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The Melin calculus for general homogeneous groups

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Abstract. The purpose of this note is to give an extension of the symbolic calculus of Melin for convolution operators on nilpotent Lie groups with dilations. Whereas the calculus of Melin is restricted to stratified nilpotent groups, the extension offered here is valid for general homogeneous groups. Another improvement concerns the L^2-boundedness theorem, where our assumptions on the symbol are relaxed. The zero-class conditions that we require are of the type

$$|D^\alpha a(\xi)| \le C_\alpha \prod_{j=1}^{R} \rho_j(\xi)^{-|\alpha_j|},$$

where ρ_j are "partial homogeneous norms" depending on the variables ξ_k for $k>j$ in the natural grading of the Lie algebra (and its dual) determined by dilations. Finally, the class of admissible weights for our calculus is substantially broader. Let us also emphasize the relative simplicity of our argument compared to that of Melin.

The purpose of this note is to give an extension of the symbolic calculus of Melin [7] for convolution operators on nilpotent Lie groups with dilations. The calculus can be viewed as a higher order generalization of the Weyl calculus for pseudodifferential operators of Hörmander [3]. In fact, the idea of such a calculus is very similar. It consists in describing the product

$$a\#b = (a^\vee \star b^\vee)^\wedge, \quad a, b \in C_c^\infty(\mathfrak{g}^\star),$$

on a homogeneous Lie group G, where $f^\wedge$ and $f^\vee$ denote the Abelian Fourier transforms on the Lie algebra $\mathfrak{g}$ and its dual $\mathfrak{g}^\star$, and its continuity in terms of suitable norms similar to those used in the theory of pseudodifferential operators. An integral part of the calculus is an L^2-boundedness theorem of the Calderón–Vaillancourt type.

This has been done by Melin whose starting point was the following formula

$$a\#b(\xi) = \mathbf{U}(a \otimes b)F(\xi, \xi),$$

where

$$\mathbf{U}(F)^{\vee}(x,y)=F^{\vee}\left(\frac{x-y+xy}{2},\frac{y-x+xy}{2}\right),\quad x,y\in\mathfrak{g}.$$

Melin shows that the unitary operator $\mathbf{U}$ can be imbedded in a one-parameter unitary group U_t with the infinitesimal generator Γ which is a differential operator on $\mathfrak{g}^\star\times\mathfrak{g}^\star$ with polynomial coefficients, and he thoroughly investigates the properties of Γ under the assumption that G is a homogeneous stratified group. From the continuity of $\mathbf{U}$ he derives a composition formula for classes of symbols satisfying the estimates

$$|D^\alpha a(\xi)|\le C_\alpha(1+|\xi|)^{m-|\alpha|}, \tag{0.1}$$

where $|\cdot|$ is the homogeneous norm on $\mathfrak{g}^\star$ and $|\alpha|$ is the homogeneous length of a multiindex α. He also proves an L^2-boundedness theorem for symbols satisfying (0.1) with $m=0$.

Our extension goes in various directions. First of all the calculus of Melin is restricted to stratified nilpotent groups, whereas the extension offered here is valid for general homogeneous groups. Another improvement concerns the L^2-boundedness theorem, where our assumptions on the symbol are less restrictive. The zero-class conditions that we require are

$$|D^\alpha a(\xi)|\le C_\alpha\prod_{j=1}^{R}\rho_j(\xi)^{-|\alpha_j|},$$

where ρ_j are "partial homogeneous norms" depending on the variables ξ_k for $k>j$ in the natural grading of the Lie algebra (and its dual) determined by dilations, and $\alpha=(\alpha_1,\alpha_2,...,\alpha_R)$ is the corresponding representation of the multiindex α relative to the grading. This direction of generalization of the boundedness theorem had been suggested by Howe [5] even before the Melin calculus was created. Finally, the class of admissible weights for our calculus is substantially broader. Let us also emphasize the relative simplicity of our argument compared to that of Melin.

Most of the techniques applied here have been already developed in a very similar context of [2]. They heavily rely on the methods of the Weyl calculus of Hörmander [3]. We take this opportunity to clarify some technical points which remained somewhat obscure in [2]. One major mistake is also corrected. Some repetition is therefore unavoidable. In [2] the reader will also find more on the background and history of various symbolic calculi on nilpotent Lie groups.

1. Preliminaries

Let X be a finite-dimensional Euclidean space. Denote by $\langle\,\cdot\,,\,\cdot\,\rangle$ and $\|\cdot\|$ the scalar product and the corresponding Euclidean norm. These are fixed throughout the paper. Whenever we identify $X^\star$ with X, it is by means of the duality determined by the scalar product. Let $X=\bigoplus_{j=1}^R X_j$ be an orthogonal sum and

$$\delta_t x_j = t^{d_j}, \quad x_j \in X_j,$$

be a family of dilations with eigenvalues $\mathcal{D}=\{d_j\}_{j=1}^R$, where

$$1 = d_1 < d_2 < \ldots < d_R.$$

Let

$$|x| = \left(\sum_{j=1}^R \|x_j\|^{2/d_j}\right)^{1/2}$$

be the corresponding homogeneous norm and

$$\rho(x) = 1+|x|.$$

Let us define a family of Euclidean norms

$$\mathbf{p}_x(z)^2 = \sum_{j=1}^R \frac{\|z_j\|^2}{\rho(x)^{2d_j}}, \quad x \in X.$$

Lemma 1.1. *We have*

$$\frac{1}{2} \le \frac{\rho(x)}{\rho(y)} \le 2 \quad \textit{if} \quad \mathbf{p}_x(x-y) < \left(\frac{1}{2\sqrt{R}}\right)^{d_R} \tag{1.2}$$

and

$$\rho(x) \le \big(\sqrt{R}+1\big)\rho(y)(1+\mathbf{p}_y(x-y)). \tag{1.3}$$

Proof. Observe that $\mathbf{p}_x(x-y) < \big(1/\big(2\sqrt{R}\big)\big)^{d_R}$ yields

$$\|x_j-y_j\|^{1/d_j} \le \frac{\rho(x)}{2R}, \quad 1 \le j \le R,$$

so

$$|x-y| \le \tfrac{1}{2}\rho(x),$$

and consequently

$$\rho(x)\le\rho(y)+|x-y|\le\rho(y)+\tfrac{1}{2}\rho(x),$$
$$\rho(y)\le\rho(x)+|x-y|\le\tfrac{3}{2}\rho(x),$$

