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Hardy type inequalities with exponential weights for a class of convolution operators

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Abstract. In this paper we consider operators of the form $H=\lambda(-i\nabla)$, with λ analytic in a strip and with some specific growth conditions at infinity, and prove Hardy type estimates in $L^2(\mathbb{R}^n)$ with exponential weights. In fact we extend our previous results [19] from bounded analytic functions on a strip to analytic functions with polynomial growth in that strip.

1. Introduction

The aim of this paper is to extend the Hardy type estimate obtained in [19] to a class of convolution operators with Fourier transforms of polynomially growing functions analytic in a strip of the form $|\mathrm{Im}\, z_j|<\delta$ for $j\in\{1, ..., n\}$. In fact most of the applications to the study of quantum Hamiltonians, that one would like to consider, deal with this type of functions.

A question one may ask when dealing with a linear equation $Lf=g$ is how regularity properties of the data g reflect themselves in regularity properties of the solution f. There are many possible senses for the word "regularity"; for us it will mean "spatial decay at infinity in L^2 sense". Given a self-adjoint operator H acting in $L^2(\mathbb{R}^n)$ and a real number E one can consider the problem of obtaining inequalities of the type

$$\|w_1 f\| \le C\|w_2(H-E)f\| \tag{1.1}$$

for f in the domain of H (supported away from the origin) and with some given weight functions w_1 and w_2. We put into evidence the real constant E only in order to have a parameter for investigating different spectral regions of H. Such estimates allow one to deduce a given decay for f once f is in the domain of H and $g=(H-E)f$ has a specific decay. In [4] and [16] such an inequality with polynomial weights is obtained for the differential operator $H=P(-i\nabla)+V$, where P is a polynomial and V a multiplication operator with a real function. In [19] we also obtained such an inequality with exponential weights, but for a convolution operator $H=\lambda(-i\nabla)$ with λ in a certain class of bounded analytic functions. In the present paper we improve the techniques of [19] and obtain an amelioration of the Theorem 14.5.2 of [16] for unbounded nonlocal convolution operators and exponential weights. This leads to a proof of the general result announced in [21].

Some of the results of the above type obtained in the literature ([7], [22], [25], [12] and [3]) may indicate that one can prove such inequalities using a special type of positivity condition associated with the existence of a *conjugate operator*. More precisely, we say that H satisfies *a Mourre estimate* with respect to the conjugate operator A, at a real value E, when (denoting by $E_J(H)$ the spectral projection of H on an interval J containing E) one has

$$E_J(H)i[H,A]\varphi_J(H)\geq \alpha E_J(H) \tag{1.2}$$

for a strictly positive constant α. Concerning the utility and significance of (1.2) in functional analysis and mathematical physics see [8].

The history of inequalities of type (1.1) is a very long one, starting probably with the classical Hardy inequality (Theorem 330 in [14]). For details on this subject see [7] and the references cited therein. Here we shall only briefly comment upon those connected with our developments. First of all, let us mention the case of second order equations with operator-valued coefficients discussed in [6] and [7], that have inspired many of our techniques. In these papers the use of a commutator inequality is also an essential ingredient. Secondly, let us recall the papers [12], [13] and [3] where the absence of positive eigenvalues for a class of Schrödinger operators (one-body resp. N-body) is proved by the unique continuation principle, starting from an a priori weighted estimate for eigenfunctions. This last estimate is proved using a conjugate operator.

Let us make some comments on the importance of inequalities of type (1.1). We mention first the two works by Agmon [1] and [2] that have been basic for many important results concerning the spectral analysis of perturbations of the Laplace operator on $\mathbb{R}^n$ and for Schrödinger operators. In [1] H is the Laplace operator on $\mathbb{R}^n$ (i.e. $\lambda(\xi)=\xi^2$) and for any $E\in\mathbb{R}\setminus\{0\}$ an inequality of type (1.1) is proved for some specific polynomial weights; this inequality leads then to the conclusion that the operator $-\Delta+V(x)$ (where $V(x)$ is a real function such that $|x|V(x)$ tends to 0 at ∞) may have only discrete positive point spectrum and rapidly decaying corresponding eigenfunctions. In [2] an estimate of type (1.1) is proven for perturbations of a second order elliptic differential operator with variable coefficients on open domains in $\mathbb{R}^n$ and negative values for E; a precise anisotropic exponential decay for the eigenfunctions corresponding to negative eigenvalues is obtained. Let us also recall the Carleman method for proving unique continuation theorems (see also [5], [9], [18] and [24]) starting from weighted estimates of type (1.1).

Moreover, as we have also shown in [19] and [20], estimates of type (1.1) can be used in order to derive a priori decay for eigenfunctions of H, a problem of much interest in the analysis of quantum Hamiltonians (see [4], [11], [12], [13], [16], [17], [23] and [26]).

In [19] we outlined a general argument leading from a Mourre estimate to a *Hardy inequality* (1.1), used it for the special case $H=\lambda(-i\nabla)$, with λ a sufficiently regular function defined on $\mathbb{R}^n$, and extended it to a large class of perturbations of such convolution operators. We used these results in [20] to study a class of perturbed periodic Schrödinger operators. In the following section we formulate an extension of our previous results (from [19]) for a class of analytic functions in a strip with a "symbol type behaviour at infinity" (without asking them to be polynomials). Our proof, in the third section of this paper, follows the main lines of [19] and we shall emphasize mainly the techniques needed to extend those arguments to the case of polynomially growing analytic functions. Let us remark at this point that considering also nonpolynomial functions λ, thus nonlocal convolution operators $\lambda(D)$ raises a series of difficulties for developing the necessary functional calculus.

2. A Hardy type inequality with exponential weights

Let us give now the framework and the precise statement of our main result. We work in the n-dimensional real space $\mathbb{R}^n$, with the Lebesgue measure denoted by $d^n x$. We write the Fourier–Lebesgue measure on $\mathbb{R}^n$, $dx:=(2\pi)^{-n/2}d^n x$.

The Hilbert space will be $\mathcal{H}:=L^2(\mathbb{R}^n;dx)\equiv L^2(\mathbb{R}^n)$.

On $L^1(\mathbb{R}^n;dx)$ we consider the Fourier transform

$$\mathcal{F}(f)(k)\equiv\hat{f}(k):=\int_{\mathbb{R}^n} e^{-ix\cdot k} f(x)\,dx \tag{2.1}$$

and extend it to an isometry of $L^2(\mathbb{R}^n)$.

Let $\mathcal{S}(\mathbb{R}^n)$ be the space of Schwarz functions on $\mathbb{R}^n$ and $\mathcal{S}'(\mathbb{R}^n)$ its dual, the space of tempered distributions. We write $\langle\,\cdot\,,\,\cdot\,\rangle\colon \mathcal{S}'(\mathbb{R}^n)\times\mathcal{S}(\mathbb{R}^n)\to\mathbb{C}$ for the canonical antiduality (antilinear in the first factor and linear in the second one). This application restricts to the usual scalar product in $L^2(\mathbb{R}^n)$. To any function $F\in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ we associate the distribution F defined by

$$\langle F,u\rangle:=\int_{\mathbb{R}^n}\overline{F(x)}u(x)\,dx \quad \text{for all } u\in C_0^\infty(\mathbb{R}^n).$$

We shall still denote by $\mathcal{F}$ the isomorphism induced on $\mathcal{S}(\mathbb{R}^n)$ by the Fourier transform and also its extension by duality to $\mathcal{S}'(\mathbb{R}^n)$. Moreover we set $f_-(x):=f(-x)$.

