{Vir}}=(\phi\psi'-\phi'\psi)\partial+\frac{c}{12}\omega(\phi,\psi),$$

(whereas $[\phi,\psi]=\phi\psi'-\phi'\psi$), where the bilinear 2-form $\omega(\phi,\psi)$ on $\mathrm{Vect}\,S^1$ is given by

$$\omega(\phi,\psi)=-\frac{1}{4\pi}\int_0^{2\pi}(\phi'+\phi''')\psi\,d\theta,$$

and a and b are real numbers. This form defines the Gelfand–Fuchs cocycle on $\operatorname{Vect} S^1$ and satisfies the Jacobi identity. The factor of $\frac{1}{12}$ is merely a matter of convention. The manifold $\mathcal{M}$ being considered as a realization $\tilde{A}$ admits affine coordinates $\{c_k\}_{k=2}^{\infty}$, where c_k is the kth coefficient of a univalent functions $f\in\tilde{A}$. Due to de Branges' theorem [5], $\mathcal{M}$ is a bounded open subset of

$$\{\{c_k\}_{k=2}^{\infty} : |c_k| < k+\varepsilon \text{ for } k=2,3,\ldots\}.$$

The Goluzin–Schiffer variational formula lifts the actions from the Lie algebra $\operatorname{Vect} S^1$ onto $\tilde{A}$. Let $f\in\tilde{A}$ and let $\nu(e^{i\theta})$ be a C^∞ real-valued function in $\theta\in(0,2\pi]$ from $\operatorname{Vect} S^1$ making an infinitesimal action as $\theta\mapsto\theta+\tau\nu(e^{i\theta})$. Let us consider a variation of f given by

$$L_\nu[f](\zeta)=\frac{f^2(\zeta)}{2\pi}\int_{S^1}\left(\frac{wf'(w)}{f(w)}\right)^2\frac{\nu(w)}{f(w)-f(\zeta)}\frac{dw}{w}. \tag{13}$$

Kirillov and Yuriev [12] and [13] have established that the variations $L_\nu[f](\zeta)$ are closed with respect to the commutator and the induced Lie algebra is the same as $\operatorname{Vect} S^1$. Moreover, Kirillov's result [10] states that there is an exponential map $\operatorname{Vect} S^1\to\operatorname{Diff} S^1$ such that the subgroup S^1 coincides with the stabilizer of the map $f(\zeta)\equiv\zeta$ from $\tilde{A}$.

It is convenient [11] to extend (13) by complex linearity to $\mathbb{C}\operatorname{Vect} S^1\to\operatorname{Vect}\tilde{A}$. Taking $\nu_k=-ie^{ik\theta}$, $k\geq 0$, from the basis of $\mathbb{C}\operatorname{Vect} S^1$, we obtain the expressions for $L_k=\delta_\nu f$, $f\in\tilde{A}\simeq\mathcal{M}$ (see formula (13)), as

$$L_0=\zeta f'(\zeta)-f(\zeta) \quad\text{and}\quad L_k=\zeta^{1+k}f' \text{ for } k>0.$$

The computation of L_k for $k<0$ is more difficult because of the poles of the integrand. For example,

$$L_{-1}=f'-1-2c_2f \quad\text{and}\quad L_{-2}=\frac{f'}{\zeta}-\frac{1}{f}-3c_2+\left(c_2^2-4c_3\right)f$$

(see, e.g., [11]).

In general, we have real vector fields in $\operatorname{Vect} S^1$. The computation of L_k must be carried out with respect to the basis $1, e^{\pm ki\theta}$ that leads also to L_k with $k\leq 0$. However, we deal with holomorphic functions and L_k with $k>0$ are to be treated as complex vector fields (see the discussions in [11, p. 738] and [1, pp. 632–634]).

In terms of the coordinates $\{c_k\}_{k=2}^{\infty}$ on $\mathcal{M}$,

$$L_0=\sum_{n=1}^{\infty}nc_n\partial_n \quad\text{and}\quad L_k=\partial_k+\sum_{n=1}^{\infty}(n+1)c_n\partial_{k+n} \text{ for } k>0,$$

where $\partial_k=\partial/\partial c_{k+1}$.

Neretin [17] introduced the sequence of polynomials P_k, in the coordinates $\{c_k\}_{k=2}^{\infty}$ on $\mathcal{M}$ by the following recurrent relations

$$L_m(P_n)=(n+m)P_{n-m}+\frac{c}{12}m(m^2-1)\delta_{n,m}, \quad P_0=P_1\equiv 0, \quad P_k(0)=0,$$

where the central charge c is fixed. This gives, for example, $P_2=c/2(c_3-c_2^2)$, $P_3=2c(c_4-2c_2c_3+c_2^3)$. In general, the polynomials P_k are homogeneous with respect to rotations of the function f. It is worth to mention that estimates of the absolute value of these polynomials have been a subject of investigations in the theory of univalent functions for a long time, e.g., for $|P_2|$ we have $|c_3-c_2^2|\leq 1$ (Bieberbach 1916 [3]), for estimates of $|P_3|$ see [7], [15], [25] and [26]. For the Neretin polynomials one can construct the generatrix function

$$P(\zeta)=\sum_{k=1}^{\infty}P_k\zeta^k=\frac{c\zeta^2}{12}S_f(\zeta),$$

where $S_f(\zeta)$ is the Schwarzian derivative of f. Let $\nu\in\mathbb{C}\,\mathrm{Vect}\,S^1$ and ν^g be the associated right-invariant tangent vector field defined at $g\in\mathrm{Diff}\,S^1$. For the basis $\nu_k=-ie^{ik\theta}\partial$, one constructs the corresponding associated right-invariant basis ν_k^g. By $\{\psi_{-k}\}_k$ we denote the dual basis of 1-forms such that the value of each form on the vector ν_k^g is given as

$$(\psi_k,\nu_n^g)=\delta_{k+n,0}.$$

Let us construct the 1-form Ψ on $\mathrm{Diff}\,S^1$ by

$$\Psi=\sum_{k=1}^{\infty}(P_k\circ\pi)\psi_k,$$

where π denotes the natural projection $\mathrm{Diff}\,S^1\to\mathcal{M}$. This form appeared in [1] and [2] in the context of the construction of a unitarizing probability measure for the Neretin representation of $\mathcal{M}$. It is invariant under the left action of S^1. If $f\in\tilde{A}$ represents g and $\nu\in\mathrm{Vect}\,S^1$, then the value of the form Ψ on the vector ν is

$$(\Psi,\nu)_f=\int_0^{2\pi}e^{2i\theta}\nu(e^{i\theta})S_f\,d\theta,$$

see [1] and [2]. So the variation of the logarithmic action given in Theorem 1 becomes

$$\frac{d}{dt}\mathcal{S}[f]=\int_0^{2\pi}\left[\mathrm{Re}\left(1+\frac{e^{i\theta}f''}{f'}\right)\right]^2\nu(e^{i\theta},t)\,d\theta+\mathrm{Re}(\Psi,\nu)_f-2\pi.$$

Taking into account the definition of the mean curvature $\varkappa(z,t)$ of the boundary of $\Omega(t)$, and the normal velocity v_n, we conclude that

$$\frac{d}{dt}\mathcal{S}[f] = 4\pi \int_{\partial\Omega(t)} (\varkappa v_n)^2 |dz| + \mathrm{Re}(\Psi, \nu)_f - 2\pi.$$

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