which implies

$$\frac{1}{2}\le\frac{\rho(x)}{\rho(y)}\le 2.$$

We also have

$$\frac{\|x_j-y_j\|^{1/d_j}}{\rho(y)}\le\frac{\|x_j-y_j\|}{\rho(y)^{d_j}}+1,$$

so

$$\frac{|x-y|}{\rho(y)}\le\sqrt{R}+\mathbf{p}_y(x-y),$$

and finally

$$\rho(x)\le\rho(y)+|x-y|=\rho(y)\left(1+\frac{|x-y|}{\rho(y)}\right)\le\big(\sqrt{R}+1\big)\rho(y)(1+\mathbf{p}_y(x-y)),$$

which completes the proof. □

For $0\le j\le R$, let

$$\rho_j(x)=\rho(x^j)=\rho\left(\sum_{k=j+1}^{R}x_k\right)$$

and let

$$\mathbf{q}_x(z)^2=\sum_{j=1}^{R}\frac{\|z_j\|^2}{\rho_j(x)^{2d_j}},\qquad x\in X,$$

be another family of norms on X determined by ρ.

Lemma 1.4. *There exist constants $C,M>0$ such that*

$$\rho_j(x)\le C\rho_j(y)(1+\mathbf{q}_y(x-y))\tag{1.5}$$

and

$$\rho_j(x)\le C\rho_j(y)(1+\mathbf{q}_x(x-y))^M.\tag{1.6}$$

Proof. Inequality (1.5) is proved in the same way as (1.3). The second inequality is proved by induction. In fact, if $j=R$, there is nothing to prove. Assume that (1.6) holds for $k>j$ with $1\le C=C_k\le C_{j+1}$ and $M=M_k\le M_{j+1}$. Then

$$\begin{aligned}\rho_j(x)&\le\rho_j(y)+|(x-y)^j|\le\rho_j(y)\left(1+\frac{|(x-y)^j|}{\rho_j(y)}\right)\\&\le\rho_j(y)\left(1+\sqrt{\sum_{k=j+1}^{R}\frac{\|x_k-y_k\|^{2/d_k}}{\rho_k(y)^2}}\right)\\&\le\sqrt{R-j}\rho_j(y)\left(1+\sqrt{\sum_{k=j+1}^{R}\frac{\|x_k-y_k\|^2}{\rho_k(y)^{2d_k}}}\right)\\&\le C_j\rho_j(y)\left(1+\sqrt{\sum_{k=j+1}^{R}\frac{\|x_k-y_k\|^2}{\rho_k(x)^{2d_k}}}\right)(1+\mathbf{q}_x(x-y))^{d_{j+1}M_{j+1}}\\&\le C_j\rho_j(y)(1+\mathbf{q}_x(x-y))^{M_j},\end{aligned}$$

which shows that (1.6) holds also for j with

$$C_j=\sqrt{R-j}C_{j+1}^{d_{j+1}}\ge C_{j+1}\ge 1,\quad M_j=d_{j+1}M_{j+1}+1\ge M_{j+1}.\quad\square$$

A family of Euclidean norms (a metric) $\mathbf{g}=\{\mathbf{g}_x\}_{x\in X}$ on X is called *slowly varying* if there exists $0<\gamma\le 1$ such that

$$\gamma\le\frac{\mathbf{g}_y}{\mathbf{g}_x}\le\frac{1}{\gamma},\quad\text{if}\quad\mathbf{g}_x(x-y)<\gamma.\tag{1.7}$$

A metric $\mathbf{g}$ on X is called *tempered* with respect to another metric $\mathbf{G}$, or briefly $\mathbf{G}$-tempered, if there exist $C,M>0$ such that

$$\left\{\frac{\mathbf{g}_x}{\mathbf{g}_y}\right\}^{\pm1}\le C(1+\mathbf{G}_x(x-y))^M,\quad\mathbf{g}_x\le\mathbf{G}_x.$$

Note that a self-tempered metric is automatically slowly varying.

Lemma 1.8. *If $\mathbf{g}$ is a self-tempered family of norms with constants C and M, then for every $x,y,z\in X$,*

$$\begin{aligned}&1+\mathbf{g}_x(x-y)\le C(1+\mathbf{g}_y(y-x))^{M+1},\\&1+\mathbf{g}_x(x-y)\le C^2(1+\mathbf{g}_x(x-z))^M(1+\mathbf{g}_y(y-z))^{M+1}.\end{aligned}$$

Proof. In fact,

$$1+\mathbf{g}_x(x-y)\le 1+C\mathbf{g}_y(y-x)(1+\mathbf{g}_y(y-x))^M\le C(1+\mathbf{g}_y(y-x))^{M+1},$$

as required. Moreover, by the first inequality,

$$\begin{aligned}1+\mathbf{g}_x(x-y) &\le 1+\mathbf{g}_x(x-z)+\mathbf{g}_x(z-y)\\ &\le 1+\mathbf{g}_x(x-z)+C\mathbf{g}_z(z-y)(1+\mathbf{g}_x(x-z))^M\\ &\le (1+\mathbf{g}_x(x-z))^M(1+C\mathbf{g}_z(z-y))\\ &\le C^2(1+\mathbf{g}_x(x-z))^M(1+\mathbf{g}_y(y-z))^{M+1},\end{aligned}$$

which completes the proof. □

Corollary 1.9. *The metrics* $\mathbf{p}$ *and* $\mathbf{q}$ *are slowly varying and* $\mathbf{q}$*-tempered.*

Proof. This follows immediately from Lemmas 1.1 and 1.4. □

A strictly positive function $\mathbf{m}$ on X is a *weight* on X with respect to the $\mathbf{G}$-tempered metric $\mathbf{g}$, if it satisfies the conditions

$$\left\{\frac{\mathbf{m}(x)}{\mathbf{m}(y)}\right\}^{\pm 1} \le C \quad \text{if } \mathbf{g}_x(x-y)\le\gamma \tag{1.10}$$

and

$$\left\{\frac{\mathbf{m}(x)}{\mathbf{m}(y)}\right\}^{\pm 1} \le C(1+\mathbf{G}(x-y))^M \tag{1.11}$$

for some $C, M>0$. The weights form a group under multiplication. A typical example of a weight for $\mathbf{p}$ or $\mathbf{q}$ is $\mathbf{m}(x)=\rho(x)$. A universal example is

$$\mathbf{m}(x)=\mathbf{g}_x(x-x_0), \tag{1.12}$$

where x_0 is fixed.