Let $BC(\mathbb{R}^n)$ be the space of bounded, continuous functions on $\mathbb{R}^n$. We shall also work with some subspaces of $C^\infty(\mathbb{R}^n)$, namely $BC^\infty(\mathbb{R}^n)$ the space of infinitely differentiable functions on $\mathbb{R}^n$ that are bounded together with all their derivatives and $C^\infty_{\text{pol}}(\mathbb{R}^n)$ the space of infinitely differentiable functions on $\mathbb{R}^n$ that have at most polynomial growth at infinity as well as their derivatives of all orders. We shall constantly use the standard multiindex notations, for $\alpha=(\alpha_1, ..., \alpha_n)\in\mathbb{N}^n$. We denote by δ_j the multiindex with 1 on position $j\in\{1, ..., n\}$ and 0 in the other positions. For a family of n commuting variables $X:=(X_1, ..., X_n)$ we use the notation $X^\alpha:=X_1^{\alpha_1}...X_n^{\alpha_n}$ and

$$\langle X\rangle := \left\{1+\sum_{j=1}^{n} X_j^2\right\}^{1/2}. \tag{2.2}$$

In $\mathcal{H}$ we shall work with two sets of commuting self-adjoint operators

$$Q := (Q_1, ..., Q_n) \quad \text{and} \quad D := (D_1, ..., D_n), \tag{2.3}$$

where Q_j and D_j are the unique self-adjoint extensions of the operators:

$$(Q_j f)(x) := x_j f(x) \quad \text{for all } f\in C_0^\infty(\mathbb{R}^n), \tag{2.4}$$

$$D_j f := -i\frac{\partial f}{\partial x_j} \quad \text{for all } f\in C_0^\infty(\mathbb{R}^n). \tag{2.5}$$

For a fixed $y\in\mathbb{R}^n$ we shall also use the notation $y\cdot D:=\sum_{j=1}^n y_j D_j$ for the self-adjoint extension of the operator defined on $C_0^\infty(\mathbb{R}^n)$. Let us recall the following simple facts

$$\mathcal{F}^{-1}Q_j\mathcal{F} = D_j \quad \text{and} \quad \mathcal{F}^{-1}D_j\mathcal{F} = -Q_j\,. \tag{2.6}$$

For any Borel function $\Phi\colon \mathbb{R}^n\to\mathbb{C}$ we denote by $\Phi(Q)$, respectively by $\Phi(D)$, the operators defined by the usual functional calculus for commuting families of self-adjoint operators, and by $\mathcal{D}(\Phi(Q))$, respectively by $\mathcal{D}(\Phi(D))$, their domains in $\mathcal{H}$. For a function $F\in C^1(\mathbb{R}^n)$ we shall denote by ∇F its gradient. If we denote by $U_D(x)$ the unitary representation of the translation group on $\mathbb{R}^n$, $(U_D(x)f)(y)=f(y+x)$, then for any function $F\colon \mathbb{R}^n\to\mathbb{C}$ that is the Fourier transform of an integrable function, we have

$$F(X) = \int_{\mathbb{R}^n} \widehat{F}(x) U_D(x)\, dx. \tag{2.7}$$

We shall mainly deal with analytic functions on a strip and use the following notation

$$\mathbb{C}^n_\delta := \{ z \in \mathbb{C}^n \,|\, |\operatorname{Im} z_j| < \delta,\ j \in \{1, ..., n\} \}$$

and let $\mathcal{O}(\mathbb{C}^n_\delta)$ be the space of analytic functions in $\mathbb{C}^n_\delta$. Our intention is to study convolution operators with functions λ of class $\mathcal{O}(\mathbb{C}^n_\delta)$ having a "symbol-type" behaviour for $|\operatorname{Re} z_j|$ going to ∞. In order to cover analytic functions that grow at infinity in the real directions we shall define the following function spaces.

Notation 2.1. For any real s we define the function spaces

$$S^s_0(\mathbb{R}^n) := \{ \rho \in C^\infty_{\mathrm{pol}}(\mathbb{R}^n) \,\big|\, |(\partial^\alpha \rho)(x)| \le C_\alpha \min\{\langle x\rangle^{s-|\alpha|}, \langle \rho(x)\rangle\} \text{ for all } \alpha \in \mathbb{N}^n \},$$
$$\mathcal{O}^s_0(\mathbb{C}^n_\delta) := \{ \lambda \in \mathcal{O}(\mathbb{C}^n_\delta) \mid \lambda(\,\cdot + iy) \in S^s_0(\mathbb{R}^n) \}$$

with uniform estimates for $\max_j |y_j| < \gamma$ for any $\gamma \in (0, \delta)$.

We denote by $\mathcal{G}$ the domain of the self-adjoint operator $\lambda(D)$ with the norm

$$\|f\|^2_{\mathcal{G}} := \|f\|^2 + \|\lambda(D) f\|^2. \tag{2.8}$$

We also set

$$\mathcal{L} := \mathcal{G} \cap L^2_{\mathrm{comp}}(\mathbb{R}^n). \tag{2.9}$$

To formulate the main result of our paper we need one more definition.

Definition 2.2. For a function $\lambda \in \mathcal{O}^s_0(\mathbb{C}^n_\delta)$ we define its set of *regular values*

$$\mathcal{E}(\lambda) := \{ t \in \mathbb{R} \mid \text{there exists } \varepsilon > 0 \text{ and } \varkappa > 0$$
$$\textit{such that } |\nabla \lambda(k) ht| \ge \varkappa \text{ for all } k \in \lambda^{-1}((t-\varepsilon, t+\varepsilon)) \}.$$

We *generalized critical value* is a point in the complementary set of $\mathcal{E}(\lambda)$ in $\mathbb{R}$.

Remark 2.3. It is obvious that $\mathcal{E}(\lambda)$ is open in $\mathbb{R}$ and that the image by λ of any point where $\nabla\lambda = 0$ is a generalized critical value; meanwhile it may happen that, due to its behaviour at infinity, λ may also have some other generalized critical values.

Theorem 2.4. *Let $\delta > 0$, $\lambda \in \mathcal{O}^s_0(\mathbb{C}^n_\delta)$ and $E \in \mathcal{E}(\lambda)$. Then there is a strictly positive constant $\gamma_0 < \delta$ such that for any $\gamma \in (0, \gamma_0)$ there is a positive constant C (depending on γ and E) for which the following estimate holds for any $f \in \mathcal{G}$:*

$$\|e^{\gamma\langle Q\rangle} f\|_{\mathcal{G}} \le C \|\sqrt{\langle Q\rangle} e^{\gamma\langle Q\rangle} (\lambda(D) - E) f\|.$$

The inequality in the above statement is understood in the sense that if the function

$$x \longmapsto \sqrt{\langle x\rangle}e^{\gamma\langle x\rangle}((\lambda(D)-E)f)(x) \tag{2.10}$$

is in $L^2(\mathbb{R}^n)$ then the function $e^{\gamma\langle x\rangle}f(x)$ is also in $L^2(\mathbb{R}^n)$ and we have the stated inequality.

Remark 2.5. In the above statement E may belong to the spectrum of the operator $\lambda(D)$ as well as to its resolvent set as long as it remains a regular value.

Remark 2.6. The conclusion of Theorem 2.4 remains true if one replaces the operator $\lambda(D)$ by $\lambda(D)+V$ for any real function V defined on $\mathbb{R}^n$ that is relatively bounded with respect to $\lambda(D)$ and satisfies the decay condition

$$\lim_{R\to\infty}\|\chi(|Q|>R)\langle Q\rangle V(Q)(\lambda(D)+i)^{-1}\|=0.$$

This fact can be proven by a straightforward extension of the argument in [19], once we have the estimate in Theorem 2.4 with the graph-norm of the operator $\lambda(D)$ in the left-hand side. From this result, by the argument in [19], one can deduce an a priori decay estimate for eigenfunctions of $\lambda(D)+V$, associated with eigenvalues $E\in\mathcal{E}(\lambda)$.

Remark 2.7. Let us observe that for the case of analytic bounded functions, considered in [19], we did not have to impose any conditions on the derivatives of λ. In fact one can prove a slightly more general form of our Theorem 2.4 for functions of the form $\lambda=\mu\rho$ with μ analytic and being the Fourier transform of a finite measure and $\rho\in\mathcal{O}_0^s(\mathbb{C}_\delta^n)$; this proof is a straightforward mixture of the arguments in [19] and those of this paper but involve rather cumbersome formulae.

