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Riemannian geometry on the diffeomorphism group of the circle

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Abstract. The topological group $\mathcal{D}^k(\mathbb{S})$ of diffeomorphisms of the unit circle $\mathbb{S}$ of Sobolev class H^k, for k large enough, is a Banach manifold modeled on the Hilbert space $H^k(\mathbb{S})$. In this paper we show that the H^1 right-invariant metric obtained by right-translation of the H^1 inner product on $T_{\mathrm{id}}\mathcal{D}^k(\mathbb{S})\simeq H^k(\mathbb{S})$ defines a smooth Riemannian metric on $\mathcal{D}^k(\mathbb{S})$, and we explicitly construct a compatible smooth affine connection. Once this framework has been established results from the general theory of affine connections on Banach manifolds can be applied to study the exponential map, geodesic flow, parallel translation, curvature etc. The diffeomorphism group of the circle provides the natural geometric setting for the Camassa–Holm equation – a nonlinear wave equation that has attracted much attention in recent years – and in this context it has been remarked in various papers how to construct a smooth Riemannian structure compatible with the H^1 right-invariant metric. We give a self-contained presentation that can serve as a detailed mathematical foundation for the future study of geometric aspects of the Camassa–Holm equation.

1. Introduction

Just like the motion of a rigid body rotating around its centre of mass may be viewed as a path in the configuration space SO(3) of rotations of $\mathbb{R}^3$, the motion of a system in continuum mechanics can be described by a path $t\mapsto\varphi(t,\cdot\,)$ in the infinite-dimensional group $\mathcal{D}(\mathcal{M})$ of diffeomorphisms of the ambient space $\mathcal{M}$; a state of the system at a certain time t is fully characterized by the position $\varphi(t,x)$ of each particle $x\in\mathcal{M}$ at the time t. To describe the motion in the configuration space one often utilizes the inherent property of nature that physical systems tend to evolve along paths of minimal length. Mathematically, the notion of distance on the configuration space is modeled by a Riemannian metric and the shortest paths are geodesics of an associated affine connection. In the case of a rigid body rotating around a fixed point, uniformity of space yields a symmetry which is captured mathematically by *left-invariance* of the metric on the Lie group SO(3). In the case of an incompressible fluid moving in a bounded smooth domain $\mathcal{M}\subset\mathbb{R}^n$,

$n=2,3$, the configuration space is the group $\mathcal{D}_\mu(\mathcal{M})$ of volume-preserving diffeomorphisms of $\mathcal{M}$, and as was first noticed by Arnold [1] and subsequently put on a rigorous mathematical foundation by Ebin and Marsden [14], endowing $D_\mu(\mathcal{M})$ with the L^2 *right-invariant* metric, the geodesics turn out to model the motion of a perfect, i.e. non-viscous, homogeneous, and incompressible, fluid. Although the left-invariance of the metric on SO(3) had to be replaced by a right-invariant metric on $D_\mu(\mathcal{M})$, this observation unveiled an important similarity between the motion of a rotating rigid body and the motion of an incompressible fluid: they could both be described as so-called Euler equations for the geodesic flow with respect to an invariant metric. In the case of SO(3) the Euler equations for the geodesic flow are the classical Euler equations for the motion of a rigid body rotating around its centre of mass (cf. [23]), and in the case of $D_\mu(\mathcal{M})$ they are the classical Euler equations for a perfect fluid.

Since then many other well-known nonlinear wave equations have been found to arise as Euler equations for the geodesic flow on diffeomorphism groups endowed with various invariant metrics. For example, the Euler equation describing the geodesics on the Virasoro group (a one-dimensional extension of the diffeomorphism group of the circle) equipped with the L^2 right-invariant metric, is the well-known Korteweg–de Vries equation [27]. On the other hand, the L^2 right-invariant metric on the diffeomorphism group of the circle gives rise to Burger's equation [2] (one of the most fundamental nonlinear partial differential equation), while the H^1 right-invariant metric yields the Camassa–Holm equation [3] – a nonlinear wave equation that has attracted much attention in recent years. Choosing a natural right-invariant metric on the quotient space of the group of diffeomorphisms of the unit circle $\mathbb{S}$ modulo the subgroup of rotations of $\mathbb{S}$, one obtains the Hunter–Saxton equation [17] (an equation modeling propagation of orientation waves in liquid crystal director fields) as the fundamental equation describing the geodesic flow.

When generalizing the theory for SO(3) to a diffeomorphism group $\mathcal{D}(\mathcal{M})$ one is faced with the following choice. One may choose to let the group $\mathcal{D}(\mathcal{M})$ consist of the smooth diffeomorphisms of $\mathcal{M}$ or, for k a sufficiently large positive integer, let $\mathcal{D}(\mathcal{M})=\mathcal{D}^k(\mathcal{M})$ incorporate all diffeomorphisms of Sobolev class H^k (or of class C^k). $\mathcal{D}(\mathcal{M})$ is a Lie group and a Fréchet manifold, whereas $\mathcal{D}^k(\mathcal{M})$ is a Banach manifold and a topological group, but not a Lie group (the group operation $(\psi,\varphi)\mapsto \psi\circ\varphi$ for $\psi,\varphi\in\mathcal{D}^k(\mathcal{M})$ is continuous but not smooth due to derivative-loss, cf. [15]).

The obvious advantage of working on $\mathcal{D}(\mathcal{M})$ rather than on $\mathcal{D}^k(\mathcal{M})$ is that composition is smooth, so that the usual operations from Lie group theory can be performed: by means of right invariance many computations can be located to the Lie algebra, there exists a well-defined Lie bracket, there are adjoint and coadjoint representations etc. Moreover, in some instances it is easier to establish smoothness of objects. For example, a right-invariant metric obtained by right-translation of an inner product at the identity is automatically smooth thanks to the smooth group operation.

On the other hand, the theory for manifolds modeled on a Fréchet space is very restricted. Whereas nearly all results familiar from finite-dimensional Riemannian geometry immediately generalize to Banach manifolds (see [21]), a transition to Fréchet manifolds introduces several technical complications. As there are no general existence and uniqueness results for differential equations in Fréchet spaces, it is intricate to study geodesic flow and parallel translation. Moreover, the inverse mapping theorem does not hold in Fréchet spaces, and its generalization, the Nash–Moser theorem, requires additional technical hypotheses to apply (see [15]). Hence, for Riemannian Fréchet manifolds neither the Lie group exponential map, nor the Riemannian exponential map, is necessarily a local diffeomorphism at the identity. Another advantage of working with the wider class $\mathcal{D}^k(\mathcal{M})$ is that when studying partial differential equations it is often preferable to work in Sobolev spaces rather than in the category of C^∞-maps.

In the present paper we are concerned with the construction of a Riemannian structure on $\mathcal{D}^k(\mathbb{S})$ compatible with the H^1 right-invariant metric. We show that the H^1 right-invariant metric is smooth on $\mathcal{D}^k(\mathbb{S})$ and provide a smooth affine connection compatible with it. Once this has been established several implications from the general theory of affine connections on Banach manifolds are stated. For example, we obtain local formulas for a smooth curvature tensor, existence of normal neighborhoods, existence and uniqueness results for the geodesic flow and parallel translation, locally length-minimizing properties of the geodesics etc. Moreover, in the last section, the affine connection on $\mathcal{D}^k(\mathbb{S})$ is extended to $\mathcal{D}(\mathbb{S})$, relating our results to those of [10], and also adding to the picture of Riemannian geometry on $\mathcal{D}(\mathbb{S})$ in the case of the H^1 right-invariant metric by providing the affine connection associated to the covariant derivative which in [10] was obtained by a Lie group approach.

In [10] it was believed that the Riemannian structure on $\mathcal{D}^k(\mathbb{S})$ was deficient due to derivative-loss. Indeed, there is an apparent loss of regularity when one, in analogy to the case of a Lie group, studies the affine connection as an object on the tangent space at the identity $T_{\mathrm{id}}\mathcal{D}^k(\mathbb{S})$ by means of right-translation. However, the loss of smoothness turns out to be introduced by the transition to $T_{\mathrm{id}}\mathcal{D}^k(\mathbb{S})$ rather than being inherent to the Riemannian structure. In fact, working directly on the manifold $\mathcal{D}^k(\mathbb{S})$ we will give a detailed proof that it carries a perfectly well-defined affine connection compatible with the H^1 right-invariant metric.

It was already remarked in [20] that the spray associated to the H^1 right-invariant metric on $\mathcal{D}^k(\mathbb{S})$ is smooth, referring to [28] and [16] where the existence of a smooth spray associated to the H^1 right-invariant metric on $\mathcal{D}^k(\mathbb{S})$ follows as a special case of a more general multi-dimensional theory. We believe it is of interest to give a self-contained direct proof of this fact and draw conclusions from it.

The Euler equation corresponding to the H^1 right-invariant metric on the diffeomorphism group of the circle is the Camassa–Holm equation as was first observed by Misiołek [25], and later studied by Kouranbaeva [20] (in the case of $\mathcal{D}^k(\mathbb{S})$) and Constantin–Kolev [10] and [11] (in the case of $\mathcal{D}(\mathbb{S})$). Our hope is that the detailed exposition of the Riemannian structure on $\mathcal{D}^k(\mathbb{S})$ presented in this paper will prove useful for the future study of qualitative aspects of the Camassa–Holm equation. Notice that in [6] (see also [8] and [24]) the geometric aspect of the Camassa–Holm equation is used to find sharp blow-up results, as well as to prove global existence of solutions.

More generally, we could, for j with $1 \le j \le k$, consider the H^j right-invariant metric on $\mathcal{D}^k(\mathbb{S})$. However, since the H^1 right-invariant metric is the only one that gives rise to a bi-Hamiltonian Euler equation [12], we choose for the sake of simplicity to restrict ourselves to the H^1 case. For the H^1 right-invariant metric the corresponding Euler equation is in fact not just bi-Hamiltonian, but is also a completely integrable infinite-dimensional Hamiltonian system (cf. [5] and [13]). Note that for the L^2 right-invariant metric there is no compatible smooth affine connection on $\mathcal{D}^k(\mathbb{S})$ (see [10] and [11]).

The manifold structure of the diffeomorphism group $\mathcal{D}^k(\mathbb{S})$ is described in Section 2. In Section 3 we define a Christoffel map Γ and we show that it is a smooth map $\mathcal{D}^k(\mathbb{S}) \to L^2_{\mathrm{sym}}(H^k(\mathbb{S}); H^k(\mathbb{S}))$. In Section 4 it is proved that the H^1 right-invariant metric $\langle \cdot, \cdot \rangle$ defines a weak Riemannian metric on $\mathcal{D}^k(\mathbb{S})$, that is, we show that it is a smooth section of the bundle $L^2_{\mathrm{sym}}(T\mathcal{D}^k(\mathbb{S}); \mathbb{R})$. We then prove, in Section 5, that the affine connection defined by Γ is compatible with the H^1 right-invariant metric in the sense that the covariant derivative induced by Γ is the unique Riemannian covariant derivative compatible with $\langle \cdot, \cdot \rangle$. In Section 6 the general theory of affine connections in Banach manifolds is adopted to obtain several results for $\mathcal{D}^k(\mathbb{S})$. This is taken further in Section 7, where we establish length-minimizing properties of the geodesics on $\mathcal{D}^k(\mathbb{S})$. In the last section we extend the definition of the affine connection to the Fréchet Lie group $\mathcal{D}(\mathbb{S})$ and relate it to the covariant derivative defined on $\mathcal{D}(\mathbb{S})$ in [10]. Finally, the appendix contains some remarks on differential calculus in Banach spaces.