Let $\mathbf{m}$ be a weight with respect to a metric $\mathbf{g}$. For $f\in C^\infty(X)$ let

$$|f|^{\mathbf{m}}_{(k)}(\mathbf{g}) = \sup_{x\in X}\frac{\mathbf{g}_x(D^k f(x))}{\mathbf{m}(x)},$$

and

$$|f|^{\mathbf{m}}_{k}(\mathbf{g}) = \sum_{j=0}^{k}|f|^{\mathbf{m}}_{(j)}(\mathbf{g}),$$

where D stands for the Fréchet derivative, and

$$\mathbf{g}_x(D^k f(x)) = \sup_{\mathbf{g}_x(y_j)\le 1}|D^k f(x)(y_1,y_2,\dots,y_k)|.$$

Let

$$S^{\mathbf{m}}(X,\mathbf{g})=\{a\in C^\infty(X):|a|_k^{\mathbf{m}}(\mathbf{g})<\infty \text{ for all } k\in\mathbf{N}\}.$$

$S^{\mathbf{m}}(X,\mathbf{g})$ is a Fréchet space with the family of seminorms $|\cdot|_k^{\mathbf{m}}(\mathbf{g})$. Thus $f\in C^\infty(X)$ belongs to $S^{\mathbf{m}}(X,\mathbf{p})$ if and only if it satisfies the estimates

$$|D^\alpha f(x)|\le C_\alpha \mathbf{m}(x)\rho(x)^{-|\alpha|},$$

where $|\alpha|$ is the homogeneous length of a multiindex α. The estimates for $f\in S^{\mathbf{m}}(X,\mathbf{q})$ are

$$|D^\alpha f(x)|\le C_\alpha \mathbf{m}(x)\prod_{j=1}^{R}\rho_j(x)^{-d_j\alpha_j}.$$

Note that every $\mathbf{p}$-weight $\mathbf{m}$ is also a $\mathbf{q}$-weight and $S^{\mathbf{m}}(X,\mathbf{p})\subset S^{\mathbf{m}}(X,\mathbf{q})$. Moreover for every k,

$$|\cdot|_k^{\mathbf{m}}(\mathbf{q})\le|\cdot|_k^{\mathbf{m}}(\mathbf{p}) \tag{1.13}$$

so the inclusion is continuous.

Apart from the Fréchet topology in the spaces $S^{\mathbf{m}}$ it is convenient to introduce a *weak topology* of the C^∞-convergence on Fréchet bounded subsets. By the Ascoli theorem, this is equivalent to the pointwise convergence of bounded sequences in $S^{\mathbf{m}}$. Following Manchon [6] we call a mapping $T\colon S^{\mathbf{m}_1}\to S^{\mathbf{m}_2}$ *double-continuous*, if it is both Fréchet continuous and weakly continuous. Moreover, $C_c^\infty(X)$ is weakly dense in $S^{\mathbf{m}}(X,\mathbf{g})$. The last assertion is a consequence of Proposition 2.1 (b) below.

2. The method of Hörmander

The following construction of a partition of unity is due to Hörmander [3]. Also the lemma that follows is an important principle of the Hörmander theory. For the convenience of the reader we include the proofs here.

Proposition 2.1. *Let* $\mathbf{g}$ *be a metric on* X.

(a) *For every* $0<r<\gamma$ *there exists a sequence* $x_\nu\in X$ *such that* X *is the union of the balls*

$$B_\nu=B_\nu(r)=\{x\in X:\mathbf{g}_{x_\nu}(x-x_\nu)<r\}$$

and no point $x\in X$ *belongs to more than* N *balls, where* N *does not depend on* x.

(b) *There exists a family of functions* $\phi_\nu \in C_c^\infty(B_\nu)$ *bounded in* $S^{\mathbf{1}}(X,\mathbf{g})$ *and such that*

$$\sum_\nu \phi_\nu(x) = 1, \quad x \in X.$$

(c) *For* $x \in X$ *let*

$$d_\nu(x) = \mathbf{g}_{x_\nu}(x - z).$$

If the metric $\mathbf{g}$ *is self-tempered, then there exist constants* $M, C_0 > 0$ *such that*

$$\sum_\nu (1 + d_\nu(x))^{-M} \le C_0, \quad x \in X.$$

All the estimates in the construction depend just on the constant γ *in* (1.7) *and the choice of* r.

Proof. (a) Let $0 < r < \gamma$. Let $\{x_\nu\}_\nu$ be a maximal sequence of points in X such that

$$\mathbf{g}_{x_\nu}(x_\mu - x_\nu) \ge \gamma r, \quad \mu \ne \nu.$$

Let $x \in X$. Note that

$$\mathbf{g}_x(x - x_\nu) < \gamma r \quad \text{implies} \quad \mathbf{g}_{x_\nu}(x - x_\nu) < r.$$

Therefore, either $\mathbf{g}_{x_\nu}(x - x_\nu) < r$ for some ν, or

$$\mathbf{g}_x(x - x_\nu) \ge \gamma r \quad \text{and} \quad \mathbf{g}_{x_\nu}(x - x_\nu) \ge r \ge \gamma r.$$

The latter contradicts the maximality of our sequence. The former implies that $X \subset \bigcup_\nu B_\nu$.

To show that the covering is uniformly locally finite suppose that $x \in B_\nu$. Then $\mathbf{g}_{x_\nu}(x - x_\nu) < r$, which implies $\mathbf{g}_x(x - x_\nu) < r/\gamma < 1$. On the other hand

$$\mathbf{g}_x(x_\mu - x_\nu) \ge \gamma r \quad \text{for } \mu \ne \nu.$$

The number of points from a uniformly discrete set in a unit ball is bounded independently of the given norm $\mathbf{g}_x$ so we are done.

(b) Let $0 < r < r_1 < \gamma$. Let $\psi \in C_c^\infty(-r_1^2, r_1^2)$ be equal to 1 on the smaller interval $[-r^2, r^2]$. If

$$\psi_\nu(x) = \psi(\mathbf{g}_{x_\nu}(x - x_\nu)^2),$$

then, by part (a), $\sum_\mu \psi_\mu(x) \geq 1$ for every $x \in X$, and it is not hard to see that

$$\phi_\nu(x) = \frac{\psi_\nu(x)}{\sum_\mu \psi_\mu}$$

has all the required properties.