Remark 2.8. The functions λ that we consider are not supposed to be polynomials (i.e. their Fourier transforms need not be supported in $\{0\}$) and this is responsible for most of the complications we encounter. Let us mention in this direction that the only case of this type appearing in the literature is the particular case $\lambda(x)=(1+|x|^2)^{1/2}$ ([10] and [15]).

Remark 2.9. A rather obvious modification of our Theorem 2 in [19] allows one to extend the result of Theorem 2.4 to perturbations of "short range type" (with differential operators with nonconstant coefficients) obtaining a generalization of Theorem 14.5.2 in [16] for nonlocal convolution operators and exponential weights. Moreover, by repeating the arguments in our proof of Theorem 2.4 for an operator of the form $\lambda(D)+V(Q,D)$ with V of "long range type" one could extend Theorem 30.2.9 in [17] for nonlocal convolution operators and exponential weights.

3. Proof of Theorem 2.4

We recall first a decay property of Fourier transforms of analytic functions, that will allow us to develop the necessary functional calculus using the ideas of [8] and [19].

Lemma 3.1. *Let $\rho\in\mathcal{O}_0^s(\mathbb{C}_\delta^n)$ for some $s>0$ and some $\delta>0$; let $m\in\mathbb{N}$ with $m>s$; then there is a positive constant $c_2(\gamma)$ such that*

$$\max_{|\alpha|=m}\sup_{0\le t\le 1}\int_{\mathbb{R}^n} e^{t\gamma|x|}|(\mathcal{F}^{-1}(\partial^\alpha\rho))(x)|\,dx\le c_2(\gamma)<\infty$$

for any $\gamma\in(0,\delta)$.

Proof. For $|\alpha|>s$ the function $\mathcal{F}^{-1}\partial^\alpha\rho$ is of class L^1 (see Proposition 1.3.6 from [8]). Then using Proposition 2.11 from [19] for some $\gamma'\in(\gamma,\delta)$ we obtain the bound

$$e^{\gamma|x|}|(\mathcal{F}\partial^\alpha\rho)(x)|\le C_{\gamma'},$$

so that

$$\int_{\mathbb{R}^n} e^{t\gamma|x|}|(\mathcal{F}\partial^\alpha\rho)(x)|\,dx\le C_{\gamma'}\int_{\mathbb{R}^n} e^{(t\gamma-\gamma')|x|}\,dx<\infty.\quad\square$$

The proof of Theorem 2.4 follows the strategy explained in [19] and through a cut-off procedure reduces the problem to a uniform weighted estimate for functions with compact support and for bounded weights of a given class. This uniform estimate for compact supports is then obtained using the positivity of the commutator with a well-suited conjugate operator. Let us begin by defining the class of weight functions that will contain the exponential weight from the Theorem 2.4 and the family of bounded weights that will approximate it. We consider the weights as being defined by their logarithms.

Definition 3.2. For any $\gamma\in(0,\delta)$ and any $m\ge 1$ we define the class of phase functions

$$\Phi_{\gamma,m}:=\{\varphi\in C^\infty([1,\infty);\mathbb{R})\,|\,0\le\varphi'\le\gamma,\ |\varphi''(t)|\le\frac{\gamma}{t},\ |\varphi^{(l)}(t)|\le\gamma \text{ for all } l\le m+1\},$$

and weight functions $w(x):=e^{\varphi(\langle x\rangle)}$ with $\varphi\in\Phi_{\gamma,m}$. We let

$$X(x):=\nabla(\varphi(x))=\frac{x}{\langle x\rangle}\varphi'(\langle x\rangle).$$

Let $\gamma>0$ and let us fix the phase function $\varphi_0(t)=\gamma t$, which is evidently in $\Phi_{\gamma,m}$. We let

$$\mathcal{M}:=\left\{f\in\mathcal{G}\;\middle|\;\sqrt{\langle Q\rangle}e^{\varphi_0(\langle Q\rangle)}(\lambda(D)-E)f\in L^2(\mathbb{R}^n)\right\} \tag{3.1}$$

and let $f\in\mathcal{G}$. We shall approximate f with functions with compact support, but in order to control the limit we shall need to work first with bounded phase functions $\varphi\in\Phi_{\gamma,m}$ that approximate φ_0. For $\lambda\in\mathcal{O}_0^s(\mathbb{C}_\delta^n)$ we shall fix $m\in\mathbb{N}$ such that $m=[s]+1$ (with $[s]$ being the integer part of s).

Let us fix $\chi\in C_0^\infty(\mathbb{R})$ such that $0\leq\chi(t)\leq 1$, $\chi(t)=0$ for $|t|\geq 1$, and $\chi(t)=1$ for $|t|\leq\frac{1}{2}$. For $x\in\mathbb{R}^n$, $f\in\mathcal{M}$ and $\theta\in(0,1]$ we set $\chi_\theta(x):=\chi(\theta\langle x\rangle)$ and $f_\theta:=\chi_\theta f$.

For $N\in\mathbb{N}$ let

$$\tilde{\eta}_N(t):=\begin{cases}\gamma, & t\leq 2N,\\ 0, & t>2N,\end{cases} \tag{3.2}$$

$$j(t):=\begin{cases}\left(\int_{\mathbb{R}}e^{-1/(1-t^2)}\,dt\right)^{-1}e^{-1/(1-t^2)}, & |t|<1,\\ 0, & |t|\geq 1,\end{cases} \tag{3.3}$$

$$j_N(t):=\frac{1}{N}j(t/N),$$

$$\eta_N:=j_N*\tilde{\eta}_N,$$

$$\varphi_N(t):=\int_0^t\eta_N(s)\,ds\quad\text{for all } t\geq 0. \tag{3.4}$$

Using Lemma 2 of Section 4 in [19] we know that given $m\in\mathbb{N}$, the phase function φ_N defined by (3.4) belongs to the class $\Phi_{\gamma,m}$ for any natural number N large enough. We shall now use the following result (which we shall prove a little bit later).

Proposition 3.3. *Let $\lambda\in\mathcal{O}_0^s(\mathbb{C}_\delta^n)$ and let $E\in\mathcal{E}(\lambda)$ be a regular value for λ (see Definition 2.2); then there exists a strictly positive constant $\gamma_0<\delta$ such that for $m:=[s]+1$ and for any $\gamma\in(0,\gamma_0)$ there exists a positive constant C_γ depending only on λ, E, γ_0 and on the class $\Phi_{\gamma,m}$, such that for any phase function $\varphi\in\Phi_{\gamma,m}$ and any $f\in\mathcal{L}$ the following estimate holds*

$$\|e^{\varphi(\langle Q\rangle)}f\|_{\mathcal{G}}\leq C_\gamma\left\|\frac{\langle Q\rangle}{\psi(Q)}e^{\varphi(\langle Q\rangle)}(\lambda(D)-E)f\right\|$$

with $\psi(Q):=\sqrt{\gamma+2\langle Q\rangle\varphi'(\langle Q\rangle)}$.