2. The diffeomorphism group

Let $\mathbb{S}$ be the circle of length one and let D_x denote differentiation with respect to x. For $X=[0,1]$ or $X=\mathbb{S}$ we define, for $n \ge 0$, $H^n(X)$ as the Sobolev space of all square integrable functions $f\in L^2(X)$ with distributional derivatives $D_x^i f\in L^2(X)$ for $i=1,...,n$. These Hilbert spaces are endowed with the inner products

$$\langle f,g\rangle_{H^n(X)}=\sum_{i=0}^{n}\int_{\mathbb{S}}(D_x^i f)(x)\,(D_x^i g)(x)\,dx.$$

By restriction of a periodic function to the unit interval, we may view $H^n(\mathbb{S})$ as a closed linear subspace of $H^n[0,1]$.

Let $k\geq 4$ be an integer and let $\mathcal{D}^k(\mathbb{S})$ denote the Banach manifold of orientation preserving diffeomorphisms of $\mathbb{S}$ of class H^k (cf. [26]). [In view of the Sobolev imbedding $H^s(\mathbb{S})\subset C^1(\mathbb{S})$ valid for $s>\frac{3}{2}$, it is to be expected that $k>\frac{3}{2}$ would be a sufficient assumption. Indeed, $k>\frac{3}{2}$ is the required assumption in order for $\mathcal{D}^k(\mathbb{S})$ to be a topological group [14]. However, for simplicity we will state the results in this paper only for $k\geq 4$, so that all derivatives exist in a classical sense. Observe that since the interesting peaked solutions of the Camassa–Holm equation [3] (cf. [13] and [22] for the periodic case), belong to $H^{3/2-\varepsilon}(\mathbb{S})$ for any $\varepsilon>0$, but not to $H^{3/2}(\mathbb{S})$, they can at any rate not be rigorously studied by means of the present approach.]

We next describe how to construct canonical charts on $T\mathcal{D}^k(\mathbb{S})$. Put $M^k=\{\varphi\in\mathcal{D}^k(\mathbb{S}):\varphi(0)=0\}$. Then the map

$$\varphi\longmapsto(\varphi(0),\varphi(\,\cdot\,)-\varphi(0))\colon\,\mathcal{D}^k(\mathbb{S})\longrightarrow\mathbb{S}\times M^k, \tag{2.1}$$

is a diffeomorphism. Note that M^k can be characterized as

$$M^k=\{\varphi\in H^k[0,1]:\varphi_x\in H^{k-1}(\mathbb{S}),\ \varphi_x>0,\ \varphi(0)=0 \text{ and } \varphi(1)=1\},$$

or equivalently

$$M^k=\{\varphi+\mathrm{id}:\varphi\in H^k(\mathbb{S}),\ \varphi_x>-1 \text{ and } \varphi(0)=0\}, \tag{2.2}$$

where $\mathrm{id}\in\mathcal{D}^k(\mathbb{S})$ is the identity map $\mathrm{id}(x)=x$ for $x\in\mathbb{S}$. If $\varphi\in M^k$ then $\varphi_x\in H^{k-1}(\mathbb{S})$ implies that $(\varphi^{-1})_x=1/(\varphi_x\circ\varphi^{-1})\in H^{k-1}(\mathbb{S})$. Hence $\varphi^{-1}\in M^k$. This proves that the inverse of any element $\varphi\in\mathcal{D}^k(\mathbb{S})$ also belongs to $\mathcal{D}^k(\mathbb{S})$.

Let $\mathbf{E}^k\subset H^k(\mathbb{S})$ be the closed linear subspace

$$\mathbf{E}^k=\{f\in H^k(\mathbb{S}):f(0)=0\}$$

with topology induced from $H^k(\mathbb{S})$. The representation (2.2) shows that M^k is an open subset of the closed hyperplane $\mathrm{id}+\mathbf{E}^k\subset H^k[0,1]$. Thus for any open interval $\mathcal{U}\subset\mathbb{S}$ of length strictly less than one, the map (2.1) provides a chart on $\mathcal{D}^k(\mathbb{S})$ with values in $\mathcal{U}\times M^k$. Moreover, we have an identification of $H^k(\mathbb{S})$ with $\mathbb{R}\times\mathbf{E}^k$ given by $u\in H^k(\mathbb{S})\mapsto(u(0),u(\,\cdot\,)-u(0))\in\mathbb{R}\times\mathbf{E}^k$. Hence, $T(\mathbb{S}\times M^k)\simeq(\mathbb{S}\times M^k)\times(\mathbb{R}\times\mathbf{E}^k)\simeq(\mathbb{S}\times M^k)\times H^k(\mathbb{S})$, so that $(\mathcal{U}\times M^k)\times H^k(\mathbb{S})$ provides a bundle chart for $T\mathcal{D}^k(\mathbb{S})$. In fact we have shown that $T\mathcal{D}^k(\mathbb{S})\simeq\mathcal{D}^k(\mathbb{S})\times H^k(\mathbb{S})$. When working with objects in $T\mathcal{D}^k(\mathbb{S})$ it will be convenient to use this representation. For example, we think of a vector field $X\colon \mathcal{D}^k(\mathbb{S})\to T\mathcal{D}^k(\mathbb{S})$ as a map $\mathcal{D}^k(\mathbb{S})\to H^k(\mathbb{S})$; the evaluation of X at $\varphi\in\mathcal{D}^k(\mathbb{S})$ is viewed as a map $\mathbb{S}\to T\mathbb{S}$ covering φ with value $(X(\varphi))(x)\in\mathbb{R}\simeq T_{\varphi(x)}\mathbb{S}$ at the point $\varphi(x)$ for $x\in\mathbb{S}$.

In the sequel, the representation $T\mathcal{D}^k(\mathbb{S})\simeq\mathcal{D}^k(\mathbb{S})\times H^k(\mathbb{S})$ will be used without further mention. If we explicitly want to point out that this representation is used, we will say that we work *locally* on $\mathcal{D}^k(\mathbb{S})$ (even though one strictly has to restrict $\mathcal{D}^k(\mathbb{S})\times H^k(\mathbb{S})$ to $(\mathcal{U}\times M^k)\times H^k(\mathbb{S})$ for it to be a chart). Observe that there are many different charts on $T\mathcal{D}^k(\mathbb{S})$, but we choose to always use the ones constructed here.

Although $\mathcal{D}^k(\mathbb{S})$ is a smooth Banach manifold it is not a Lie group. Indeed, the group operation $(\psi,\varphi)\mapsto\psi\circ\varphi\colon \mathcal{D}^k(\mathbb{S})\times\mathcal{D}^k(\mathbb{S})\to\mathcal{D}^k(\mathbb{S})$ is continuous but not C^1; right multiplication $R_\varphi\colon \psi\mapsto\psi\circ\varphi$ is smooth whereas left multiplication $L_\psi\colon \varphi\mapsto\psi\circ\varphi$ is continuous but not C^1 due to derivative-loss (see [15]).

3. A smooth Christoffel map

Let $k\geq 4$ be an integer and let $A=1-D_x^2$. A Fourier series argument shows that A is an isomorphism $H^j(\mathbb{S})\to H^{j-2}(\mathbb{S})$ for any integer j (see [18]). We define a symmetric bilinear map Γ on $T_{\mathrm{id}}\mathcal{D}^k(\mathbb{S})\simeq H^k(\mathbb{S})$ by

$$\Gamma_{\mathrm{id}}(u,v)=-A^{-1}\bigl(uv+\tfrac{1}{2}u_xv_x\bigr)_x,\quad u,v\in H^k(\mathbb{S}),\tag{3.1}$$

and extend it to a bilinear map $\Gamma_\varphi\colon T_\varphi\mathcal{D}^k(\mathbb{S})\times T_\varphi\mathcal{D}^k(\mathbb{S})\to T_\varphi\mathcal{D}^k(\mathbb{S})$ for any $\varphi\in\mathcal{D}^k(\mathbb{S})$ by right invariance, i.e.

$$\Gamma_\varphi(T_{\mathrm{id}}R_\varphi(u),T_{\mathrm{id}}R_\varphi(v))=T_{\mathrm{id}}R_\varphi(\Gamma_{\mathrm{id}}(u,v)).\tag{3.2}$$

Being a linear map, the derivative of R_φ is $TR_\varphi(V)=V\circ\varphi$. Locally, for $U,V\in T_\varphi\mathcal{D}^k(\mathbb{S})\simeq H^k(\mathbb{S})$, we get

$$\Gamma_\varphi(U,V)=-\bigl(A^{-1}\bigl((U\circ\varphi^{-1})(V\circ\varphi^{-1})+\tfrac{1}{2}(U\circ\varphi^{-1})_x(V\circ\varphi^{-1})_x\bigr)_x\bigr)\circ\varphi.$$

As composition is not smooth on $\mathcal{D}^k(\mathbb{S})$ it is not clear whether Γ possesses any smoothness properties. The main result of this section is that $\varphi\mapsto\Gamma_\varphi$ is a smooth mapping $\mathcal{D}^k(\mathbb{S})\to L^2_{\mathrm{sym}}(H^k(\mathbb{S});H^k(\mathbb{S}))$, where $L^2_{\mathrm{sym}}(H^k(\mathbb{S});H^k(\mathbb{S}))$ denotes the Banach space of symmetric continuous bilinear maps $H^k(\mathbb{S})\to H^k(\mathbb{S})$.

Remark 3.1. The motivation for the definition of Γ comes in Section 5 where we will see that Γ defines a Riemannian connection on $\mathcal{D}^k(\mathbb{S})$ compatible with the

H^1 right-invariant metric. Indeed, the map Γ corresponds to the Christoffel symbols Γ^i_{jk} well-known from finite-dimensional Riemannian geometry. When working on a Banach manifold $\mathcal{M}$ the Christoffel symbols are replaced by a family of locally defined symmetric bilinear maps $\Gamma_m\colon \mathbf{E}\times\mathbf{E}\to\mathbf{E}$; one for each chart $\mathcal{U}\times\mathbf{E}$ on $T\mathcal{M}$ (cf. [21]). The reason we can view our map Γ as a globally defined object on $\mathcal{D}^k(\mathbb{S})$ is due to the implicit identification $T\mathcal{D}^k(\mathbb{S})\simeq\mathcal{D}^k(\mathbb{S})\times H^k(\mathbb{S})$ described in Section 2.

To prove that Γ is smooth we need a couple of lemmas.

Lemma 3.2. *Let, for an integer j with $0\le j\le k$ and smooth functions $f_0, ..., f_j\colon \mathbb{S}\to\mathbb{R}$,*

$$\mathcal{P}=\sum_{i=0}^{j} f_i(x)D_x^i$$

be a linear differential operator. Then the map

$$(\varphi,U)\longmapsto(\mathcal{P}(U\circ\varphi^{-1}))\circ\varphi\colon \mathcal{D}^k(\mathbb{S})\times H^k(\mathbb{S})\longrightarrow H^{k-j}(\mathbb{S}) \tag{3.3}$$

is smooth.