(c) Let $r<\gamma$. Let $x\in X$. For $k\in\mathbf{N}$ let

$$M_k = \{\nu : d_\nu(x) < k\}.$$

It is sufficient to show that the number $|M_k|$ of elements in M_k is bounded by a polynomial in k. Let $\nu \in M_k$ and let

$$V_\nu = \{z \in X : \mathbf{g}_x(z-x_\nu) < r_k\},$$

where

$$r_k = \frac{r}{C(1+k)^M}.$$

Observe that V_ν is contained both in B_ν (see part (a)) and in the ball

$$V = \{z \in X : \mathbf{g}_x(z-x) < R_k\},$$

where $R_k = r_k + C(1+k)^{M+1}$. In fact, if $\mathbf{g}_x(z-x_\nu) < r_k$, then

$$\mathbf{g}_{x_\nu}(z-x_\nu) \leq C\mathbf{g}_x(z-x_\nu)(1+k)^M < r,$$

and

$$\begin{aligned} \mathbf{g}_x(z-x) \leq \mathbf{g}_x(z-x_\nu) + \mathbf{g}_x(x_\nu - x) &< r_k + C\mathbf{g}_{x_\nu}(x_\nu - x)(1+\mathbf{g}_{x_\nu}(x_\nu - x))^M \\ &< r_k + C(1+1+\mathbf{g}_{x_\nu}(x_\nu - x))^{M+1} < r_k + C(1+k)^{M+1} \end{aligned}$$

Hence

$$C_1 |M_k| r_k^{\dim X} \leq \sum_{\nu\in M_k} |V_\nu| \leq N \Big| \bigcup_{\nu \in M_k} V_\nu \Big| \leq N|V| \leq C_1 N R_k^{\dim X},$$

which immediately implies the desired estimate

$$|M_k| \leq N \left(1 + \frac{R_k}{r_k}\right)^{\dim X}. \quad \square$$

Lemma 2.2. *Let X be a finite-dimensional vector space with a Euclidean norm $\|\cdot\|$. Let $r_1>r>0$. Let L be an affine function such that $L(x)\neq 0$ for $x\in B(x_0, r_1)$.*

Then for every $k\in\mathbf{N}$,

$$\left\|D^k\frac{1}{L}(x)\right\|\le\frac{k!r_1}{(r_1-r)^{k+1}|L(x_0)|},\quad x\in B(x_0,r).$$

The estimate does not depend on the choice the norm.

Proof. We may assume that $x_0=0$ and $L(0)=1$. Let ξ be a linear functional on X such that $L(x)=\langle\xi,x\rangle+1$. Since

$$L(x)=\langle\xi,x\rangle+1>0,\quad \|x\|<r_1,$$

it follows that $\|\xi\|\le 1/r_1$ and

$$L(x)\ge 1-\frac{r}{r_1},\quad x\in B(0,r).$$

Consequently,

$$\begin{aligned}\left\|D^k\frac{1}{L}(x)\right\|&\le\frac{k!\|\xi\|^k}{|L(x)|^{k+1}}\le\frac{k!(1/r_1)^k}{(r_1-r/r_1)^{k+1}}\\&\le\frac{k!r_1}{(r_1-r)^{k+1}}\end{aligned}$$

for x in $B(0,r)$. $\square$

For the general theory of slowly varying metrics and its applications to the theory of pseudodifferential calculus the reader is referred to Hörmander [4], vol. I and III.

3. The Melin operator U

Let $\mathfrak{g}$ be a nilpotent Lie algebra with a fixed scalar product. The dual vector space $\mathfrak{g}^\star$ will be identified with $\mathfrak{g}$ by means of the scalar product. We shall also regard $\mathfrak{g}$ as a Lie group with the Campbell–Hausdorff multiplication

$$x_1\circ x_2=x_1+x_2+r(x_1,x_2),$$

where

$$r(x_1,x_2)=\tfrac{1}{2}[x_1,x_2]+\tfrac{1}{12}([x_1,[x_1,x_2]]+[x_2,[x_2,x_1]])+\tfrac{1}{24}[x_2,[x_1,[x_2,x_1]]]+\dots$$

is the (finite) sum of terms of order at least 2 in the Campbell–Hausdorff series for $\mathfrak{g}$. Let $\{\delta_t\}_{t>0}$, be a family of group dilations on $\mathfrak{g}$ and let

$$\mathfrak{g}_j=\left\{x\in\mathfrak{g}:\delta_t x=t^{d_j}x\right\},\quad 1\le j\le R,$$

where $1=d_1<d_2\leq\ldots<d_R$. Then

$$\mathfrak{g}=\mathfrak{g}_1\oplus\mathfrak{g}_2\oplus\ldots\oplus\mathfrak{g}_R \tag{3.1}$$

and

$$[\mathfrak{g}_i,\mathfrak{g}_j]\subset\begin{cases}\mathfrak{g}_k, & \text{if } d_i+d_j=d_k,\\ \{0\}, & \text{if } d_i+d_j\notin\mathcal{D},\end{cases}$$

where $\mathcal{D}=\{d_j:1\leq j\leq R\}$. Let $x\mapsto|x|$ be the homogeneous norm on $\mathfrak{g}$ as defined in Section 1.

For a function $f\in C_c^\infty(\mathfrak{g}\times\mathfrak{g})$ let

$$\mathbf{U}f(\mathbf{y})=\iint_{\mathfrak{g}\times\mathfrak{g}}e^{-i\langle\mathbf{x},\mathbf{y}\rangle}f^\vee(\mathbf{x})e^{-i\langle r(\mathbf{x}),\tilde{\mathbf{y}}\rangle}\,d\mathbf{x},$$

where $\mathbf{x}=(x_1,x_2)$, $\mathbf{y}=(y_1,y_2)\in\mathfrak{g}\times\mathfrak{g}$, and $\tilde{\mathbf{y}}=(y_1+y_2)/2$. We shall refer to $\mathbf{U}$ as the *Melin operator* on $\mathfrak{g}$. The importance of $\mathbf{U}$ consists in

$$\widehat{f\star g}\,(y)=\mathbf{U}(\hat{f}\otimes\hat{g})(y,y),\quad y\in\mathfrak{g}, \tag{3.2}$$

which is checked directly.