Thus if we fix $\gamma\leq\gamma_0$ we get the above inequality for $f=f_\theta$ and $\varphi=\varphi_N$ for any $f\in\mathcal{M}$, $\theta\in(0,1]$ and $N\in\mathbb{N}$ large enough, so that φ_N belongs to the class $\Phi_{\gamma,m}$. Now the removal of the cut-off in f and φ goes exactly along the lines explained in Section 4 of [19] by using the Fatou lemma on the left-hand side and the dominated convergence theorem on the right-hand side. In fact we shall prove the following result

Lemma 3.4. *With the above notation, for any $f \in \mathcal{M}$ and $N \in \mathbb{N}$ large enough we have*

$$\lim_{\theta \to 0} \left\| \frac{\langle Q \rangle}{\psi_N(Q)} e^{\varphi_N} [\lambda(D), \chi_\theta(Q)] f \right\| = 0$$

As Lemma 3 from Section 4 of [19] remains evidently true, we obtain that for $f \in \mathcal{M}$ and γ small enough we have

$$\frac{\langle x \rangle e^{\varphi_N(\langle x \rangle)}}{\psi_N(x)} \leq \frac{C}{\sqrt{\gamma}} \sqrt{\langle x \rangle} e^{\gamma \langle x \rangle}$$

and thus

$$\|e^{\varphi_N(\langle Q \rangle)} f\|_{\mathcal{G}} \leq C \left\| \frac{\langle Q \rangle}{\psi_N(Q)} e^{\varphi_N(\langle Q \rangle)} (\lambda(D) - E) f \right\| \leq C \| \sqrt{\langle Q \rangle} e^{\gamma \langle Q \rangle} (\lambda(D) - E) f \|.$$

Using Fatou Lemma for the left-hand side we finish the proof of Theorem 2.4.

In order to prove Proposition 3.3 and Lemma 3.4 we need to extend the functional calculus elaborated in [19] to the case of polynomially growing functions λ. We recall some facts on tempered distributions that we shall use in this context. Let $S \in \mathcal{S}'(\mathbb{R}^{2n})$ and for any $u \in \mathcal{S}(\mathbb{R}^n)$ let us denote by $\langle S, \overline{u} \rangle$ the tempered distribution defined by

$$\mathcal{S}(\mathbb{R}^n) \ni v \longmapsto \langle \langle S, \overline{u} \rangle, v \rangle := \langle S, v \otimes u \rangle \in \mathbb{C}. \tag{3.5}$$

We shall denote by $\mathcal{S}(\mathbb{R}^n) \otimes \mathcal{S}'(\mathbb{R}^n)$ the class of distributions $S \in \mathcal{S}'(\mathbb{R}^{2n})$ that have the property that for any $u \in \mathcal{S}(\mathbb{R}^n)$ the distribution $\langle S, \overline{u} \rangle$ is in fact defined by a function of class $\mathcal{S}(\mathbb{R}^n)$. Then we can extend the application from (3.5) to an application

$$\mathcal{S}'(\mathbb{R}^n) \ni T \longmapsto \langle \langle S, \overline{u} \rangle, T \rangle := \overline{\langle T, \langle S, \overline{u} \rangle \rangle} \in \mathbb{C}. \tag{3.6}$$

Thus for any tempered distribution $S \in \mathcal{S}(\mathbb{R}^n) \otimes \mathcal{S}'(\mathbb{R}^n)$ and any tempered distribution T in $\mathcal{S}'(\mathbb{R}^n)$ the above argument allows us to define a distribution $\langle S, \overline{T} \rangle \in \mathcal{S}'(\mathbb{R}^n)$ by

$$\langle \langle S, \overline{T} \rangle, u \rangle := \langle \langle S, \overline{u} \rangle, T \rangle. \tag{3.7}$$

Let us mention the following simple fact that we shall need in connection with the above analysis.

Lemma 3.5. *If* $F\in C^\infty_{\rm pol}(\mathbb{R}^n)$ *has the property that for all* $p\in\mathbb{N}$, $\alpha\in\mathbb{N}^n$ *and* $\beta\in\mathbb{N}^n$, *there exists* $r\in\mathbb{N}$ *and* $C_{\alpha,\beta,p}<\infty$ *such that*

$$\sup_{y\in\mathbb{R}^n}\sup_{x\in\mathbb{R}^n}\langle y\rangle^{-r}\langle x\rangle^p\left|\partial_x^\alpha\partial_y^\beta F(x,y)\right|\le C_{\alpha,\beta,p},$$

then F *defines a distribution of class* $\mathcal{S}(\mathbb{R}^n)\otimes\mathcal{S}'(\mathbb{R}^n)$ *and for any* $T\in\mathcal{S}'(\mathbb{R}^n)$ *the distribution* $\langle F,\overline{T}\rangle$ *is defined by a* $C^\infty(\mathbb{R}^n)$ *function that satisfies*

$$\partial_y^\alpha\langle F,\overline{T}\rangle=\left\langle\partial_y^\alpha F,\overline{T}\right\rangle.$$

Sometimes we indicate by a subscript the variable in which a distribution acts. We extend (2.7) for any distribution $F\in\mathcal{S}'(\mathbb{R}^n)$ in weak sense on $\mathcal{S}(\mathbb{R}^n)$.

As in [19] we have to work with a class of pseudodifferential operators that formally look like

$$(\lambda\diamond G)(Q,D)=\int_{\mathbb{R}^n}\hat{\lambda}(y)G(y,Q)U_D(y)\,dy, \tag{3.8}$$

with $\hat{\lambda}$ being a tempered distribution with exponential decay and for functions G of class C^∞ having some specific growth properties at infinity (coming from the conditions on the weight functions).

Let us consider first a function $G\in BC^\infty(\mathbb{R}^n\times\mathbb{R}^n)$ and try to define $\rho\diamond G$. We choose $g\in C_0^\infty(\mathbb{R}^n)$ and observe that for any $z\in\mathbb{R}^n$ the function

$$\mathbb{R}^n\ni y\longmapsto G(y,z)g(y+z)\in\mathbb{C}$$

is of class $C_0^\infty(\mathbb{R}^n)$ so that the following equality holds

$$\left\langle\bar{\bar{\rho}}_y,G(y,z)g(y+z)\right\rangle=(\rho(D)[G(Q,z)U_D(z)g])(0). \tag{3.9}$$

In order to estimate the L^2 norm of this function (as a function of $z\in\mathbb{R}^n$) we shall commute $\rho(D)$ with $G(Q,z)$ and use a Taylor expansion of $G(y,z)$ in the variable y. Taking $N>s$ we get

$$\begin{aligned}&(\rho(D)[G(Q,z)U_D(z)g])(x)\\ &\quad=\sum_{|\alpha|\le N-1}c_\alpha\left(\partial_1^\alpha G\right)(x,z)[\rho^{(\alpha)}(D)g](x+z)\\ &\qquad+\sum_{|\alpha|=N}c'_\alpha\int_0^1(1-t)^{N-1}dt\int_{\mathbb{R}^n}\widehat{\rho^{(\alpha)}}(y)(\partial_1^\alpha G)(x+ty,z)g(x+y+z)\,dy.\end{aligned} \tag{3.10}$$

For $|\alpha|>s$ we observe that $\rho^{(\alpha)}$ has a Fourier transform in $L^1(\mathbb{R}^n)$ due to Proposition 1.3.6 from [8]. Let us bound now the L^2 norm of $(\rho\diamond G)(Q,D)g$ by using (3.10) and (3.9) and estimating each term in the sum.

$$\|(\partial_1^\alpha G)(0,Q)\rho^{(\alpha)}(D)g\|_{L^2}^2 \leq \sup_z |\,\partial_1^\alpha G(0,z)\,|^2\, \|\rho^{(\alpha)}(D)g\|_{L^2}^2,$$

(3.11)

$$\left\|\int_{\mathbb{R}^n} \widehat{\rho^{(\alpha)}}(y)(\partial_1^\alpha G)(ty,z)g(y+z)\,dy\right\|_{L^2(\mathbb{R}^n_z)}^2 \leq \|\partial_1^\alpha G\|_{L^\infty(\mathbb{R}^n\times\mathbb{R}^n)}^2 \|\widehat{\rho^{(\alpha)}}\|_{L^1}^2 \|g\|_{L^2}^2.$$

In consequence, we have the following bound:

$$\begin{aligned}\|(\rho\diamond G)(Q,D)g\|_{L^2} \leq &\Big(\max_{|\alpha|\leq N} \|\partial_1^\alpha G\|_{L^\infty(\mathbb{R}^n\times\mathbb{R}^n)}\Big)\\ &\times\Big(\max_{|\alpha|\leq N-1} \|\rho^{(\alpha)}(D)g\|_{L^2} + \max_{|\alpha|=N} \|\widehat{\rho^{(\alpha)}}\|_{L^1}\|g\|_{L^2}\Big).\end{aligned} \tag{3.12}$$

The following statement follows easily from the above calculus with a Taylor expansion up to order $m-1$ and estimating the remainder by using Proposition 1.3.6 from [8].