Proof. By linearity it is enough to check that

$$(\varphi,U)\longmapsto f\circ\varphi\cdot(D_x^j(U\circ\varphi^{-1}))\circ\varphi\colon \mathcal{D}^k(\mathbb{S})\times H^k(\mathbb{S})\longrightarrow H^{k-j}(\mathbb{S})$$

is smooth for a smooth function f and $j\ge 0$ with $j\le k$. Since f is smooth the left composition operator $L_f\colon \varphi\mapsto f\circ\varphi$ is smooth $\mathcal{D}^k(\mathbb{S})\to H^k(\mathbb{S})$. Indeed,

$$DL_f(\varphi)V=f_x\circ\varphi\cdot V,$$

and a similar formula clearly holds for D^iL_f for any $i\ge 0$. On the other hand,

$$(\varphi,U)\longmapsto(\varphi,(D_x^j(U\circ\varphi^{-1}))\circ\varphi)\colon \mathcal{D}^k(\mathbb{S})\times H^k(\mathbb{S})\longrightarrow\mathcal{D}^k(\mathbb{S})\times H^{k-j}(\mathbb{S})$$

is the composition of j maps of the form

$$(\varphi,U)\longmapsto(\varphi,((U\circ\varphi^{-1})_x)\circ\varphi)\colon \mathcal{D}^k(\mathbb{S})\times H^{k-i}(\mathbb{S})\longrightarrow\mathcal{D}^k(\mathbb{S})\times H^{k-i-1}(\mathbb{S}) \tag{3.4}$$

for $0\le i\le j-1$. But

$$((U\circ\varphi^{-1})_x)\circ\varphi=\frac{U_x}{\varphi_x}.$$

Since D_x is a continuous linear operator $H^{k-i}(\mathbb{S})\to H^{k-i-1}(\mathbb{S})$ and

$$\varphi\longmapsto 1/\varphi_x\colon \mathcal{D}^k(\mathbb{S})\longrightarrow H^{k-1}(\mathbb{S})$$

is a smooth map, we conclude that (3.4) is a smooth map for any i with $0\le i\le j-1$. This proves the lemma. □

For two operators A and B we denote by $[A,B]$ the commutator $AB-BA$.

Lemma 3.3. *Let $A=1-D_x^2$. The map*

$$Q_1\colon (u,v)\longmapsto [v,A^{-1}]u = vA^{-1}u - A^{-1}(vu)$$

is a continuous bilinear map $H^{k-3}(\mathbb{S})\times H^k(\mathbb{S})\to H^k(\mathbb{S})$, and

$$Q_2\colon (u,v,w)\longmapsto [w,[v,A^{-1}]]u$$

is a continuous trilinear map $H^{k-4}(\mathbb{S})\times H^k(\mathbb{S})\times H^k(\mathbb{S})\to H^k(\mathbb{S})$.

Proof. We give a direct proof without Fourier analysis. If we can show that the composition $A\circ Q_1$ is a continuous map $H^{k-3}(\mathbb{S})\times H^k(\mathbb{S})\to H^{k-2}(\mathbb{S})$ the result will follow since A is an isomorphism $H^k(\mathbb{S})\to H^{k-2}(\mathbb{S})$. We compute

$$A([v,A^{-1}]u) = A(vA^{-1}u)-vu = vA^{-1}u - v_{xx}A^{-1}u - 2v_xA^{-1}u_x - vA^{-1}u_{xx} - vu.$$

Using that $vA^{-1}u-vA^{-1}u_{xx}=vu$, we get

$$A([v,A^{-1}]u) = -v_{xx}A^{-1}u - 2v_xA^{-1}u_x. \tag{3.5}$$

From this expression it follows that $A\circ Q_1$ is indeed continuous $H^{k-3}(\mathbb{S})\times H^k(\mathbb{S}) \to H^{k-2}(\mathbb{S})$.

Similarly, for $(u,v,w)\in H^{k-4}(\mathbb{S})\times H^k(\mathbb{S})\times H^k(\mathbb{S})$, we consider

$$A(Q_2(u,v,w)) = A([w,[v,A^{-1}]]u) = A(w[v,A^{-1}]u - [v,A^{-1}](wu)). \tag{3.6}$$

Using the identity $Aw=wA-[w,A]$ together with formula (3.5) and the fact that $[w,A]u=2w_xu_x+w_{xx}u$, we simplify the first term on the right-hand side as

$$\begin{aligned} A(w[v,A^{-1}]u) &= wA[v,A^{-1}]u - [w,A]([v,A^{-1}]u) \\ &= -wv_{xx}A^{-1}u - 2wv_xA^{-1}u_x - 2w_x([v,A^{-1}]u)_x - w_{xx}[v,A^{-1}]u. \end{aligned}$$

On the other hand, employing (3.5) and the identity $A^{-1}w=wA^{-1}-[w,A^{-1}]$, the second term on the right-hand side of (3.6) becomes

$$\begin{aligned} -A([v,A^{-1}](wu)) &= v_{xx}A^{-1}(wu) + 2v_xA^{-1}(wu_x) + 2v_xA^{-1}(w_xu) \\ &= v_{xx}A^{-1}(wu) + 2wv_xA^{-1}u_x - 2v_x[w,A^{-1}](u_x) + 2v_xA^{-1}(w_xu). \end{aligned}$$

Thus

$$\begin{aligned} A(Q_2(u,v,w)) = {} & -wv_{xx}A^{-1}u - 2w_x([v,A^{-1}]u)_x - w_{xx}[v,A^{-1}]u \\ & + v_{xx}A^{-1}(wu) - 2v_x[w,A^{-1}](u_x) + 2v_xA^{-1}(w_xu). \end{aligned}$$

By the result for Q_1 this is a continuous map of $(u,v,w)\in H^{k-4}(\mathbb{S})\times H^k(\mathbb{S})\times H^k(\mathbb{S})$ into $H^{k-2}(\mathbb{S})$. Again since $A\colon H^k(\mathbb{S})\to H^{k-2}(\mathbb{S})$ is an isomorphism, this proves the statement for Q_2. □

Now we are in a position to establish smoothness of Γ. In the proof of Theorem 3.4 the identity

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} U\circ(\varphi+\varepsilon V)^{-1} = -(U\circ\varphi^{-1})_x V\circ\varphi^{-1}, \quad \varphi\in\mathcal{D}^k(\mathbb{S}),\ U,V\in H^k(\mathbb{S}), \tag{3.7}$$

will be used repeatedly. It is a consequence of

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} (\varphi+\varepsilon V)^{-1} = -\frac{V\circ\varphi^{-1}}{\varphi_x\circ\varphi^{-1}},$$

and

$$\frac{U_x\circ\varphi^{-1}}{\varphi_x\circ\varphi^{-1}} = (U\circ\varphi^{-1})_x,$$

where the first formula follows by differentiating the identity, true for all small enough ε,

$$(\varphi+\varepsilon V)\circ(\varphi+\varepsilon V)^{-1} = \mathrm{id},$$

with respect to ε.

Theorem 3.4. *The map*

$$\varphi\longmapsto\Gamma_\varphi\colon \mathcal{D}^k(\mathbb{S})\longrightarrow L^2_{\mathrm{sym}}(H^k(\mathbb{S});H^k(\mathbb{S}))$$

is smooth, where Γ *is defined by* (3.1)–(3.2).

Proof. By the remarks following Proposition A.3 in the appendix, it is enough to show that the map

$$(\varphi,U,V)\longmapsto\Gamma_\varphi(U,V)\colon \mathcal{D}^k(\mathbb{S})\times H^k(\mathbb{S})\times H^k(\mathbb{S})\longrightarrow H^k(\mathbb{S}) \tag{3.8}$$

is smooth. Recall that

$$\Gamma_\varphi(U,V) = -\Big(A^{-1}\Big(U\circ\varphi^{-1}\cdot V\circ\varphi^{-1}+\tfrac{1}{2}(U\circ\varphi^{-1})_x(V\circ\varphi^{-1})_x\Big)_x\Big)\circ\varphi.$$

Hence the map

$$(\varphi,U,V)\longmapsto(\varphi,-\Gamma_\varphi(u,v))\colon \mathcal{D}^k(\mathbb{S})\times H^k(\mathbb{S})\times H^k(\mathbb{S})\longrightarrow\mathcal{D}^k(\mathbb{S})\times H^k(\mathbb{S})$$

is the composition of the three maps

$$(\varphi,U,V)\longmapsto\Big(\varphi,UV+\tfrac{1}{2}\big((U\circ\varphi^{-1})_x(V\circ\varphi^{-1})_x\big)\circ\varphi\Big)\colon \mathcal{D}^k(\mathbb{S})\times H^k(\mathbb{S})\times H^k(\mathbb{S})\longrightarrow\mathcal{D}^k(\mathbb{S})\times H^{k-1}(\mathbb{S}), \tag{3.9}$$

$$(\varphi,U)\longmapsto(\varphi,((U\circ\varphi^{-1})_x)\circ\varphi)\colon \mathcal{D}^k(\mathbb{S})\times H^{k-1}(\mathbb{S})\longrightarrow\mathcal{D}^k(\mathbb{S})\times H^{k-2}(\mathbb{S}), \tag{3.10}$$

and

$$(3.11)\quad P\colon (\varphi,U)\longmapsto(\varphi,(A^{-1}(U\circ\varphi^{-1}))\circ\varphi)\colon \mathcal{D}^k(\mathbb{S})\times H^{k-2}(\mathbb{S})\longrightarrow\mathcal{D}^k(\mathbb{S})\times H^k(\mathbb{S}).$$

We will show that they are all smooth maps.

Smoothness of the maps (3.9) and (3.10) follows from Lemma 3.2. As for the third map P, note that its inverse is

$$(3.12)\qquad (\varphi,U)\longmapsto(\varphi,(A(U\circ\varphi^{-1}))\circ\varphi)\colon \mathcal{D}^k(\mathbb{S})\times H^{k}(\mathbb{S})\longrightarrow\mathcal{D}^k(\mathbb{S})\times H^{k-2}(\mathbb{S}),$$

which is smooth by Lemma 3.2. Hence, if we can show that $P=(P_1,P_2)$ is Gateaux-C^2, then Proposition A.2 implies that P is C^1, and so smoothness will follow automatically by Proposition A.4.

Since P_1 is trivially smooth and P_2 is linear in U, it is enough to prove that $D_1P_2\colon \mathcal{D}^k(\mathbb{S})\times H^{k-2}(\mathbb{S})\times H^k(\mathbb{S})\to H^k(\mathbb{S})$ exists and is Gateaux-C^1.