Let

$$\mathfrak{g}'=\mathfrak{g}_1\oplus\mathfrak{g}_2\oplus\ldots\oplus\mathfrak{g}_{R-1}. \tag{3.3}$$

The commutator

$$\mathfrak{g}'\times\mathfrak{g}'\ni(x_1,x_2)\longmapsto[x_1,x_2]'\in\mathfrak{g}',$$

where $'$ stands for the orthogonal projection onto $\mathfrak{g}'$, makes $\mathfrak{g}'$ into a Lie algebra isomorphic to $\mathfrak{g}/\mathfrak{g}_R$ with $x\mapsto x'$ playing the role of the canonical quotient homomorphism. The group multiplication in $\mathfrak{g}'$ is

$$x_1\circ' x_2=x_1+x_2+r(x_1,x_2)'.$$

Proposition 3.4. *For $f\in C_c^\infty(\mathfrak{g}\times\mathfrak{g})$,*

$$\mathbf{U}f(\mathbf{y},\lambda)=\mathbf{U}'(P_\lambda f(\,\cdot\,,\lambda))(\mathbf{y}),\quad \mathbf{y}\in\mathfrak{g}',\lambda\in\mathfrak{g}_R, \tag{3.5}$$

where

$$P_\lambda g(\mathbf{y})=\iint_{\mathfrak{g}'\times\mathfrak{g}'}e^{-i\langle\mathbf{x},\mathbf{y}\rangle}g^\vee(\mathbf{x})e^{-i\langle r(\mathbf{x}),\tilde{\lambda}\rangle}\,d\mathbf{x},\quad g\in C_c^\infty(\mathfrak{g}'\times\mathfrak{g}'),$$

is an integral operator on $C_c^\infty(\mathfrak{g}')$ invariant under Abelian translations, and $\mathbf{U}'$ stands for the Melin operator on $\mathfrak{g}'$.

Proof. In fact,

$$\begin{aligned}\mathbf{U}f(\mathbf{y},\lambda) &= \iint_{\mathfrak{g}\times\mathfrak{g}} e^{-i\langle \mathbf{x},\mathbf{y}\rangle} e^{-i\langle t,\lambda\rangle} f^{\vee}(\mathbf{x},t) e^{-i\langle r'(\mathbf{x}),\tilde{\mathbf{y}}\rangle} e^{-i\langle r(\mathbf{x}),\tilde{\lambda}\rangle}\, d\mathbf{x}\, dt \\ &= \iint_{\mathfrak{g}'\times\mathfrak{g}'} e^{-i\langle \mathbf{x},\mathbf{y}\rangle} (f(\mathbf{x}^{\vee},\lambda) e^{-i\langle r(\mathbf{x}),\tilde{\lambda}\rangle}) e^{-i\langle r'(\mathbf{x}),\tilde{\mathbf{y}}\rangle}\, d\mathbf{x} \\ &= \iint_{\mathfrak{g}'\times\mathfrak{g}'} e^{-\langle \mathbf{x},\mathbf{y}\rangle} (P_\lambda f(\,\cdot\,,\lambda))^{\vee}(\mathbf{x}) e^{-i\langle r'(\mathbf{x}),\tilde{\mathbf{y}}\rangle}\, d\mathbf{x} \\ &= \mathbf{U}'(P_\lambda f(\,\cdot\,,\lambda))(\mathbf{y})\end{aligned}$$

for all $f\in C_c^\infty(\mathfrak{g}\times\mathfrak{g})$, $\mathbf{y}\in\mathfrak{g}'\times\mathfrak{g}'$, $\lambda\in\mathfrak{g}_R\times\mathfrak{g}_R$. □

The reader who is familiar with our previous paper may be surprised that (3.5) differs from an analogous formula in [2] by the order of the operators P_λ and $\mathbf{U}'$. As a matter of fact, (3.4) in [2] is false and we take this opportunity to correct it. Fortunately only the proof of Proposition 5.1 in [2] requires some obvious corrections. This mistake has been brought to my attention by Jacek Dziubański.

For the background on homogeneous groups we recommend Folland–Stein [1].

4. The inductive step

In what follows we apply the notation of Section 1 among others to $X=\mathfrak{g}$ and $X=\mathfrak{g}\times\mathfrak{g}$. In the latter case we employ the product norm $\|\mathbf{x}\|^2=\|x_1\|^2+\|x_2\|^2$, the product dilations $\delta_t\mathbf{x}=\delta_t x_1+\delta_t x_2$, and the product homogeneous norm $|\mathbf{x}|^2=|x_1|^2+|x_2|^2$. In addition, $\rho(\mathbf{x})=1+|\mathbf{x}|$.

From now on, $\mathbf{g}$ always denotes a metric of the form

$$\mathbf{g}_{\mathbf{x}}(\mathbf{z})^2=\sum_{j=1}^{R}\frac{\|\mathbf{z}_j\|^2}{g_j(\mathbf{x})^{2d_j}},$$

where $g_j(\mathbf{x})\geq\rho_j(\mathbf{x})$. We also assume that $\mathbf{g}$ is $\mathbf{q}$-tempered. Of course, what we focus on is $\mathbf{g}=\mathbf{p}$ or $\mathbf{g}=\mathbf{q}$.

Let $\lambda\in\mathfrak{g}_R\times\mathfrak{g}_R$ (see (3.1) and (3.3)). It is easily seen that the family of metrics

$$\mathbf{g}^{\lambda}_{\mathbf{x}}(\mathbf{y})=\mathbf{g}_{(\mathbf{x},\lambda)}(\mathbf{y},0),\quad \mathbf{x},\mathbf{y}\in\mathfrak{g}',$$

is uniformly slowly varying and uniformly $\mathbf{q}^\lambda$-tempered. In particular $\mathbf{q}^\lambda$ are uniformly slowly varying. Let γ be a joint constant for all the metrics $\mathbf{g}^\lambda$ and $\mathbf{q}^\lambda$. Let $B_\nu=B_\nu(\mathbf{x}_\nu,r)\subset\mathfrak{g}'\times\mathfrak{g}'$ be the common covering of Lemma 2.1 for all these metrics.