Proposition 3.6. *Let $\rho\in S_0^s(\mathbb{R}^n)$ and $F\in BC^\infty(\mathbb{R}^n)$. For $m\in\mathbb{N}$ and $m>s$ we have*

(1) $$[\rho(D),F(Q)]g = \sum_{1\leq|\alpha|\leq m-1} \frac{(-i)^{|\alpha|}}{\alpha!}(\partial^\alpha F)(Q)(\partial^\alpha\rho)(D)g + R_m(F,\rho)g$$

for any $g\in C_0^\infty(\mathbb{R}^n)$, with $R_m(F,\rho)\in\mathcal{B}(\mathcal{H})$ satisfying

$$\|R_m(F,\rho)\| \leq C(F)\max_{|\alpha|=m} \|\mathcal{F}(\partial^\alpha\rho)\|_{L^1};$$

(2) $\|\rho(D)F(Q)g\|_{L^2} \leq C(\rho)(\max_{|\alpha|\leq m} \|\partial^\alpha F\|_{L^\infty})\|\langle\rho(D)\rangle g\|_{L^2}$.

We recall the following well-known result.

Proposition 3.7. *For $\lambda\in\mathcal{O}_0^s(\mathbb{C}^n_\delta)$ the operator $\lambda(D)$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^n)$.*

The following step in developing our calculus is to allow the function G to grow at infinity in the first variable uniformly with respect to the second one. In order to control the behaviour of G at infinity we shall introduce two weight functions W and Ω with some specific hypothesis (similar to the one in [19]). In the following we shall take $N=2n$ or $N=n$.

Hypothesis 3.8.
(1) W and Ω are even, positive functions in $C^\infty(\mathbb{R}^N)$ and satisfy
(a) there exists $\varkappa>0$ with $\varkappa\le W(x)$ and $\varkappa\le\Omega(x)$ for all $x\in\mathbb{R}^N$;
(b) there is $C<\infty$ so that $W(x+y)\le CW(x)W(y)$ and $\Omega(x+y)\le C\Omega(x)\Omega(y)$.
(2) $G\in C^\infty(\mathbb{R}^N\times\mathbb{R}^n)$ such that for all $m\in\mathbb{N}$, there exists $C_m<\infty$ with

$$\max_{|\alpha|\le m}|(\partial_1^\alpha G)(y,z)|\le C_m W(y)\Omega^{-1}(z). \tag{3.13}$$

In our developments usually $W(x)=e^{\gamma|x|}$ and $\Omega(x)$ is the constant function 1 or the function $\langle x\rangle^r$ for some $r>0$. We set $W_t(x):=W(tx)$.

If we ask $W\widehat{\rho^{(\alpha)}}$ to be in $L^1(\mathbb{R}^n)$ for any α with $|\alpha|=m$ then the integral with respect to y in (3.11) is still convergent and we get a similar bound for the L^2 norm of $(\rho\diamond G)(Q,D)g$ as in (3.12), but with the norm $\|\widehat{\rho^{(\alpha)}}\|_{L^1}$ replaced by $\|W\widehat{\rho^{(\alpha)}}\|_{L^1}$ and the supremum norm of G replaced by the supremum norm of $W^{-1}G$. One only has to observe that in the sum with $|\alpha|\le m-1$ the supremum on the derivatives of the function G is taken only with respect to the second variable, the first one being fixed at zero (see (3.11)). We shall now prove two propositions (analogs of those in [19]) which contain the main technical facts for our proofs. We shall underline variables $\underline{y}=(y,y')$ in $\mathbb{R}^{2n}$ and the corresponding families of operators $\underline{Q}$ and $\underline{D}$.

Notation 3.9. For W and Ω as in Hypothesis 3.8 and $\lambda\in\mathcal{O}_0^s(\mathbb{C}_\delta^n)$ let

$$|\!|\!|\lambda|\!|\!|_{(1),m}:=\max_{|\alpha|=m}\sup_{0\le t\le 1}\big\|W_t\Omega\widehat{\lambda^{(\alpha)}}\big\|_{L^1}<\infty.$$

Proposition 3.10. *Suppose $W\in C^\infty(\mathbb{R}^{2n})$ and $\Omega\in C^\infty(\mathbb{R}^n)$ satisfy Hypothesis* 3.8 *and $Z\in C^\infty(\mathbb{R}^{2n}\times\mathbb{R}^n)$ satisfies* (2) *of Hypothesis* 3.8; *suppose also given $\lambda\in\mathcal{O}_0^s(\mathbb{C}^n)$ and $m>s$. Then for f and g in $C_0^\infty(\mathbb{R}^n)$ one has*

$$\int_{\mathbb{R}^n}\Big|\Big\langle\overline{(\hat\lambda_-\otimes\hat\lambda)_{\underline{y}}},Z(\underline{y},z)(U_{\underline{D}}(z,z)(\overline{f}\otimes g))(\underline{y})\Big\rangle\Big|\,dz\le C_m(Z)\mathbf{N}(f)\mathbf{N}(g),$$
$$\mathbf{N}(f):=\max_{|\alpha|<m}\|\Omega^{-1}(Q)\lambda^{(\alpha)}(D)f\|_{L^2}+|\!|\!|\lambda|\!|\!|_{(1),m}\|\Omega^{-1}(Q)f\|_{L^2},$$

with $C_m(Z)$ being a constant depending only on Z and its derivatives up to order m with respect to the first variable.

Proof. We remark that for any fixed $z\in\mathbb{R}^n$ the application

$$\mathbb{R}^{2n}\ni\underline{y}\longmapsto Z(\underline{y},z)\overline{f}(z+y)g(z+y')\in\mathbb{C} \tag{3.14}$$

is of class $C_0^\infty(\mathbb{R}^{2n})$ and we have

(3.15)
$$\Big\langle\overline{(\hat{\lambda}_-\otimes\hat{\lambda})_{\underline{y}}}, Z(\underline{y},z)\bar{f}(z+y)g(z+y')\Big\rangle=\big((\lambda_-\otimes\lambda)(\underline{D})Z(\underline{Q},z)U_{\underline{D}}(z,z)(\bar{f}\otimes g)\big)(0).$$

We use the Taylor expansion of the function Z in both variables y and y' and the conditions of the Hypothesis 3.8 in order to first obtain for any $z\in\mathbb{R}^n$,

$$\Big|\Big\langle\overline{(\hat{\lambda}_-\otimes\hat{\lambda})_{\underline{y}}}, Z(\underline{y},z)\bar{f}(z+y)g(z+y')\Big\rangle\Big|\le C_m(Z)N(f;z)N(g;z),$$

where

$$N(f;z):=\sum_{|\alpha|<m}|(\Omega^{-1}(Q)\lambda^\alpha(D)f)(z)|+|\!|\!|\lambda|\!|\!|_{(1),m}|\Omega^{-1}(z)f(z)|.$$

Then, by integrating with respect to z and using the Cauchy–Schwarz inequality and the hypothesis on the function λ, we get the stated result. □

Proposition 3.11. *Suppose $W\in C^\infty(\mathbb{R}^n)$ and $\Omega\in C^\infty(\mathbb{R}^n)$ satisfy Hypothesis* 3.8 *and $G\in C^\infty(\mathbb{R}^n\times\mathbb{R}^n)$ satisfies* (2) *from Hypothesis* 3.8; *suppose also that $\lambda\in\mathcal{O}_0^s(\mathbb{C}^n)$ and $m>s$. Then for any function g in $C_0^\infty(\mathbb{R}^n)$ one has*

$$\|(\lambda\diamond G)(Q,D)g\|_{L^2}\le C_m(G)\mathbf{N}(g),$$
$$\mathbf{N}(g):=\max_{|\alpha|<m}\|\Omega^{-1}(Q)\lambda^{(\alpha)}(D)g\|_{L^2}+|\!|\!|\lambda|\!|\!|_{(1),m}\|\Omega^{-1}(Q)g\|_{L^2},$$

with $C_m(G)$ being a constant depending only on G and its derivatives up to order m with respect to the first variable.