A computation, using linearity of A^{-1} together with (3.7), gives

$$D_1P_2(\varphi,U)V=-(A^{-1}((U\circ\varphi^{-1})_xV\circ\varphi^{-1}))\circ\varphi+V\cdot(A^{-1}(U\circ\varphi^{-1})_x)\circ\varphi,$$

that is,

$$(3.13)\qquad D_1P_2(\varphi,U)V=([V\circ\varphi^{-1},A^{-1}](U\circ\varphi^{-1})_x)\circ\varphi.$$

Thus

$$D_1P_2(\varphi,U)V=R_\varphi\circ Q_1\circ((D_x\circ R_{\varphi^{-1}})\times R_{\varphi^{-1}})(U,V),$$

where Q_1 is the continuous map $H^{k-3}(\mathbb{S})\times H^k(\mathbb{S})\to H^k(\mathbb{S})$ from Lemma 3.3. Since composition is continuous on $\mathcal{D}^k(\mathbb{S})$, we deduce that

$$D_1P_2\colon \mathcal{D}^k(\mathbb{S})\times H^{k-2}(\mathbb{S})\times H^k(\mathbb{S})\longrightarrow H^k(\mathbb{S})$$

is continuous. Furthermore, differentiation of (3.13) gives, for

$$(\varphi,U,V,W)\in\mathcal{D}^k(\mathbb{S})\times H^{k-2}(\mathbb{S})\times H^k(\mathbb{S})\times H^k(\mathbb{S}),$$

that

$$\begin{aligned}D_1^2P_2(\varphi,U)(V,W)=&-([(V\circ\varphi^{-1})_xW\circ\varphi^{-1},A^{-1}](U\circ\varphi^{-1})_x)\circ\varphi\\&-([V\circ\varphi^{-1},A^{-1}]((U\circ\varphi^{-1})_xW\circ\varphi^{-1})_x)\circ\varphi\\&+W\cdot([V\circ\varphi^{-1},A^{-1}](U\circ\varphi^{-1})_x)_x\circ\varphi.\end{aligned}$$

It is straightforward to check that this can be written as

$$\begin{aligned} D_1^2P_2(\varphi,U)(V,W) = (&-[W\circ\varphi^{-1},A^{-1}]((U\circ\varphi^{-1})_x(V\circ\varphi^{-1})_x) \\ &-[V\circ\varphi^{-1},A^{-1}]((U\circ\varphi^{-1})_x(W\circ\varphi^{-1})_x) \\ &+[W\circ\varphi^{-1},[V\circ\varphi^{-1},A^{-1}]](U\circ\varphi^{-1})_{xx})\circ\varphi, \end{aligned}$$

so that employing Lemma 3.3 again, we see that $D_1^2P_2$ is continuous $\mathcal{D}^k(\mathbb{S})\times H^{k-2}(\mathbb{S})\times H^k(\mathbb{S})\times H^k(\mathbb{S})\to H^k(\mathbb{S})$. This proves that D_1P_2 is Gateaux-C^1 and completes the proof of the theorem. □

4. The H^1 right-invariant metric

The H^1 metric on $H^k(\mathbb{S})$ defines a positive definite symmetric bilinear form $\langle\cdot,\cdot\rangle_{\mathrm{id}}$ on $T_{\mathrm{id}}\mathcal{D}^k(\mathbb{S})\simeq H^k(\mathbb{S})$ by

$$\langle u,v\rangle_{\mathrm{id}} = \int_{\mathbb{S}} uAv\,dx = \int_{\mathbb{S}}(uv+u_xv_x)\,dx, \quad u,v\in T_{\mathrm{id}}\mathcal{D}^k(\mathbb{S}),$$

where, as before, $A=1-D_x^2$ and $k\geq 4$. We define a positive definite symmetric bilinear form $\langle\cdot,\cdot\rangle_\varphi$ on each tangent space $T_\varphi\mathcal{D}^k(\mathbb{S})$ by right translation

$$\langle T_{\mathrm{id}}R_\varphi(u), T_{\mathrm{id}}R_\varphi(v)\rangle_\varphi = \langle u,v\rangle_{\mathrm{id}}.$$

Locally, for $U,V\in T_\varphi\mathcal{D}^k(\mathbb{S})\simeq H^k(\mathbb{S})$, we have

$$\langle U,V\rangle_\varphi = \int_{\mathbb{S}} U\circ\varphi^{-1}A(V\circ\varphi^{-1})\,dx.$$

We call $\langle\cdot,\cdot\rangle$ the H^1 *right-invariant metric* on $\mathcal{D}^k(\mathbb{S})$. Just like for the Christoffel map Γ it is not a priori clear whether $\varphi\mapsto\langle\cdot,\cdot\rangle_\varphi$ is a smooth map since the group operation $\varphi\mapsto\varphi^{-1}$ on $\mathcal{D}^k(\mathbb{S})$ is not smooth. The next result establishes that the H^1 right-invariant metric on $\mathcal{D}^k(\mathbb{S})$ is indeed a Riemannian metric.

Theorem 4.1. *The map*

$$\varphi\longmapsto\langle\cdot,\cdot\rangle_\varphi\colon \mathcal{D}^k(\mathbb{S})\longrightarrow L^2_{\mathrm{sym}}(T_\varphi\mathcal{D}^k(\mathbb{S});\mathbb{R})$$

is a smooth section of the bundle $L^2_{\mathrm{sym}}(T\mathcal{D}^k(\mathbb{S});\mathbb{R})$.

Proof. Let $g\colon\mathcal{D}^k(\mathbb{S})\to L^2_{\mathrm{sym}}(H^k(\mathbb{S});\mathbb{R})$ be the local representative for $\langle\cdot,\cdot\rangle$, that is, for $U,V\in T_\varphi\mathcal{D}^k(\mathbb{S})\simeq H^k(\mathbb{S})$,

$$g(\varphi)(U,V) = \int_{\mathbb{S}} U\circ\varphi^{-1}A(V\circ\varphi^{-1})\,dx.$$

Let $P(\varphi,U,V)=g(\varphi)(U,V)$. By the remarks following Proposition A.3 in the appendix, it is enough to show that P is smooth $\mathcal{D}^k(\mathbb{S})\times H^k(\mathbb{S})\times H^k(\mathbb{S})\to\mathbb{R}$.

For $W\in H^k(\mathbb{S})$ we get

$$\begin{aligned} D_1P(\varphi,U,V)W &= \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0}\int_{\mathbb{S}} A(U\circ(\varphi+\varepsilon W)^{-1})V\circ(\varphi+\varepsilon W)^{-1}\,dx \\ &= -\int_{\mathbb{S}} A((U\circ\varphi^{-1})_x W\circ\varphi^{-1})V\circ\varphi^{-1}\,dx \\ &\quad -\int_{\mathbb{S}} A(U\circ\varphi^{-1})(V\circ\varphi^{-1})_x W\circ\varphi^{-1}\,dx. \end{aligned}$$

By symmetry we consider only the second integral. The substitution $y=\varphi^{-1}(x)$ yields, as $(V\circ\varphi^{-1})_x(x)\,dx=V_x\circ\varphi^{-1}(x)(\varphi^{-1})_x(x)\,dx=V_x(y)\,dy$,

$$\int_{\mathbb{S}} A(U\circ\varphi^{-1})(V\circ\varphi^{-1})_x W\circ\varphi^{-1}\,dx=\int_{\mathbb{S}}(A(U\circ\varphi^{-1}))\circ\varphi\cdot V_x W\,dy.$$

Since

$$(\varphi,U)\longmapsto(A(U\circ\varphi^{-1}))\circ\varphi\colon \mathcal{D}^k(\mathbb{S})\times H^k(\mathbb{S})\longrightarrow H^{k-2}(\mathbb{S})$$

is a smooth map by Lemma 3.2, we see that D_1P is smooth $\mathcal{D}^k(\mathbb{S})\times H^k(\mathbb{S})\times H^k(\mathbb{S})\times H^k(\mathbb{S})\to\mathbb{R}$. As P is continuous and linear in both U and V, P is smooth. □

5. Covariant derivative

We first make a general definition. Let $\mathcal{M}$ be a Banach manifold endowed with a Riemannian metric $\langle\cdot,\cdot\rangle$ and let $\mathfrak{X}(\mathcal{M})$ denote the space of smooth vector fields on $\mathcal{M}$. Recall that for $X,Y\in\mathfrak{X}(\mathcal{M})$ the Lie bracket $[X,Y]$ is defined locally by

$$[X,Y](m)=DY(m)\cdot X(m)-DX(m)\cdot Y(m).$$

Definition 5.1. An $\mathbb{R}$-bilinear operator $(X,Y)\mapsto\nabla_X Y:\mathfrak{X}(\mathcal{M})\times\mathfrak{X}(\mathcal{M})\to\mathfrak{X}(\mathcal{M})$ is a *Riemannian covariant derivative* if it satisfies

(1) $X(m)=0$ implies $(\nabla_X Y)(m)=0$ for $m\in\mathcal{M}$ and $X,Y\in\mathfrak{X}(\mathcal{M})$ (punctual dependence on X),

(2) $\nabla_X Y-\nabla_Y X=[X,Y]$ for $X,Y\in\mathfrak{X}(\mathcal{M})$ (torsion-free),

(3) $\nabla_X(fY)=(\mathcal{L}_X f)Y+f\nabla_X Y$ for $f\in C^\infty(\mathcal{M})$ and $X,Y\in\mathfrak{X}(\mathcal{M})$ (derivation in Y),

(4) $\mathcal{L}_X\langle Y,Z\rangle=\langle\nabla_X Y,Z\rangle+\langle Y,\nabla_X Z\rangle$ for $X,Y,Z\in\mathfrak{X}(\mathcal{M})$ (compatible with the metric).

Remark 5.2. The $\mathbb{R}$-linearity in X together with (1) shows that ∇ is $C^\infty(\mathcal{M})$-linear in X, i.e. $\nabla_{fX}Y=f\nabla_X Y$ for a smooth function $f\colon \mathcal{M}\to\mathbb{R}$. In finite dimensions punctual dependence on X and $C^\infty(\mathcal{M})$-linearity in X are equivalent properties of ∇, but this is not true in the infinite-dimensional case (cf. [21]).

We define the operator $\nabla\colon \mathfrak{X}(\mathcal{D}^k(\mathbb{S}))\times\mathfrak{X}(\mathcal{D}^k(\mathbb{S}))\to\mathfrak{X}(\mathcal{D}^k(\mathbb{S}))$ locally by

$$(\nabla_X Y)(\varphi)=DY(\varphi)\cdot X(\varphi)-\Gamma_\varphi(Y(\varphi),X(\varphi)), \tag{5.1}$$

where $X,Y\colon \mathcal{U}\times M^k\to H^k(\mathbb{S})$ are the local representatives of vector fields $X,Y\in \mathfrak{X}(\mathcal{D}^k(\mathbb{S}))$ (see Section 2 for a detailed description of a bundle chart of the form $(\mathcal{U}\times M^k)\times H^k(\mathbb{S})$ on $T\mathcal{D}^k(\mathbb{S})$).