Let

$$d_\nu^\lambda(\mathbf{y})=\mathbf{q}_{\mathbf{x}_\nu}^\lambda(\mathbf{y}-\mathbf{x}_\nu)$$

(cf. Proposition 2.1). Let $\rho^\lambda(\mathbf{x})=\rho(\mathbf{x},\lambda)$ and $g_j^\lambda(\mathbf{x})=g_j(\mathbf{x},\lambda)$.

Here comes the crucial step in our argument.

Lemma 4.1. *For every N, there exist C and k such that*

$$|P_\lambda f(\mathbf{y})|\le C|f|_k^{\mathbf{1}}(\mathbf{g}^\lambda)(1+d_\nu^\lambda(\mathbf{y}))^{-N} \tag{4.2}$$

for $f\in C_c^\infty(B_\nu)$ uniformly in $\lambda\in\mathfrak{g}_R\times\mathfrak{g}_R$ and $\nu\in\mathbf{N}$.

Proof. Let $f\in C_c^\infty(\mathfrak{g}'\times\mathfrak{g}')$ be supported in B_ν. There exist C and k such that

$$|P_\lambda f(\mathbf{y})|\le\iint_{\mathfrak{g}'\times\mathfrak{g}'}|f^\vee(\mathbf{x})|\,d\mathbf{x}=\|f\|_{A(\mathfrak{g}'\times\mathfrak{g}')}=\|f_\lambda\|_{A(\mathfrak{g}'\times\mathfrak{g}')}\le C|f|_k^{\mathbf{1}}(\mathbf{g}^\lambda), \tag{4.3}$$

where

$$f_\lambda(\mathbf{y})=f\left(\sum_{j=1}^{R-1}g_j^\lambda(\mathbf{x}_\nu)^{d_j}\mathbf{y}_j\right),$$

and $\|\cdot\|_{A(\mathfrak{g}'\times\mathfrak{g}')}$ stands for the Fourier algebra norm. The last inequality is achieved by the Sobolev inequality

$$\|f\|_{A(\mathfrak{g}'\times\mathfrak{g}')}\le C(s)\sum_{|\alpha|\le s}\|D^\alpha f\|_2$$

applied to f_λ which is supported in a ball of radius 1 with respect to the norm $\|\cdot\|$.

Assume now that (4.2) is true for some N. Let $d_\nu^\lambda(\mathbf{y})=a>1$. Note that otherwise the estimate is easy. Therefore there exists $\xi\in(\mathfrak{g}'\times\mathfrak{g}')^\star$ of unit length with respect to the norm dual to $\mathbf{q}_{\mathbf{x}_\nu}^\lambda$ such that $\xi(\mathbf{y}-\mathbf{x})\ge a$ for $\mathbf{x}\in B(\mathbf{x}_\nu,r_1)$, where $0<r<r_1<\gamma$. The norm one condition reads

$$1=(\mathbf{q}_{\mathbf{x}_\nu}^\lambda)^\star(\xi)^2=\sum_{1\le j\le R-1}(\rho_j)_\lambda(\mathbf{x}_\nu)^{2d_j}\,\|\xi_j\|^2\ge\sum_{1\le j\le R-1}(1+\|\lambda\|^{1/d_R})^{2d_j}\,\|\xi_j\|^2.$$

Then $L(\mathbf{x})=\langle\mathbf{x}-\mathbf{y},\xi\rangle$ does not vanish on $B(\mathbf{x}_\nu,r_1)$ so, by Lemma 2.2,

$$\mathbf{g}_{\mathbf{x}_\nu}^\lambda\left(D^k\frac{1}{L}(\mathbf{x})\right)\le\frac{C_k(r,r_1)}{a},\quad \mathbf{x}\in B(\mathbf{x}_\nu,r).$$

Note that $L(\mathbf{y})=0$. Therefore,

$$P_\lambda(Lf)(\mathbf{y})=[P_\lambda,L]f(\mathbf{y})=\sum_{j=1}^{R-1}\xi_j(1+\|\lambda\|^{1/d_R})^{d_j}P_\lambda(f_{\lambda,j})(\mathbf{y}),$$

where

$$f_{\lambda,j}(\mathbf{z}) = \frac{1}{i}(1+\|\lambda\|^{1/d_R})^{d_R-d_j}\left\langle r_j(iD), \frac{\tilde{\lambda}}{(1+\|\lambda\|^{1/d_R})^{d_R}}\right\rangle f(\mathbf{z}),$$

and

$$r_j(\mathbf{x},\lambda) = D_{(x_1)_j} r(\mathbf{x},\lambda) + D_{(x_2)_j} r(\mathbf{x},\lambda)$$

is a homogeneous polynomial of degree $d_R - d_j$. It follows that

$$|P_\lambda(f)(\mathbf{y})| \le \left(\sum_{j=1}^{R-1}\left|P_\lambda\left(\left(\frac{1}{L}f\right)_{\lambda,j}\right)(\mathbf{y})\right|^2\right)^{1/2},$$

and consequently, by Lemma 2.2 and the induction hypothesis,

$$|P_\lambda f(\mathbf{y})| \le \frac{C_k(r,r_1)}{a}|f|_k^{\mathbf{1}}(\mathbf{g}^\lambda)(1+d_\nu(\mathbf{y}))^{-N} \le C_k(r,r_1)|f|_k^{\mathbf{1}}(\mathbf{g}^\lambda)(1+d_\nu(\mathbf{y}))^{-N-1},$$

which completes the proof of (4.2). □

Let $\mathbf{m}$ be a $\mathbf{g}$-weight. Then $\mathbf{m}_\lambda(\mathbf{x}) = \mathbf{m}(\mathbf{x},\lambda)$ is a weight on $\mathfrak{g}'\times\mathfrak{g}'$ with respect to $\mathbf{g}^\lambda$ (which is $\mathbf{q}^\lambda$-tempered), and the family of weights is uniform in λ. Let $\phi_\nu^\lambda \in C_c^\infty(B_\nu)$ be the partition of unity on $\mathfrak{g}'\times\mathfrak{g}'$ for $\mathbf{g}^\lambda$. By Proposition 2.1, ϕ_ν^λ are bounded in $S^{\mathbf{1}}(\mathfrak{g}'\times\mathfrak{g}',\mathbf{g}^\lambda)$ uniformly in ν and λ.