Proof. The proof of this proposition goes exactly as the above one if one observes that

$$\Big\langle\overline{\hat{\lambda}_y}, G(y,z)g(z+y)\Big\rangle=(\rho(D)G(Q,z)U_D(z)g)(0). \qquad (3.16)$$ □

Let us first use these results in order to prove Lemma 3.4.

Proof of Lemma 3.4. We observe that $(\langle x\rangle/\psi_N(x))e^{\varphi_N(\langle x\rangle)}$ is a bounded function for each N. We set

$$\zeta_\theta(y,z;t):=\langle z\rangle y\cdot(\nabla\chi)(\theta\langle z+ty\rangle) \qquad (3.17)$$

and observe that we can write

$$\langle Q\rangle[\lambda(D),\chi_\theta(Q)]f=\int_0^1(\lambda\diamond\zeta_\theta(t))(Q,D)f\,dt. \qquad (3.18)$$

Taking into account that $\chi'(t)$ has support in the set $\{t: \frac{1}{2}\leq t\leq 1\}$ and that the function $\langle z\rangle$ is submultiplicative, we see easily that $|\zeta_\theta(y,z;t)|\leq c\langle y\rangle^2$ for any $z\in\mathbb{R}^n$, any $t\in[0,1]$ and any $\theta\in(0,1]$, with c depending only on the function χ. Moreover, some long but straightforward calculations proves that

$$|\partial_y^\gamma(\chi'(\theta\langle z+ty\rangle))|\leq Ct^{|\gamma|}h_\theta(\langle z+ty\rangle), \tag{3.19}$$

where h_θ is the characteristic function of $[1/2\theta, 1/\theta]$. We finally get that for $\alpha\in\mathbb{N}^n$,

$$\left|\partial_y^\alpha \zeta_\theta(y,z;t)\right|\leq Ch_\theta(\langle z+ty\rangle)\langle y\rangle^2. \tag{3.20}$$

When we compute $(\lambda\diamond\zeta_\theta(t))(Q,D)f$ as in Proposition 3.11, we obtain an expansion similar to (3.10) in which we can use (3.20) in order to get

$$\begin{aligned}&\left|\left\langle \overline{\hat{\lambda}_y}, (\zeta_\theta(y,z;t)f(y+z))\right\rangle\right|\\ &\qquad\leq C\Bigg(\sum_{|\alpha|<m} h_\theta(\langle z\rangle)|(\lambda^{(\alpha)}(D)f)(z)|\\ &\qquad\qquad+\sum_{|\alpha|=m}\int_0^1(1-\tau)^{m-1}\,d\tau\int_{\mathbb{R}^n}|\widehat{\lambda^{(\alpha)}}(y)|h_\theta(\langle z+t\tau y\rangle)\langle\tau y\rangle^2|f(z+y)|\,dy\Bigg).\end{aligned}$$

Considering the L^2 norms of these terms with respect to the variable z, we use Proposition 3.11 in order to obtain an upper bound uniform in θ and use the dominated convergence theorem in order to show that they converge to zero for $\theta\to 0$, due to the fact that the function h_θ converges pointwise to zero. We have thus proved that

$$\lim_{\theta\to 0}\|\langle Q\rangle[\lambda(D),\chi_\theta(Q)]f\|=0. \quad\square \tag{3.21}$$

Proof of Proposition 3.3. We use the same strategy as in [19] and compute the commutator of $\lambda(D)$ with a well-suited conjugate operator (which need not be self-adjoint in $L^2(\mathbb{R}^n)$ but just symmetric). We fix $\varphi\in\Phi_{\gamma,m}$, a function $\lambda\in\mathcal{O}_0^s(\mathbb{C}_\delta^n)$ with $\gamma<\delta$ and define

$$G_j(y,z):=\int_0^1 e^{sy\cdot X(z)}z_j\,ds,\quad F_j(y,z):=\int_0^1 e^{-sy\cdot X(z-y)}(z_j-y_j)\,ds,$$

so that, if we take $W(x)=e^{\gamma|x|}$ and $\Omega(x)=\langle x\rangle$, we can apply Proposition 3.11 to define the operator

$$A:=\frac{1}{2}\sum_{j=1}^n((\partial_j\lambda)\diamond G_j+(\partial_j\lambda)\diamond F_j) \tag{3.22}$$

on $C_0^\infty(\mathbb{R}^n)$ and extend it by continuity to the domain $\mathcal{L}$ (see (2.9)). Let us observe that for $g\in\mathcal{L}$ we have

$$\|\langle Q\rangle^{-1}Ag\|_{L^2}\le C\|g\|_{\mathcal{G}}. \tag{3.23}$$

We define

$$B_0:=i[\lambda(D),A_0]=\sum_{j=1}^{n}(\partial_j\lambda)(D)^2 \tag{3.24}$$

and observe that it is relatively bounded with respect to $\lambda(D)^2$.

Let us now fix a test function $f\in\mathcal{L}$. We begin by showing that for $f\in\mathcal{L}$ we have $e^\varphi f\in\mathcal{L}$. In fact the support condition is obvious and if we choose $\eta\in C_0^\infty(\mathbb{R}^n)$ with the property $\eta f=f$ and let $\zeta:=\eta e^\varphi\in C_0^\infty$ we obtain (using Proposition 3.6 and the conditions satisfied by the functions in $\Phi_{\gamma,m}$ and by λ)

$$\|\lambda(D)\zeta(Q)f\|\le c(\lambda)C(\zeta)\|f\|_{\mathcal{G}} \tag{3.25}$$

so that the multiplication with the function ηe^φ leaves $\mathcal{G}$ invariant.

As in [19] we have to compute the commutator

$$\begin{aligned}(-i)\langle e^\varphi f,[A,\lambda(D)]e^\varphi f\rangle=\langle e^\varphi f,\phi_J B_0\phi_J e^\varphi f\rangle&\\ +(\langle e^\varphi f,B_0\phi_J^\perp e^\varphi f\rangle+\langle e^\varphi f,\phi_J^\perp B_0\phi_J e^\varphi f\rangle)&\\ +\langle e^\varphi f,\mathrm{R}e^\varphi f\rangle,&\end{aligned} \tag{3.26}$$

where J is an interval containing the value E and contained in the set $\mathcal{E}(\lambda)$, ϕ_J is the characteristic function of the interval J, $\phi_J^\perp:=1-\phi_J$ and we also denote by ϕ_J the operator $\phi_J(\lambda(D))$. In order to estimate the remainder

$$R:=(-i)[A,\lambda(D)]-B_0 \tag{3.27}$$

we can define the sesquilinear form

$$\begin{gathered}\mathcal{L}\times\mathcal{L}\ni(g,f)\longmapsto\mathbf{B}(g,f)\in\mathbb{C},\\ i\mathbf{B}(g,f):=\langle Ag,(\lambda(D)-E)f\rangle-\langle(\lambda(D)-E)g,Af\rangle.\end{gathered} \tag{3.28}$$