Theorem 5.3. *The bilinear map* ∇ *defined by* (5.1) *is a Riemannian covariant derivative on* $\mathcal{D}^k(\mathbb{S})$ *compatible with the* H^1 *right-invariant metric.*

Proof. Properties (1)–(3) are immediate from the local formula defining ∇. To establish (4) we write locally, for vector fields $X,Y,Z\in\mathfrak{X}(\mathcal{D}^k(\mathbb{S}))$,

$$\begin{aligned}
&(\mathcal{L}_X\langle Y,Z\rangle)(\varphi)\\
&\quad=\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0}\int_{\mathbb{S}} A(Y(\varphi+\varepsilon X(\varphi))\circ(\varphi+\varepsilon X(\varphi))^{-1})Z(\varphi+\varepsilon X(\varphi))\circ(\varphi+\varepsilon X(\varphi))^{-1}\,dx\\
&\quad=\int_{\mathbb{S}} A((DY(\varphi)\cdot X(\varphi))\circ\varphi^{-1}-(Y(\varphi)\circ\varphi^{-1})_x X(\varphi)\circ\varphi^{-1})Z(\varphi)\circ\varphi^{-1}\,dx\\
&\qquad+\int_{\mathbb{S}} A((DZ(\varphi)\cdot X(\varphi))\circ\varphi^{-1}-(Z(\varphi)\circ\varphi^{-1})_x X(\varphi)\circ\varphi^{-1})Y(\varphi)\circ\varphi^{-1}\,dx,
\end{aligned}$$

where we used formula (3.7) to carry out the differentiation. With $u=X(\varphi)\circ\varphi^{-1}$, $v=Y(\varphi)\circ\varphi^{-1}$, and $w=Z(\varphi)\circ\varphi^{-1}$, we get

$$\begin{aligned}
(\mathcal{L}_X\langle Y,Z\rangle)(\varphi)=&\int_{\mathbb{S}} A((DY(\varphi)\cdot X(\varphi))\circ\varphi^{-1})w\,dx\\
&+\int_{\mathbb{S}} A((DZ(\varphi)\cdot X(\varphi))\circ\varphi^{-1})v\,dx-\int_{\mathbb{S}} A(v_x u)w\,dx\\
&-\int_{\mathbb{S}} A(w_x u)v\,dx.
\end{aligned} \tag{5.2}$$

On the other hand

$$\langle\nabla_X Y,Z\rangle_\varphi=\int_{\mathbb{S}}(DY(\varphi)\cdot X(\varphi)-\Gamma_\varphi(Y(\varphi),X(\varphi)))\circ\varphi^{-1}A(Z(\varphi)\circ\varphi^{-1})\,dx.$$

Since $\Gamma_\varphi(Y(\varphi),X(\varphi))=-\big(A^{-1}(vu+\tfrac12 v_x u_x)_x\big)\circ\varphi$, we get

$$\langle\nabla_X Y,Z\rangle_\varphi=\int_{\mathbb{S}}(DY(\varphi)\cdot X(\varphi))\circ\varphi^{-1}A(w)\,dx+\int\big(vu+\tfrac12 v_x u_x\big)_x w\,dx. \tag{5.3}$$

Now it is easy to check that

$$-\int_{\mathbb{S}} A(v_x u)w\,dx - \int_{\mathbb{S}} A(w_x u)v\,dx = \int \left(vu + \tfrac{1}{2}v_x u_x\right)_x w\,dx + \int \left(wu + \tfrac{1}{2}w_x u_x\right)_x v\,dx$$

so by (5.2) and (5.3) we obtain

$$(\mathcal{L}_X\langle Y, Z\rangle)(\varphi) = \langle \nabla_X Y, Z\rangle_\varphi + \langle Y, \nabla_X Z\rangle_\varphi.$$

This proves that ∇ also satisfies (4). □

In the finite-dimensional case, given a Riemannian metric $\langle\,\cdot\,,\cdot\,\rangle$ on a manifold $\mathcal{M}$ there automatically exists a Riemannian covariant derivative ∇ compatible with $\langle\,\cdot\,,\cdot\,\rangle$. For vector fields X, Y and Z on $\mathcal{M}$, $\nabla_X Y$ is defined as the unique vector field such that

$$\begin{aligned} 2\langle \nabla_X Y, Z\rangle = &-\langle [Y, X], Z\rangle - \langle X, [Y, Z]\rangle - \langle Y, [X, Z]\rangle \\ &+ \mathcal{L}_X\langle Y, Z\rangle + \mathcal{L}_Y\langle Z, X\rangle - \mathcal{L}_Z\langle X, Y\rangle. \end{aligned} \tag{5.4}$$

Indeed, the bracket $\langle\,\cdot\,,\cdot\,\rangle$ establishes an isomorphism $T_m\mathcal{M}\to T_m^*\mathcal{M}$ for each $m\in\mathcal{M}$, so since the right-hand side is a continuous linear functional of $Z(m)$, existence of $(\nabla_X Y)(m)$ follows immediately.

This approach does not apply to $\mathcal{D}^k(\mathbb{S})$ endowed with the H^1 right-invariant metric. The right-hand side of (5.4) is a continuous linear functional of $Z(\varphi)$ for each $\varphi\in\mathcal{D}^k(\mathbb{S})$. But the topology of $T_\varphi\mathcal{D}^k(\mathbb{S})\simeq H^k(\mathbb{S})$ induced by the H^k inner product is stronger than the topology defined by the H^1 right-invariant metric $\langle\,\cdot\,,\cdot\,\rangle_\varphi$ – the H^1 right-invariant metric is a *weak Riemannian metric* on $\mathcal{D}^k(\mathbb{S})$. Therefore there are elements in $T_\varphi^*\mathcal{D}^k(\mathbb{S})$ that can not be expressed as $\langle V,\cdot\,\rangle_\varphi$ for some $V\in T_\varphi\mathcal{D}^k(\mathbb{S})$; the spaces $T_\varphi\mathcal{D}^k(\mathbb{S})$ and $T_\varphi^*\mathcal{D}^k(\mathbb{S})$ are in duality with respect to the H^k inner product; not with respect to $\langle\,\cdot\,,\cdot\,\rangle_\varphi$. The explicit formula for Γ gave us a way to circumvent this difficulty.

On the contrary, even for weak Riemannian metrics *uniqueness* of the Riemannian covariant derivative can be deduced from (5.4). For if ∇ satisfies (1)–(4) of Definition 5.1, then writing down the property (4) for the cyclic permutations of $X, Y, Z\in\mathfrak{X}(M)$ we get

$$\begin{aligned} \mathcal{L}_X\langle Y, Z\rangle &= \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle, \\ \mathcal{L}_Y\langle Z, X\rangle &= \langle \nabla_Y Z, X\rangle + \langle Z, \nabla_Y X\rangle, \\ \mathcal{L}_Z\langle X, Y\rangle &= \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle. \end{aligned}$$

Adding the first two and subtracting the third of these relations, (5.4) drops out. Since $\langle\,\cdot\,,\cdot\,\rangle$ is non-degenerate, (5.4) shows the uniqueness of ∇.

6. Riemannian geometry on $\mathcal{D}^k(\mathbb{S})$

Now that we proved that the map $\varphi \mapsto \Gamma_\varphi$ defined in Section 3 is a smooth Christoffel map (Theorem 3.4), all the usual constructions for affine connections on Banach manifolds can be carried out without additional effort. In this section we state some relevant results. The general theory for Banach manifolds that we here apply to $\mathcal{D}^k(\mathbb{S})$ can be found in [21].

We also show that if $t \mapsto \varphi(t)$ is a C^2-curve in $\mathcal{D}^k(\mathbb{S})$, then φ is a geodesic if and only if $u(t) = \varphi_t(t) \circ \varphi(t)^{-1}$ solves the Camassa–Holm equation, establishing the relation between geodesics for the H^1 right-invariant metric on $\mathcal{D}^k(\mathbb{S})$ and the Camassa–Holm equation that was hinted at in the introduction.

Remark 6.1. One might argue that the fact that $\langle \cdot , \cdot \rangle$ is only a weak Riemannian metric (see the discussion following Theorem 5.3) would prevent the general results from Banach manifold theory from applying. However, the results presented in this section depend only on the existence of an affine connection derived from a family of smooth Christoffel maps – there is no metric involved. We showed in Theorem 3.4 that Γ is a smooth global Christoffel map. Hence, as far as this section is concerned, it is irrelevant whether Γ is compatible with any metric or not.

6.1. Affine connection

The *horizontal subbundle* $\mathrm{Hor} \subset TT\mathcal{D}^k(\mathbb{S})$ is defined locally by

$$\mathrm{Hor} = \{(\varphi, U, V, W) \in \mathcal{D}^k(\mathbb{S}) \times H^k(\mathbb{S}) \times H^k(\mathbb{S}) \times H^k(\mathbb{S}) : W = \Gamma_\varphi(U, V)\},$$

whereas the *vertical subbundle* $\mathrm{Ver} \subset TT\mathcal{D}^k(\mathbb{S})$ is given by $\mathrm{Ver} = \ker T\pi$, where $\pi \colon T\mathcal{D}^k(\mathbb{S}) \to \mathcal{D}^k(\mathbb{S})$ is the canonical projection. For each $U_\varphi \in T_\varphi \mathcal{D}^k(\mathbb{S})$, Hor_{U_φ} defines a complementary subspace of horizontal vectors to Ver_{U_φ}, that is, $\mathrm{Hor}_{U_\varphi} \oplus \mathrm{Ver}_{U_\varphi} = T_{U_\varphi} T\mathcal{D}^k(\mathbb{S})$, where Hor_{U_φ} and Ver_{U_φ} denote the fibers of Hor respectively Ver over U_φ. This defines an *affine connection* on $\mathcal{D}^k(\mathbb{S})$.

The *horizontal lift* $\mathrm{hor}_U V$ of a vector $V \in T_\varphi \mathcal{D}^k(\mathbb{S})$ with respect to $U \in T_\varphi \mathcal{D}^k(\mathbb{S})$ is the unique element in Hor_U such that its image under $T\pi$ equals V. Locally,

$$\mathrm{hor}_U V = (\varphi, U, V, \Gamma_\varphi(U, V)), \quad \varphi \in \mathcal{D}^k(\mathbb{S}),\ U, V \in H^k(\mathbb{S}).$$

6.2. Spray

The *spray* Z associated to Γ is the second order vector field $Z \colon T\mathcal{D}^k(\mathbb{S}) \to TT\mathcal{D}^k(\mathbb{S})$ defined by $Z(U) = \mathrm{hor}_U U$. Locally,

$$Z(\varphi, U) = (\varphi, U, U, \Gamma_\varphi(U, U)), \quad \varphi \in \mathcal{D}^k(\mathbb{S}),\ U \in H^k(\mathbb{S}).$$

6.3. Covariant derivative along a curve

Let $J\subset\mathbb{R}$ be an open interval and let $\varphi\colon J\to\mathcal{D}^k(\mathbb{S})$ be a C^1-curve. A *lift* of φ to $T\mathcal{D}^k(\mathbb{S})$ is a C^1-map $V\colon J\to T\mathcal{D}^k(\mathbb{S})$ such that $\pi\circ V=\varphi$. Let $\mathrm{Lift}(\varphi)$ be the space of lifts of φ.

Define the operator $\nabla_{\varphi_t}\colon \mathrm{Lift}(\varphi)\to\mathrm{Lift}(\varphi)$ by

$$\nabla_{\varphi_t}V=V_t-\Gamma_\varphi(V,\varphi_t),\quad V\in\mathrm{Lift}(\varphi). \tag{6.1}$$

The map ∇_{φ_t} satisfies the derivation property

$$(\nabla_{\varphi_t}(fV))(t)=f'(t)V(t)+f(t)(\nabla_{\varphi_t}V)(t)$$

for a C^1-function $f\colon J\to\mathbb{R}$. Moreover, by the chain rule, ∇_{φ_t} is the unique linear map $\mathrm{Lift}(\varphi)\to\mathrm{Lift}(\varphi)$ such that if X and Y are vector fields on $\mathcal{D}^k(\mathbb{S})$ with $V(t)=Y(\varphi(t))$ for $t\in J$ and $\varphi_t(t_0)=X(\varphi(t_0))$ for some $t_0\in J$, then

$$(\nabla_{\varphi_t}V)(t_0)=(\nabla_XY)(\varphi(t_0)).$$

6.4. Parallel translation

Let $J\subset\mathbb{R}$ be an open interval and let $\varphi\colon J\to\mathcal{D}^k(\mathbb{S})$ be a C^2-curve. A lift $V\colon J\to T\mathcal{D}^k(\mathbb{S})$ of φ is *φ-parallel* if $V_t(t)\in TT\mathcal{D}^k(\mathbb{S})$ is horizontal for all $t\in J$. Locally, V is φ-parallel if and only if

$$V_t=\Gamma_\varphi(V,\varphi_t),\quad t\in J,$$

which is equivalent to $\nabla_{\varphi_t}V\equiv 0$. Let $\mathrm{Par}(\varphi)$ denote the set of φ-parallel lifts of φ. Applying the theory for Banach manifolds we get the following result, cf. [21].