Observe that

$$\mathbf{m}_\lambda(\mathbf{y}) \le C_1\mathbf{m}_\lambda(\mathbf{x}_\nu)(1+\mathbf{q}^\lambda_{\mathbf{x}_\nu}(\mathbf{y}-\mathbf{x}_\nu))^M \le C_1\mathbf{m}_\lambda(\mathbf{x}_\nu)(1+d_\nu^\lambda(\mathbf{y}))^M. \tag{4.4}$$

Proposition 4.5. *For every λ there exists a unique double-continuous extension of P_λ to a mapping*

$$P_\lambda\colon S^{\mathbf{m}_\lambda}(\mathfrak{g}'\times\mathfrak{g}',\mathbf{g}^\lambda) \longrightarrow S^{\mathbf{m}_\lambda}(\mathfrak{g}'\times\mathfrak{g}',\mathbf{g}^\lambda).$$

All the estimates hold uniformly in λ.

Proof. By Lemma 4.1 and (4.4),

$$|P_\lambda(\phi_\nu^\lambda f)(\mathbf{y})| \le C_1|\phi_\nu^\lambda f|_k^{\mathbf{1}}(\mathbf{g}^\lambda)(1+d_\nu^\lambda(\mathbf{y}))^{-N}$$

and

$$\begin{aligned}\mathbf{m}_\lambda(\mathbf{y})^{-1}|P_\lambda(\phi_\nu^\lambda f)(\mathbf{y})| &\le C_2\mathbf{m}_\lambda(\mathbf{x}_\nu)^{-1}(1+d_\nu^\lambda(\mathbf{y}))^M|P_\lambda(\phi_\nu^\lambda f)(\mathbf{y})|\\ &\le C_3|f|_k^{\mathbf{m}_\lambda}(1+d_\nu^\lambda(\mathbf{y}))^{-N+M}.\end{aligned}$$

Let N be so large that

$$\sum_{\nu}(1+d_{\nu}^{\lambda}(\mathbf{y}))^{-N+M}<\infty,$$

see Proposition 2.1 (c). Then our estimate which remains valid for f in a bounded subset of $S^{\mathbf{m}_\lambda}(\mathfrak{g}'\times\mathfrak{g}',\mathbf{g}^\lambda)$ without any restriction on the support implies that for every $\mathbf{y}\in\mathfrak{g}'$,

$$f\longmapsto\sum_{\nu}P_\lambda(\phi_\nu^\lambda f)(\mathbf{y})$$

defines a weakly continuous linear form on $S^{\mathbf{m}}(\mathfrak{g}'\times\mathfrak{g}',\mathbf{g}^\lambda)$. Consequently, P_λ admits a (unique) weakly continuous extension to the whole of $S^{\mathbf{m}}(\mathfrak{g}'\times\mathfrak{g}',\mathbf{g}^\lambda)$, and

$$|P_\lambda(f)(\mathbf{y})|=\Big|\sum_{\nu}P_\lambda\big(\phi_\nu^\lambda f\big)(\mathbf{y})\Big|\le C_5|f|_k^{\mathbf{m}_\lambda}\mathbf{m}_\lambda(\mathbf{y}).$$

The estimates for the derivatives of $P_\lambda f$ follow from the fact that P_λ commutes with translations, and hence with differentiations. □

5. Continuity of U and symbolic calculus

Recall from Section 4 that the Melin operator $\mathbf{U}$ has been defined for $f\in C_c^\infty(\mathfrak{g}\times\mathfrak{g})$.

Theorem 5.1. *Let* $\mathbf{m}$ *be a* $\mathbf{g}$*-weight on* $\mathfrak{g}\times\mathfrak{g}$. *There exists a double-continuous extension of the Melin operator to*

$$\mathbf{U}\colon S^{\mathbf{m}}(\mathfrak{g}\times\mathfrak{g},\mathbf{g})\longrightarrow S^{\mathbf{m}}(\mathfrak{g}\times\mathfrak{g},\mathbf{g}).$$

Proof. Suppose that $\mathfrak{g}$ is as in (3.1) and proceed by induction. If $R=1$, $\mathfrak{g}$ is Abelian and $\mathbf{U}=I$ so the assertion is obvious. Assume that our theorem is true for $\mathfrak{g}'$ as in (3.3) and $\mathbf{U}=\mathbf{U}'$. For $\lambda\in\mathfrak{g}_R$ and $f\in S^{\mathbf{m}}(\mathfrak{g}\times\mathfrak{g},\mathbf{g})$ let $f_\lambda(\mathbf{y})=f(\mathbf{y},\lambda)$, $(\rho_\lambda)(\mathbf{y})=\rho(\mathbf{y},\lambda)$, and $\mathbf{m}_\lambda(\mathbf{y})=\mathbf{m}(\mathbf{y},\lambda)$.

By hypothesis, $f_\lambda=f(\,\cdot\,,\lambda)\in S^{\mathbf{m}_\lambda}(\mathfrak{g}'\times\mathfrak{g}',\mathbf{g}_\lambda)$ uniformly in λ (cf. the previous section). Now Proposition 4.5 yields

$$P_\lambda f_\lambda\in S^{\mathbf{m}_\lambda}(\mathfrak{g}'\times\mathfrak{g}',\mathbf{g}^\lambda)$$

uniformly in λ so, by the induction hypothesis,

$$\mathbf{U}'P_\lambda f_\lambda \in S^{\mathbf{m}_\lambda}(\mathfrak{g}'\times\mathfrak{g}', \mathbf{g}^\lambda)$$

uniformly in λ. The same holds true for the derivatives $(\partial/\partial\lambda)^j\mathbf{U}'P_\lambda f_\lambda$ which is checked directly. Thus, by (3.5), we get the desired estimate: There exists $C>0$ such that for every $k_1\in\mathbf{N}$, there exists $k_2\in\mathbf{N}$ such that

$$|\mathbf{U}f|^{\mathbf{m}}_{k_1}(\mathbf{g}) \le C|f|^{\mathbf{m}}_{k_2}(\mathbf{g}), \quad f\in S^{\mathbf{m}}(\mathfrak{g}\times\mathfrak{g}, \mathbf{g}).$$