As remarked after (3.24), B_0 defines a bounded sesquilinear form on $\mathcal{L}$ that we shall denote $\mathbf{B_0}(f,g):=\langle g,B_0f\rangle$. Thus for $f\in\mathcal{L}$ we have to estimate the difference $\langle e^\varphi f,\mathrm{R}e^\varphi f\rangle=\mathbf{B}(e^\varphi f,e^\varphi f)-\mathbf{B_0}(e^\varphi f,e^\varphi f)$. We shall approximate $f\in\mathcal{L}$ with functions in $C_0^\infty(\mathbb{R}^n)$ with respect to the norm of $\mathcal{G}$ (due to Proposition 3.7). We fix f and g in $C_0^\infty(\mathbb{R}^n)$ and observe that in computing the commutator $i[\lambda(D),A]$ one has two types of terms: those containing the commutator $i[\lambda(D),Q]=\nabla\lambda(D)$ and the difference $(1-e^{syX(Q)})$ that we write using a Taylor expansion of first order and the terms containing the commutator $i[\lambda(D), e^{syX(Q)}]$ for which we use Proposition 3.6. After some calculations we obtain

$$
\begin{aligned}
\langle f, (i[\lambda(D), A] - (\nabla\lambda)^2(D))g\rangle = &-\frac{i}{2}\sum_{j=1}^{n}\langle((\partial_j\lambda)_-\diamond G_j^+)(Q,D)^* f, (\partial_j\lambda)(D)g\rangle \\
&-\frac{i}{2}\sum_{j=1}^{n}\langle(\partial_j\lambda)(D)f, ((\partial_j\lambda)\diamond G_j^-)(Q,D)g\rangle \\
&-\frac{1}{2}\int_{\mathbb{R}^n}\langle\hat{\lambda}_-\otimes\hat{\lambda}, Z_+(\underline{Q}, z)U_{\underline{D}}(z)(\overline{f}\otimes g)\rangle\, dz \\
&+\frac{1}{2}\int_{\mathbb{R}^n}\langle\hat{\lambda}\otimes\hat{\lambda}_-, Z_-(\underline{Q}, z)U_{\underline{D}}(z)(\overline{f}\otimes g)\rangle\, dz,
\end{aligned}
\tag{3.29}
$$

where the functions

$$
G_j(y,z)^{\pm} := \sum_{l=1}^{n}\int_0^1 s\int_0^1 y_j y_l X_l(z) e^{\pm tsy\cdot X(z)}\, dt\, ds
$$

$$
Z_{\pm}(-y, y', z) := \sum_{j,k,l=1}^{n}\int_0^1\int_0^1 y'_l y_k y_l (\partial_l X_k)(z+ty') e^{\pm y\cdot X(z+ty')}(z_j \pm y'_j)\, dt\, ds
$$

evidently satisfy the conditions of Hypothesis 3.8 with $W(y) := e^{\gamma\langle y\rangle}$ and $\Omega(z) := 1$ and with constants $C(G_j^{\pm})$ and $C(Z_{\pm})$ which are proportional to γ (due to the properties of the functions in $\Phi_{\gamma,m}$). Using Propositions 3.10 and 3.11 we obtain the bound

$$
|\langle e^{\varphi} f, \mathrm{R} e^{\varphi} f\rangle| \leq \gamma C \|e^{\varphi} f\|^2.
$$

Let us come back to formula (3.26) and treat the terms containing the projectors $\phi_J^{\perp}$. We recall that B_0 defines a bounded sesquilinear form on $\mathcal{G}$ and we denote by $\|B_0\|$ its norm. Then for any $\theta > 0$,

$$
\begin{aligned}
|\langle e^{\varphi} f, B_0\phi_J^{\perp} e^{\varphi} f\rangle + \langle e^{\varphi} f, \phi_J^{\perp} B_0 \phi_J e^{\varphi} f\rangle| &\leq \|B_0\| \|\phi_J^{\perp} e^{\varphi} f\|_{\mathcal{G}} \{\|e^{\varphi} f\|_{\mathcal{G}} + \|\phi_J e^{\varphi} f\|_{\mathcal{G}}\} \\
&\leq \|B_0\| (\theta \|\phi_J^{\perp} e^{\varphi} f\|_{\mathcal{G}}^2 + \tfrac{1}{\theta}\|e^{\varphi} f\|_{\mathcal{G}}^2).
\end{aligned}
\tag{3.30}
$$

For the main contribution in (3.26) we use the Mourre type inequality (which explains partially the form we have chosen for our operator A)

$$
\phi_J(\lambda(D)) B_0 \phi_J(\lambda(D)) \geq \big(\inf_{\lambda(x)\in J} |(\nabla\lambda)(x)|^2\big)\phi_J(\lambda(D)),
\tag{3.31}
$$

the constant multiplying $\phi(\lambda(D))$ on the right-hand side being strictly positive; we denote it

$$a := \inf_{\lambda(x)\in J} |(\nabla\lambda)(x)|^2. \tag{3.32}$$

Putting everything together we obtain

$$\begin{aligned}(-i)\langle e^{\varphi} f, [A,\lambda(D)]e^{\varphi} f\rangle \\ \geq a\big\|\phi_J e^{\varphi} f\big\|^2 - \|\!|B_0|\!\|(\theta\|\phi_J^{\frac{1}{2}} e^{\varphi} f\|_{\mathcal{G}}^2 + \tfrac{1}{\theta}\|e^{\varphi} f\|_{\mathcal{G}}^2) - \gamma C\|e^{\varphi} f\|_{\mathcal{G}}^2.\end{aligned} \tag{3.33}$$

On the left-hand side we proceed as in [19] and write

$$\begin{aligned}&\langle Ae^{\varphi} f,(\lambda(D)-E)e^{\varphi} f\rangle \\ &\quad = \langle Ae^{\varphi} f, e^{\varphi}(\lambda(D)-E)f\rangle + \langle Ae^{\varphi} f, (\lambda(D)-e^{\varphi}\lambda(D)e^{-\varphi})e^{\varphi} f\rangle.\end{aligned} \tag{3.34}$$

In estimating the second term of the right-hand side we encounter the second instance in which the form of the operator A plays a crucial role. In fact we repeat the argument in [19] and decompose $S(f,g) := \langle f,\lambda(D)g\rangle - \langle f, e^{\varphi}\lambda(D)e^{-\varphi} g\rangle$ into its formally hermitian and antihermitian parts. For the hermitian part $S_1(f,g)$ we write the difference $\varphi(\langle Q+y\rangle) - \varphi(\langle Q\rangle) - y\cdot X(Q)$ using a Taylor expansion of first order

$$\begin{aligned}\xi(y) &:= \varphi'(y)\langle y\rangle^{-1}, \\ Y(Q;s,y) &:= s(\varphi(\langle Q\rangle) - \varphi(\langle Q+y\rangle) + y\cdot X(Q)) - y\cdot X(Q)\end{aligned}$$

and after some calculations we get

$$\operatorname{Im} S_1(Ag,g) = -2\left\langle Ag, \frac{\varphi'(Q)}{\langle Q\rangle} Ag\right\rangle + \operatorname{Im}\langle Ag, \widetilde{R}_1 g\rangle \tag{3.35}$$

with

$$\begin{aligned}\widetilde{R}_1 :=& \int_0^1\!\!\int_0^1 (1-t)((\lambda\diamond G_-(s,t))(Q,D) + (\lambda\diamond G_+(s,t))(Q,D))\,dt\,ds \\ &+ \int_0^1\!\!\int_0^1 (\lambda\diamond F(s,t))(Q,D)\,dt\,ds,\end{aligned} \tag{3.36}$$

$$\begin{aligned}G_-(s,t;y,z) &:= \sum_{j,k=1}^n c_{jk} y_j y_k e^{Y(z;s,-y)}(\partial_j X_k)(z-ty), \\ G_+(s,t;y,z) &:= \sum_{j,k=1}^n c_{jk} y_j y_k (\partial_j X_k)(z-(1-t)y)e^{Y(z-y;s,y)}, \\ F(s,t;y,z) &:= \sum_{j,k=1}^n y_j y_k (\partial_k \xi)(z-(1-t)y) z_j e^{sy\cdot X(z)}.\end{aligned} \tag{3.37}$$