Theorem 6.2. *Let $t_0\in J$. Given $V_0\in T_{\varphi(0)}\mathcal{D}^k(\mathbb{S})$, there exists a unique φ-parallel lift $t\mapsto V(t;V_0)\colon J\to T\mathcal{D}^k(\mathbb{S})$ such that $V(t_0;V_0)=V_0$. The map $V_0\mapsto V(\,\cdot\,;V_0)$ is a linear isomorphism $T_{\varphi(t_0)}\mathcal{D}^k(\mathbb{S})\to\mathrm{Par}(\varphi)$.*

Moreover, for each $t\in J$, the map $P_t\colon T_{\varphi(t_0)}\mathcal{D}^k(\mathbb{S})\to T_{\varphi(t)}\mathcal{D}^k(\mathbb{S})$ defined by $P_t(V_0)=V(t;V_0)$ is a linear isomorphism.

Hence, the map P_t gives a well-defined *parallel translation* along any C^2-curve in $\mathcal{D}^k(\mathbb{S})$.

Define two continuous maps $u,v\colon J\to T_{\mathrm{id}}\mathcal{D}^k(\mathbb{S})\simeq H^k(\mathbb{S})$ by

$$u(t)=T_{\varphi(t)}R_{\varphi(t)^{-1}}(\varphi_t(t))=\varphi_t(t)\circ\varphi(t)^{-1},$$

and

$$v(t)=T_{\varphi(t)}R_{\varphi(t)^{-1}}(V(t))=V(t)\circ\varphi(t)^{-1}.$$

We can express the equation defining parallel translation as an equation in u and v. Note that u and v are not C^1-maps as

$$u_t=\varphi_{tt}\circ\varphi^{-1}+\frac{\varphi_{tx}\circ\varphi^{-1}}{\varphi_x\circ\varphi^{-1}}$$

is in general an element in $H^{k-1}(\mathbb{S})$ but not in $H^k(\mathbb{S})$. Nevertheless, we see that

$$u,v\in C(J;H^k(\mathbb{S}))\cap C^1(J;H^{k-1}(\mathbb{S})).$$

Theorem 6.3. *Let $\varphi\colon J\to\mathcal{D}^k(\mathbb{S})$ be a C^2-curve and $V\colon J\to T\mathcal{D}^k(\mathbb{S})$ be a lift of φ. Define $u,v\colon J\to H^k(\mathbb{S})$ by*

$$v(t)=V(t)\circ\varphi(t)^{-1}\quad\textit{and}\quad u(t)=\varphi_t(t)\circ\varphi(t)^{-1},$$

so that $u,v\in C(J;H^k(\mathbb{S}))\cap C^1(J;H^{k-1}(\mathbb{S}))$. Then V is φ-parallel if and only if u and v satisfy the equation

$$v_t+v_xu=\Gamma_{\mathrm{id}}(v,u)\quad\textit{in } H^{k-1}(\mathbb{S}). \tag{6.2}$$

Proof. First note that

$$v_xu=(V\circ\varphi^{-1})_x\varphi_t\circ\varphi^{-1}=\frac{V_x\circ\varphi^{-1}}{\varphi_x\circ\varphi^{-1}}\varphi_t\circ\varphi^{-1}=-V_x\circ\varphi^{-1}\cdot(\varphi^{-1})_t. \tag{6.3}$$

Suppose V is φ-parallel. Using (6.3) we compute in $H^{k-1}(\mathbb{S})$,

$$v_t=V_t\circ\varphi^{-1}+V_x\circ\varphi^{-1}\cdot(\varphi^{-1})_t=\Gamma_\varphi(V,\varphi_t)\circ\varphi^{-1}-v_xu=\Gamma_{\mathrm{id}}(v,u)-v_xu,$$

where we used the right invariance of Γ. Conversely, if (6.2) holds, then (6.3) yields

$$V_t\circ\varphi^{-1}=v_t-V_x\circ\varphi^{-1}\cdot(\varphi^{-1})_t=\Gamma_{\mathrm{id}}(v,u)-v_xu-V_x\circ\varphi^{-1}\cdot(\varphi^{-1})_t=\Gamma_\varphi(V,\varphi_t)\circ\varphi^{-1},$$

showing that V is φ-parallel. □

6.5. Curvature

The *curvature tensor* $\mathcal{R}$ is the trilinear map $\mathfrak{X}(\mathcal{D}^k(\mathbb{S}))\times\mathfrak{X}(\mathcal{D}^k(\mathbb{S}))\times\mathfrak{X}(\mathcal{D}^k(\mathbb{S}))\to\mathfrak{X}(\mathcal{D}^k(\mathbb{S}))$ defined by

$$\mathcal{R}(X,Y)Z=\nabla_X\nabla_YZ-\nabla_Y\nabla_XZ-\nabla_{[X,Y]}Z.$$

$\mathcal{R}$ is tensorial and a long computation shows that locally, for $U,V,W\in T_\varphi\mathcal{D}^k(\mathbb{S})\simeq H^k(\mathbb{S})$, it holds that

$$\begin{aligned}\mathcal{R}(U,V)W&=D_1\Gamma(\varphi,W,U)V-D_1\Gamma(\varphi,W,V)U\\&\quad+\Gamma(\varphi,\Gamma(\varphi,W,V),U)-\Gamma(\varphi,\Gamma(\varphi,W,U),V),\end{aligned} \tag{6.4}$$

where we write $\Gamma(\varphi, U, V)$ for $\Gamma_\varphi(U,V)$. Since $\varphi \mapsto \Gamma_\varphi$ is a smooth map $\mathcal{D}^k(\mathbb{S}) \to L^2_{\mathrm{sym}}(H^k(\mathbb{S}); H^k(\mathbb{S}))$, the local formula (6.4) for the curvature tensor $\mathcal{R}$ immediately shows that $\mathcal{R}$ is a smooth section of the bundle $L^3(T\mathcal{D}^k(\mathbb{S}); T\mathcal{D}^k(\mathbb{S}))$.

6.6. Geodesics

A C^2-map $\varphi\colon J \to \mathcal{D}^k(\mathbb{S})$ is a *geodesic* if φ_t is an integral curve of the spray Z. This is equivalent to $\nabla_{\varphi_t}\varphi_t \equiv 0$ on J. Locally the geodesic equation is

$$\varphi_{tt} = \Gamma_\varphi(\varphi_t, \varphi_t).$$

For a vector $V_0 \in T\mathcal{D}^k(\mathbb{S})$ we let $t \mapsto V(t; V_0)\colon J \to T\mathcal{D}^k(\mathbb{S})$ be the integral curve of Z with initial data V_0 defined on some maximal interval J. Let $\mathfrak{D}$ be the set of vectors $V_0 \in T\mathcal{D}^k(\mathbb{S})$ such that $V(\,\cdot\,; V_0)$ is defined at least on the interval $[0,1]$. We get the following result for geodesics and the exponential map on $\mathcal{D}^k(\mathbb{S})$ (cf. [21]).

Theorem 6.4. *The set $\mathfrak{D} \subset T\mathcal{D}^k(\mathbb{S})$ is open and the map*

$$V_0 \longmapsto V(1; V_0)\colon \mathfrak{D} \longrightarrow T\mathcal{D}^k(\mathbb{S})$$

is smooth. Also, the exponential map $\exp\colon \mathfrak{D} \to \mathcal{D}^k(\mathbb{S})$ *defined by*

$$\exp(V_0) = \pi(V(1; V_0))$$

is smooth, and if $\exp_\varphi$ denotes the restriction of $\exp$ to $T_\varphi\mathcal{D}^k(\mathbb{S})$, then the derivative of $\exp_\varphi$ at $0 \in T_\varphi\mathcal{D}^k(\mathbb{S})$ is the identity. By the inverse mapping theorem $\exp_\varphi$ is a diffeomorphism from a neighborhood of 0 in $T_\varphi\mathcal{D}^k(\mathbb{S})$ to a neighborhood of φ in $\mathcal{D}^k(\mathbb{S})$.

Just like for parallel translation we can express the geodesic equation as an equation for $u = \varphi_t \circ \varphi^{-1}$. If $\mathcal{D}^k(\mathbb{S})$ were a Lie group, the equation for u would be the Euler equation on the Lie algebra $T_{\mathrm{id}}\mathcal{D}^k(\mathbb{S})$ for the H^1 right-invariant metric.

Theorem 6.5. *Let $\varphi\colon J \to \mathcal{D}^k(\mathbb{S})$ be a C^2-curve and define $u\colon J \to H^k(\mathbb{S})$ by $u(t) = \varphi_t(t) \circ \varphi(t)^{-1}$ so that $u \in C(J; H^k(\mathbb{S})) \cap C^1(J; H^{k-1}(\mathbb{S}))$. Then φ is a geodesic if and only if u solves the Camassa–Holm equation*

$$u_t - u_{txx} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \quad t \in J, \ x \in \mathbb{S}. \tag{6.5}$$

Proof. The curve φ is a geodesic if and only if φ_t is φ-parallel. By Theorem 6.3 this holds if and only if

$$u_t = \Gamma_{\mathrm{id}}(u,u) - uu_x.$$

By the definition (3.1) of Γ we rewrite this as

$$u_t = -A^{-1}\big(u^2 + \tfrac{1}{2}u_x^2\big)_x - uu_x. \tag{6.6}$$

Finally, applying the isomorphism $A = 1 - D_x^2$ to both sides of (6.6) shows that (6.6) and (6.5) are equivalent. □

Let us point out that some of these geodesics exist for all times whereas others do not. Indeed, it is known (see [4] and [8]) that smooth initial data u_0 for the Camassa–Holm equation either develop into smooth global solutions, or singularities in finite time can arise in the form of wave breaking (the solution u, representing the waters free surface, remains bounded while its slope becomes unbounded in finite time). In the first case, the geodesic $\varphi\colon J \to \mathcal{D}^k(\mathbb{S})$ with $\varphi(0) = \mathrm{id}$ and $\varphi_t(0) = u_0$ is defined for all times, while in the second case the geodesic flow breaks down at the blow-up time (see [10]). For example, any geodesic starting at the identity in a nonzero direction $u_0 \in T_{\mathrm{id}}\mathcal{D}^k(\mathbb{S}) \simeq H^k(\mathbb{S})$ satisfying $\int_{\mathbb{S}} u_0\, dx = 0$ has a finite existence time in view of the blow-up result presented in [7].

6.7. Normal neighborhoods

Let $\varphi_0 \in \mathcal{D}^k(\mathbb{S})$. Locally, arbitrarily small neighborhoods of a point $(\varphi_0, 0) \in \mathcal{D}^k(\mathbb{S}) \times H^k(\mathbb{S})$ are of the form $\mathcal{U}_0 \times B_\varepsilon(0)$, for some neighborhood $\mathcal{U}_0$ of φ_0 in $\mathcal{D}^k(\mathbb{S})$ and some ball $B_\varepsilon(0)$ of radius ε around $0 \in H^k(\mathbb{S})$. The next result establishes the existence of *normal neighborhoods* $(\mathcal{V}, \mathcal{W})$ on $\mathcal{D}^k(\mathbb{S})$ (see [21]).