This proves our assertion. □

Corollary 5.2. *Let* $\mathbf{m}_1$ *and* $\mathbf{m}_2$ *be* $\mathbf{g}$*-weights on* $\mathfrak{g}$. *Then*

$$C_c^\infty(\mathfrak{g})\times C_c^\infty(\mathfrak{g}) \ni (a,b) \longmapsto a\#b = (a^\vee \star b^\vee)^\wedge \in \mathcal{S}(\mathfrak{g})$$

extends uniquely to a double-continuous mapping

$$S^{\mathbf{m}_1}(\mathfrak{g},\mathbf{g})\times S^{\mathbf{m}_2}(\mathfrak{g},\mathbf{g}) \longrightarrow S^{\mathbf{m}_1\mathbf{m}_2}(\mathfrak{g},\mathbf{g}).$$

Proof. This is a straightforward consequence of (3.2) and Theorem 5.1 applied to the metric $\mathbf{g}\oplus\mathbf{g}$ on $\mathfrak{g}\times\mathfrak{g}$. □

Let ϕ_ν be the standard partition of unity for the metric $\mathbf{q}$ on $\mathfrak{g}$. Let $\Phi_{\mu\nu}(\mathbf{x}) = \phi_\mu(x_1)\phi_\nu(x_2)$, where $\mathbf{x}=(x_1,x_2)\in\mathfrak{g}\times\mathfrak{g}$. Let $\mathbf{Q}=\mathbf{q}\oplus\mathbf{q}$. Note that, by Lemma 1.8,

$$1+\mathbf{q}_{x_\nu}(x_\nu - x_\mu) \le C^2(1+\mathbf{q}_{x_\mu}(x_\mu - y))^M(1+\mathbf{q}_{x_\nu}(x_\nu - y))^{M+1} \tag{5.3}$$

for every $y\in\mathfrak{g}$ and every μ and ν.

Corollary 5.4. *Let* $f\in S^{\mathbf{1}}(\mathfrak{g}\times\mathfrak{g},\mathbf{Q})$. *Let*

$$f_{\mu\nu}(y) = \mathbf{U}(\Phi_{\mu\nu}f)(y,y)$$

be a function on $\mathfrak{g}$. *Then, for every* N, *there exists a norm* $|\cdot|^{\mathbf{1}}(\mathbf{Q})$ *in* $S^{\mathbf{1}}(\mathfrak{g},\mathbf{Q})$ *such that for every* μ *and* ν,

$$\|f_{\mu\nu}\|_{A(\mathfrak{g})} \le |f|^{\mathbf{1}}(\mathbf{Q})(1+\mathbf{q}_{x_\nu}(x_\mu - x_\nu))^{-N}.$$

Proof. If

$$\mathbf{m}_{\mu\nu}(\mathbf{y}) = (1+\mathbf{q}_{x_\nu}(x_\mu - y_1))^{-N(M+1)}(1+\mathbf{q}_{x_\nu}(x_\nu - y_2))^{-NM},$$

where $\mathbf{y}=(y_1,y_2)$, then, of course, $\mathbf{m}_{\mu\nu}$ is a weight (see (1.12)), and

$$\Phi_{\mu\nu}f \in S^{\mathbf{m}_{\mu\nu}}(\mathfrak{g}\times\mathfrak{g},\mathbf{Q})$$

uniformly so, by Proposition 5.1,

$$\mathbf{U}(\Phi_{\mu\nu}f)\in S^{\mathbf{m}_{\mu\nu}}(\mathfrak{g}\times\mathfrak{g},\mathbf{Q})$$

uniformly in μ and ν. Hence, by (5.3),

$$|D_y^\alpha f_{\mu\nu}(y)|\le|f|_k^{\mathbf{m}_{\mu\nu}}\mathbf{m}_{\mu\nu}(y,y)\le|f|_k^{\mathbf{1}}(1+\mathbf{q}_{x_\nu}(x_\mu-x_\nu))^{-N}$$

for $|\alpha|\le k$. If k is large enough, our assertion follows by the Sobolev inequality. □

We conclude with the L^2-boundedness theorem for $a\in S^{\mathbf{1}}(\mathfrak{g},\mathbf{q})$. Recall that $S^{\mathbf{1}}(\mathfrak{g},\mathbf{p})\subset S^{\mathbf{1}}(\mathfrak{g},\mathbf{q})$ and, by (1.13), the inclusion is continuous.

Theorem 5.5. *Let $a\in S^{\mathbf{1}}(\mathfrak{g},\mathbf{q})$. The linear operator $f\mapsto Af=f\star a^\vee$ defined initially on the dense subspace $C_c^\infty(\mathfrak{g})$ of $L^2(\mathfrak{g})$ extends to a bounded mapping of $L^2(\mathfrak{g})$. To be more specific, there exists a norm $|\cdot|^{\mathbf{1}}(\mathbf{q})$ in $S^{\mathbf{1}}(\mathfrak{g},\mathbf{q})$ such that*

$$\|Af\|_{L^2(\mathfrak{g})}\le|a|^{\mathbf{1}}\|f\|_{L^2(\mathfrak{g})},\quad f\in C_c^\infty(\mathfrak{g}).$$

Proof. Let

$$A_vf=f\star(\phi_va)^\vee,\quad f\in L^2(\mathfrak{g}).$$

Since $\phi_v\in C_c^\infty(\mathfrak{g})$, the operators A_v are bounded. Furthermore, by (3.2) with the notation of Corollary 5.4,

$$A_u^\star A_vf(y)=(\bar a\otimes a)_{u,v}^\vee\star f,\quad A_uA_v^\star f(y)=(a\otimes\bar a)_{u,v}^\vee\star f,$$

so that, by Corollary 5.4,

$$\|A_u^\star A_v\|+\|A_uA_v^\star\|\le(|a|^{\mathbf{1}}(\mathbf{q}))^2(1+g_{x_\nu}(x_\nu-x_\mu))^{-N},$$

where N can be taken as large, as we wish, and $|\cdot|^{\mathbf{1}}(\mathbf{q})$ is a norm in $S^{\mathbf{1}}(\mathfrak{g},\mathbf{q})$ depending only on N.

On the other hand,

$$a=\sum_u\phi_ua,$$

where the series is weakly convergent in $S^{\mathbf{1}}(\mathfrak{g},\mathbf{q})$ so that

$$Af=\sum_uA_uf,\quad f\in C_c^\infty(\mathfrak{g}).$$

Thus, the sequence of operators A_u satisfies the hypothesis of Cotlar's Lemma (see e.g. Stein [8]), and therefore the series $\sum_u A_u$ is strongly convergent to the extension of our operator A whose norm is bounded by $C_0|a|^{\mathbf{1}}$ (see Proposition 2.1). □

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