Let us observe that all the terms appearing in the above expressions contain at least first order derivatives of the functions X_k, which decay at infinity as $\langle z\rangle^{-1}$, so that we can use Proposition 3.10 with $W(y){=}e^{\gamma'\langle y\rangle}$ (with $\gamma'{>}\gamma$) and $\Omega(z){=}\langle z\rangle$ in order to obtain the bound

$$|\langle Ag, \widetilde{R}_1 g\rangle| = |\langle\langle Q\rangle^{-1}Ag, \langle Q\rangle\widetilde{R}_1 g\rangle| \leq \gamma C\|g\|^2_{\mathcal{G}}. \tag{3.38}$$

In dealing with the formally antihermitian part $S_2(f,g)$, we let

$$\begin{aligned} \theta(y,z;s) &:= \varphi(\langle z{+}sy\rangle) - \varphi(\langle z\rangle) = sy\cdot\int_0^1 X(z{+}tsy)\,dt, \\ G(y,z;s) &:= \sum_{j=1}^n y_j X_j(z-(1-s)y)(e^{-\theta(y,z-y;s)} - e^{\theta(y,z-y;s)}) \end{aligned} \tag{3.39}$$

and observe that $|\theta(y,z;s)|{\leq}\gamma\langle y\rangle$ and $|G(y,z;s)|{\leq}\gamma e^{\gamma'\langle y\rangle}$ for $\gamma'{>}\gamma$. After some calculations we get

$$\begin{aligned} \operatorname{Im} S_2(Ag,g) &= -\frac{i}{2}\int_0^1 \langle g, [A, (\lambda\diamond G(s))(Q,D)]g\rangle\,ds \\ &= \big\langle \overline{\hat{\lambda}_y}, \langle g, U_D(y)[A, G(y,Q;s)]g\rangle\big\rangle \\ &\quad + \big\langle \overline{\hat{\lambda}_y}, \langle g, [A, U_D(y)]G(y,Q;s)g\rangle\big\rangle. \end{aligned} \tag{3.40}$$

In order to estimate the different contributions to (3.40) we remark that they can be written in the form

$$\int_{\mathbb{R}^n} \big\langle \overline{(\hat{\lambda}_-\otimes\hat{\lambda})_{\underline{y}}}, (Z(\underline{Q},z)U_{\underline{D}}(z)(\overline{g}\otimes g))(y)\big\rangle\,dz.$$

We remark that in $[A, U_D(y')]$ each term contains derivatives of $X_j(z)$ of order at least one, and these behave like $\langle z\rangle^{-1}$ for z large. Thus, one can prove the conditions from (2) of Hypothesis 3.8 for $Z(\underline{y},z)$ with $W(\underline{y}){=}e^{\gamma'\langle\underline{y}\rangle}$ (with $\gamma'{>}\gamma$) and $\Omega(z){\equiv}1$. Then using Proposition 3.10 we conclude that

$$|\operatorname{Im} S_2(Ag,g)| \leq \gamma C\|g\|^2_{\mathcal{G}}. \tag{3.41}$$

The conclusion of this calculations is that

$$\begin{aligned} &\operatorname{Im}\langle Ae^{\varphi}f, (\lambda(D) - e^{\varphi}\lambda(D)e^{-\varphi})e^{\varphi}f\rangle \\ &\qquad = -\left\langle Ae^{\varphi}f, \frac{\varphi'(\langle Q\rangle)}{\langle Q\rangle}Ae^{\varphi}f\right\rangle + \operatorname{Im}\langle Ae^{\varphi}f, \widetilde{R}e^{\varphi}f\rangle, \end{aligned} \tag{3.42}$$

with the remainder $\widetilde{R}$ satisfying

$$|\operatorname{Im}\langle Ae^{\varphi}f, \widetilde{R}e^{\varphi}f\rangle| \leq \gamma C\|e^{\varphi}f\|^2_{\mathcal{G}}. \tag{3.43}$$

For the first term on the right-hand side of (3.34) we get

$$\begin{aligned}&2|\operatorname{Im}\langle Ae^{\varphi}f,e^{\varphi}(\lambda(D)-E)f\rangle|\\ &\qquad\le 2\left\|\frac{\psi(Q)}{\langle Q\rangle}Ae^{\varphi}f\right\|\left\|\frac{\langle Q\rangle}{\psi(Q)}e^{\varphi}(\lambda(D)-E)f\right\|\\ &\qquad\le\left\langle Ae^{\varphi}f,\left(\frac{\psi(Q)}{\langle Q\rangle}\right)^2Ae^{\varphi}f\right\rangle+\left\|\frac{\langle Q\rangle}{\psi(Q)}e^{\varphi}(\lambda(D)-E)f\right\|^2\end{aligned}\tag{3.44}$$

for any nonvanishing function ψ for which the right-hand side is bounded. With the choice for the function ψ made in the statement of Proposition 3.3 we observe that

$$\left|\left\langle Ae^{\varphi}f,\left(2\frac{\varphi'(\langle Q\rangle)}{\langle Q\rangle}-\frac{\psi(Q)^2}{\langle Q\rangle^2}\right)Ae^{\varphi}f\right\rangle\right|\le\gamma C\|e^{\varphi}f\|^2_{\mathcal{G}}.\tag{3.45}$$

Let us now still observe that

$$c(J)\|\phi_Je^{\varphi}f\|^2\ge\|e^{\varphi}f\|^2_{\mathcal{G}}-\|\phi_J^{\perp}e^{\varphi}f\|^2_{\mathcal{G}}\tag{3.46}$$

$$\begin{aligned}\|\phi_J^{\perp}e^{\varphi}f\|^2_{\mathcal{G}}&=\|\phi_J^{\perp}e^{\varphi}f\|^2+\|\phi_J^{\perp}\lambda(D)e^{\varphi}f\|^2\\ &\le(1+2E^2)\|(\lambda(D)-E)^{-1}\phi_J^{\perp}(\lambda(D)-E)e^{\varphi}f\|^2\\ &\quad+2\|(\lambda(D)-E)e^{\varphi}f\|^2\\ &\le c(E,J)\|(\lambda(D)-E)e^{\varphi}f\|^2\\ &\le c(E,J)(\|e^{\varphi}(\lambda(D)-E)f\|^2+\|R_2e^{\varphi}f\|^2)\end{aligned}\tag{3.47}$$

with $c(J):=\sup_{t\in J}\langle t\rangle^2$ and $c(E,J):=2+(1+2E^2)/d(E,J^c)^2$. The remainder $R_2:=e^{\varphi}\lambda(D)e^{-\varphi}-\lambda(D)$ in the last term of (3.47) is estimated by

$$\begin{aligned}\|R_2f\|&:=\|(e^{\varphi}\lambda(D)e^{-\varphi}-\lambda(D))f\|=\left\|\langle\overline{\hat{\lambda}_y},(e^{\varphi(\langle Q\rangle)-\varphi(\langle Q+y\rangle)}-1)U_D(y)f\rangle\right\|\\ &\le\left\|\langle\overline{\hat{\lambda}_y},y\cdot X(Q+sy)e^{\theta(Q;s,y)}U_D(y)f\rangle\right\|\le\gamma C\|f\|_{\mathcal{G}}.\end{aligned}$$

Finally we have

$$\begin{aligned}\left(\frac{a}{c(J)}-\left(\widetilde{C}\gamma+\frac{1}{\theta}|\!|\!|B_0|\!|\!|\right)\right)\|e^{\varphi}f\|^2_{\mathcal{G}}&\le\left\|\frac{\langle Q\rangle}{\psi(Q)}e^{\varphi}(\lambda(D)-E)f\right\|^2\\ &\quad+\left(\frac{a}{c(J)}+\theta|\!|\!|B_0|\!|\!|\right)c(E,J)\|e^{\varphi}(\lambda(D)-E)f\|^2.\end{aligned}\tag{3.48}$$

Choosing γ small enough and θ large enough, we can assure the strict positivity of the coefficient of the left-hand side of (3.48) and thus we get Proposition 3.3. Let us strengthen the fact that the choice of γ and θ only depends on the value of the constant a in (3.32), on E and on the function λ. Moreover, the constant C_γ that we obtain depends only on the class $\Phi_{\gamma,m}$ and not on the specific phase function φ. □

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