Theorem 6.6. *Let $\varphi_0 \in \mathcal{D}^k(\mathbb{S})$. Given an open neighborhood $\mathcal{V} = \mathcal{U}_0 \times B_\varepsilon(0)$ of $(\varphi_0, 0)$ in $T\mathcal{D}^k(\mathbb{S})$, there is an open neighborhood $\mathcal{W} \subset \mathcal{U}_0$ of φ_0 in $\mathcal{D}^k(\mathbb{S})$ such that any two points $\varphi, \psi \in \mathcal{W}$ can be joined by a unique geodesic lying in $\mathcal{U}_0$, and such that for each $\varphi \in \mathcal{W}$ the exponential $\exp_\varphi$ maps the open set in $T_\varphi\mathcal{D}^k(\mathbb{S})$ represented by $(\varphi, B_\varepsilon(0))$ diffeomorphically onto an open set $\mathcal{U}(\varphi)$ containing $\mathcal{W}$.*

Note that the geodesic lies in $\mathcal{U}_0$ but not necessarily in $\mathcal{W}$.

One also has the following smoothness result. Let $t \mapsto \exp_\varphi(tV_0)\colon [0,1] \to \mathcal{D}^k(\mathbb{S})$ be the unique geodesic in $\mathcal{U}_0$ joining φ and ψ, then the correspondence

$$(\varphi, V_0) \longleftrightarrow (\varphi, \exp_\varphi(V_0))$$

is smooth.

7. Length-minimizing properties of geodesics

In this section we show that geodesics in $\mathcal{D}^k(\mathbb{S})$ are locally length minimizing. We also prove the global converse that any length-minimizing curve in $\mathcal{D}^k(\mathbb{S})$

is a geodesic. Since the H^1 right-invariant metric $\langle\,\cdot\,,\cdot\,\rangle$ on $\mathcal{D}^k(\mathbb{S})$ is only a weak Riemannian metric (i.e. the topology induced by $\langle\,\cdot\,,\cdot\,\rangle_\varphi$ on $T_\varphi\mathcal{D}^k(\mathbb{S})$ is weaker than the original topology on $T_\varphi\mathcal{D}^k(\mathbb{S})\simeq H^k(\mathbb{S})$ – see Section 5), we give the results with full proofs. In essence, however, the proofs are just a repetition of the arguments presented in [21] for the general situation of Riemannian metrics on Banach manifolds.

We first need to establish Gauss' lemma for the H^1 right-invariant metric on $\mathcal{D}^k(\mathbb{S})$.

7.1. Gauss' lemma

Let $J_1, J_2\subset\mathbb{R}$ be open intervals, and let $\sigma\colon J_1\times J_2\to\mathcal{D}^k(\mathbb{S})$ be a C^2-map. A *lift* of σ is a C^1-map $Q\colon J_1\times J_2\to T\mathcal{D}^k(\mathbb{S})$ such that $\pi\circ Q=\sigma$. For each fixed $t\in J_2$ we may form the curve $\sigma^t(r)=\sigma(r,t)$. Let $\partial_1\sigma\colon J_1\times J_2\to T\mathcal{D}^k(\mathbb{S})$ denote the partial derivatives of σ with respect to the first variable, that is,

$$\partial_1\sigma(r,t)=\frac{d\sigma^t}{dr}(r).$$

We define $\partial_2\sigma$ similarly. For a lift Q of σ we let D_1Q be the lift of σ obtained by taking the covariant derivative of Q^t along the curve σ^t. Similarly, we have D_2Q.

Lemma 7.1. *Let $J_1, J_2\subset\mathbb{R}$ be open intervals, and let $\sigma\colon J_1\times J_2\to\mathcal{D}^k(\mathbb{S})$ be a C^2-map. Then*

(1) *$D_1\partial_2\sigma=D_2\partial_1\sigma$ and*

(2) *$\partial_2\langle\partial_1\sigma,\partial_1\sigma\rangle_\sigma=2\langle D_1\partial_2\sigma,\partial_1\sigma\rangle_\sigma$.*

Proof. Locally,

$$D_1\partial_2\sigma=\partial_1\partial_2\sigma-\Gamma_\sigma(\partial_2\sigma,\partial_1\sigma).$$

Symmetry of Γ proves (1). Moreover, since the covariant derivative is compatible with the metric, we have

$$\partial_2\langle\partial_1\sigma,\partial_1\sigma\rangle_\sigma=2\langle D_2\partial_1\sigma,\partial_1\sigma\rangle_\sigma,$$

so that (2) follows from (1). □

For $V\in T_\varphi\mathcal{D}(\mathbb{S})$ we write $\|V\|_\varphi$ for the norm of V, that is, $\|V\|_\varphi=\langle V,V\rangle_\varphi^{1/2}$. Whenever defined, we let the *shell* $\mathrm{Sh}(\varphi,c)\subset\mathcal{D}^k(\mathbb{S})$ be the image of $\{V\in T_\varphi\mathcal{D}^k(\mathbb{S})\colon \|V\|_\varphi=c\}$ under $\exp_\varphi$.

Lemma 7.2. (Gauss' lemma) *Let $\varphi_0\in\mathcal{D}^k(\mathbb{S})$ and let $(\mathcal{V},\mathcal{W})$ be a normal neighborhood of φ_0. Let $\varphi\in\mathcal{W}$. Then the geodesics through φ are orthogonal to $\mathrm{Sh}(\varphi,c)$ for c sufficiently small and positive.*

Proof. For $c>0$ sufficiently small, $\exp_\varphi$ is defined on an open ball in $T_\varphi\mathcal{D}^k(\mathbb{S})$ of radius slightly larger than c. The assertion amounts to proving that for any curve $U\colon J\to T_\varphi\mathcal{D}^k(\mathbb{S})$ with $\|U(t)\|_\varphi=1$ for all $t\in J$, and $0<r<c$, if we define

$$\sigma(r,t)=\exp_\varphi(rU(t)),$$

then the two curves

$$t\longmapsto \exp_\varphi(r_0U(t)) \quad \text{and} \quad r\longmapsto \exp_\varphi(rU(t_0))$$

are orthogonal for any given value (r_0,t_0), that is, we have to show that $\langle\partial_1\sigma,\partial_2\sigma\rangle_\sigma\equiv0$. For each fixed t, $\sigma^t\colon r\mapsto\sigma(r,t)$ is a geodesic. Hence $D_1\partial_1\sigma\equiv0$ and

$$\partial_1\langle\partial_1\sigma,\partial_1\sigma\rangle_\sigma=2\langle D_1\partial_1\sigma,\partial_1\sigma\rangle_\sigma=0,$$

so that the function

$$r\longmapsto\langle\partial_1\sigma(r,t),\partial_1\sigma(r,t)\rangle_{\sigma(r,t)} \tag{7.1}$$

is constant for each t. Since $\partial_1\sigma(0,t)=U(t)$ and $U(t)$ has length 1 we infer that $\langle\partial_1\sigma,\partial_1\sigma\rangle_\sigma\equiv1$. Therefore, using Lemma 7.1,

$$\partial_1\langle\partial_1\sigma,\partial_2\sigma\rangle_\sigma=\langle D_1\partial_1\sigma,\partial_2\sigma\rangle_\sigma+\tfrac12\partial_2\langle\partial_1\sigma,\partial_1\sigma\rangle_\sigma\equiv0.$$

Consequently

$$r\longmapsto\langle\partial_1\sigma(r,t),\partial_2\sigma(r,t)\rangle_{\sigma(r,t)}$$

is a constant function of r for each fixed t. But for $r=0$ we have $\sigma(0,t)=\exp_\varphi(0)=\varphi$ for every t, so that $\partial_2\sigma(0,t)=0$ for all t. We conclude that $\langle\partial_1\sigma,\partial_2\sigma\rangle_\sigma\equiv0$. □

7.2. Length-minimizing geodesics

For a C^1-curve $\gamma\colon J\to\mathcal{D}^k(\mathbb{S})$ we define the length $L(\gamma)$ by

$$L(\gamma)=\int_J\langle\gamma_t(t),\gamma_t(t)\rangle_{\gamma(t)}^{1/2}\,dt.$$

The length of a piecewise C^1-path is defined as the sum of the lengths of its C^1-segments. Let $\varphi_0\in\mathcal{D}^k(\mathbb{S})$ and let $(\mathcal{V},\mathcal{W})$ be a normal neighborhood of φ_0 as in Theorem 6.6. Also, let $\varphi\in\mathcal{W}$ and let $\mathcal{U}(\varphi)$ be a neighborhood containing $\mathcal{W}$ as in the second half of Theorem 6.6. Using that $\exp_\varphi$ is a diffeomorphism $B_\varepsilon(0)\subset T_\varphi\mathcal{D}^k(\mathbb{S})\to\mathcal{U}(\varphi)$, we see that for each piecewise C^1-path $\gamma\colon[a,b]\to\mathcal{U}(\varphi)\setminus\{\varphi\}$

there exists a unique map $t \mapsto U(t)\colon [a,b] \to T_\varphi \mathcal{D}^k(\mathbb{S})$ such that $\|U(t)\|_\varphi = 1$ for $t \in [a,b]$ and

$$\gamma(t) = \exp_\varphi(r(t)U(t)) \quad \text{with } 0 < r(t) < \varepsilon.$$

Since, locally, $r(t)$ and $U(t)$ are obtained by the inverse of the exponential map followed by the smooth maps

$$V \longmapsto \|V\|_\varphi\colon H^k(\mathbb{S}) \longrightarrow \mathbb{R} \quad \text{respectively} \quad V \longmapsto \frac{V}{\|V\|_\varphi}\colon H^k(\mathbb{S}) \longrightarrow H^k(\mathbb{S}),$$

we deduce that r and U are piecewise C^1.

Lemma 7.3. *For a piecewise C^1-curve $\gamma\colon [a,b] \to \mathcal{U}(\varphi) \setminus \{\varphi\}$ as above, we have the inequality*

$$L(\gamma) \geq |r(b) - r(a)|.$$

Equality holds if and only if the function $t \mapsto r(t)$ is monotone and the map $t \mapsto U(t)$ is constant.

Proof. Let $\sigma(r,t) = \exp_\varphi(rU(t))$. Then $\gamma(t) = \sigma(r(t),t)$. We have

$$\gamma'(t) = \partial_1 \sigma(r(t),t) r'(t) + \partial_2 \sigma(r(t),t).$$

By Lemma 7.2, $\partial_1 \sigma$ and $\partial_2 \sigma$ are orthogonal. Repeating the argument leading up to (7.1) we find that $\|\partial_1 \sigma\|_\sigma \equiv 1$. Hence

$$\|\gamma'(t)\|^2 = |r'(t)|^2 + \|\partial_2 \sigma\|^2 \geq |r'(t)|^2,$$

with equality holding if and only if $\partial_2 \sigma = 0$, i.e. if and only if $U'(t) = 0$. Therefore

$$L(\gamma) = \int_a^b \|\gamma'(t)\| dt \geq \int_a^b |r'(t)| dt \geq |r(b) - r(a)|,$$

with equality if and only if $t \mapsto U(t)$ is constant and $t \mapsto r(t)$ is monotone. □

The next theorem says that geodesics in $\mathcal{D}^k(\mathbb{S})$ are locally length minimizing.

Theorem 7.4. *Let $(\mathcal{V}, \mathcal{W})$, $\mathcal{V} = \mathcal{U}_0 \times B_\varepsilon(0)$, constitute a normal neighborhood of an element $\varphi_0 \in \mathcal{D}^k(\mathbb{S})$. Let $\alpha\colon [0,1] \to \mathcal{D}^k(\mathbb{S})$ be the unique geodesic joining two points $\varphi, \psi \in \mathcal{W}$. Then, for any other piecewise C^1-path $\gamma\colon [0,1] \to \mathcal{D}^k(\mathbb{S})$ joining φ and ψ, it holds that*

$$L(\alpha) \leq L(\gamma).$$

If equality holds, then a reparametrization of γ is equal to α.

Proof. Let $\varphi, \psi \in \mathcal{W}$ and let $\psi = \exp_\varphi(rU_0)$ with $0 < r < \varepsilon$, $U_0 \in T_\varphi \mathcal{D}^k(\mathbb{S})$, and $\|U_0\|_\varphi = 1$. For each $\delta > 0$, $0 < \delta < r$, the path γ contains a segment joining the shells $\mathrm{Sh}(\varphi, \delta)$ and $\mathrm{Sh}(\varphi, r)$, and lying between the two shells. By Lemma 7.3 the length of this segment is at least $r - \delta$. Letting $\delta \to 0$ shows that $L(\gamma) \geq r$. Now assume equality holds. The same lemma proves that in this case the polar component $t \mapsto U(t)$ of γ is constant and the radial component $t \mapsto r(t)$ is monotone. Hence we may reparametrize γ so that it becomes a geodesic. Assume this has been done so that $\gamma\colon [0, r] \to \mathcal{D}^k(\mathbb{S})$ is the curve $t \mapsto \exp_\varphi(tV_0)$ with $\exp_\varphi(rV_0) = \psi$, for some $V_0 \in T_\varphi \mathcal{D}^k(\mathbb{S})$ with $\|V_0\|_\varphi = 1$. Since, by Theorem 6.6, $\exp_\varphi$ is a diffeomorphism $B_\varepsilon(0) \to \mathcal{U}(\varphi)$ with $\mathcal{W} \subset \mathcal{U}(\varphi)$, we infer that $V_0 = U_0$. Thus α is equal to γ. □

Conversely, it holds globally that any length-minimizing curve is a geodesic.

Theorem 7.5. *If $\alpha\colon [0, 1] \to \mathcal{D}^k(\mathbb{S})$ is a piecewise C^1-path parametrized by arc-length such that $L(\alpha) \leq L(\gamma)$ for all paths γ in $\mathcal{D}^k(\mathbb{S})$ joining $\alpha(0)$ and $\alpha(1)$, then α is a geodesic.*

Proof. We can find a partition of $[0, 1]$ such that the image under α of each small interval in the partition is contained in some neighborhood $\mathcal{W}$ as in Theorem 7.4. Since α is globally length-minimizing it is also locally length-minimizing. As α was assumed to be parametrized by arc-length, the second half of Theorem 7.4 shows that α is a geodesic on each such small interval. Hence the entire path is a geodesic, as was to be shown. □

8. The Fréchet Lie group $\mathcal{D}(\mathbb{S})$

In this section we comment on the relationship between $\mathcal{D}^k(\mathbb{S})$ and the Fréchet Lie group $\mathcal{D}(\mathbb{S})$ of orientation-preserving smooth diffeomorphisms of the circle. $\mathcal{D}(\mathbb{S})$ is an infinite-dimensional Lie group modeled on the Fréchet space $C^\infty(\mathbb{S})$ (see [9] and [15]). The Lie bracket on the Lie algebra $T_{\mathrm{id}}\mathcal{D}(\mathbb{S}) \simeq C^\infty(\mathbb{S})$ induced from right-invariant vector fields is given by (cf. [10])

$$[u, v] = uv_x - u_x v, \quad u, v \in C^\infty(\mathbb{S}).$$

The fact that $\mathcal{D}(\mathbb{S})$ is a Lie group makes it sometimes easier to establish smoothness of objects for $\mathcal{D}(\mathbb{S})$ than for $\mathcal{D}^k(\mathbb{S})$. For example, we can define our map Γ on $\mathcal{D}(\mathbb{S})$ just like we did for $\mathcal{D}^k(\mathbb{S})$ (see Section 3) by

$$\Gamma_\varphi(U, V) = -\left(A^{-1}\left((U \circ \varphi^{-1})(V \circ \varphi^{-1}) + \tfrac{1}{2}(U \circ \varphi^{-1})_x (V \circ \varphi^{-1})_x\right)_x\right) \circ \varphi,$$

where $\varphi \in \mathcal{D}(\mathbb{S})$, $U, V \in T_\varphi \mathcal{D}(\mathbb{S}) \simeq C^\infty(\mathbb{S})$, and $A = 1 - D_x^2$. In this case, as composition is smooth, it is immediate that Γ is smooth and so defines an affine connection on $\mathcal{D}(\mathbb{S})$. Similarly, the H^1 right-invariant metric

$$\langle U, V\rangle_\varphi = \int_{\mathbb{S}} A(U\circ\varphi^{-1})V\circ\varphi^{-1}dx, \quad U, V\in T_\varphi\mathcal{D}(\mathbb{S})\simeq C^\infty(\mathbb{S}),$$

is trivially smooth([1]). However, as we remarked in the introduction, the absence of general existence and uniqueness results for differential equations makes it involved to study geodesic flow, parallel translation etc. on $\mathcal{D}(\mathbb{S})$. In fact, when studying the geodesic flow on $\mathcal{D}(\mathbb{S})$ one starts with $\mathcal{D}^k(\mathbb{S})$ and in the limit as $k\to\infty$ results are obtained for $\mathcal{D}(\mathbb{S})$ [10]. In contrast, as we saw in Sections 6–7, once the smooth Riemannian structure has been established on $\mathcal{D}^k(\mathbb{S})$, the other aspects of Riemannian geometry come neatly packaged.

In [10] existence of a Riemannian covariant derivative on $\mathcal{D}(\mathbb{S})$ compatible with the H^1 right-invariant metric was proved by means of the following result.

Theorem 8.1. ([10]) *Let $\langle\cdot,\cdot\rangle$ be a right-invariant metric on $\mathcal{D}(\mathbb{S})$. Assume that there exists a bilinear operator $B\colon C^\infty(\mathbb{S})\times C^\infty(\mathbb{S})\to C^\infty(\mathbb{S})$ such that*

$$\langle B(u,v), w\rangle_{\mathrm{id}} = \langle u, [v,w]\rangle_{\mathrm{id}}, \quad u, v, w\in C^\infty(\mathbb{S}). \tag{8.1}$$

Then there exists a unique Riemannian covariant derivative ∇ on $\mathcal{D}(\mathbb{S})$ compatible with $\langle\cdot,\cdot\rangle$, given by

$$(\nabla_X Y)(\varphi) = [X, Y-Y_\varphi^R](\varphi)+\tfrac{1}{2}([X_\varphi^R, Y_\varphi^R](\varphi)-B(X_\varphi^R, Y_\varphi^R)(\varphi)-B(Y_\varphi^R, X_\varphi^R)(\varphi)),$$

where for $X\in\mathfrak{X}(\mathcal{D}(\mathbb{S}))$, we denote by X_φ^R the right-invariant vector field whose value at φ is $X(\varphi)$ and we extend B to a bilinear map on the family $\mathfrak{X}^R(\mathcal{D}(\mathbb{S}))$ of right-invariant vector fields, $B\colon\mathfrak{X}^R(\mathcal{D}(\mathbb{S}))\times\mathfrak{X}^R(\mathcal{D}(\mathbb{S}))\to\mathfrak{X}^R(\mathcal{D}(\mathbb{S}))$ by

$$B(Z,W)(\varphi) = B(Z(\mathrm{id}), W(\mathrm{id}))\circ\varphi \quad \textit{for } \varphi\in\mathcal{D}(\mathbb{S}) \textit{ and } Z, W\in\mathfrak{X}^R(\mathcal{D}(\mathbb{S})).$$

The condition (8.1) is satisfied in the case of the H^1 right-invariant metric by

$$B(u,v) = -A^{-1}\big(2v_x Au + vAu_x\big), \tag{8.2}$$

so that the existence of a compatible Riemannian covariant derivative is established for $\mathcal{D}(\mathbb{S})$.

This approach is not applicable in the case of $\mathcal{D}^k(\mathbb{S})$ as left composition is not smooth. Since $\mathcal{D}^k(\mathbb{S})$ is not a Lie group, we were forced to work directly on $\mathcal{D}^k(\mathbb{S})$ rather than first translate objects to $T_{\mathrm{id}}\mathcal{D}^k(\mathbb{S})$ by means of right invariance.

([1]) Note that when dealing with Fréchet manifolds, smoothness is always defined as Gateaux-smoothness (see the appendix) – the space $L(E,F)$ is in general not a Fréchet space even though E and F are Fréchet spaces, preventing the definition for Banach spaces from generalizing without modification (see [15] and [19]).

To relate the two approaches we will now show that our map Γ is the Christoffel map corresponding to the covariant derivative ∇ obtained in [10]. In fact, if we define a covariant derivative $\overline{\nabla}$ on $\mathcal{D}(\mathbb{S})$ locally, for $X,Y\in\mathfrak{X}(\mathcal{D}(\mathbb{S}))$, by

$$(\overline{\nabla}_X Y)(\varphi)=DY(\varphi)\cdot X(\varphi)-\Gamma_\varphi(Y(\varphi),X(\varphi)),$$

then the same proof that we used to show that Γ gives rise to a Riemannian metric compatible with the H^1 right-invariant metric in the case of $\mathcal{D}^k(\mathbb{S})$ (see Theorem 5.3), works unchanged for $\mathcal{D}(\mathbb{S})$. Hence $\overline{\nabla}$ and ∇ coincide in view of the uniqueness of the Riemannian covariant derivative. This shows that Γ is the Christoffel map for the Riemannian covariant derivative obtained in [10]. We state this as a theorem and also provide a direct proof for the sake of clarity.

Theorem 8.2. *Let ∇ be the Riemannian covariant derivative on $\mathcal{D}(\mathbb{S})$ compatible with the H^1 right-invariant metric derived from Theorem* 8.1 *and formula* (8.2). *Then the map Γ defined by*

$$\Gamma_\varphi(U,V)=-\big(A^{-1}\big((U\circ\varphi^{-1})(V\circ\varphi^{-1})+\tfrac{1}{2}(U\circ\varphi^{-1})_x(V\circ\varphi^{-1})_x\big)_x\big)\circ\varphi,$$

for $\varphi\in\mathcal{D}(\mathbb{S})$ and $U,V\in T_\varphi\mathcal{D}(\mathbb{S})\simeq C^\infty(\mathbb{S})$, is the Christoffel map corresponding to ∇, that is, locally

$$(\nabla_X Y)(\varphi)=DY(\varphi)\cdot X(\varphi)-\Gamma_\varphi(Y(\varphi),X(\varphi)). \tag{8.3}$$

Direct proof. Let $R_\psi\colon \mathcal{D}(\mathbb{S})\to\mathcal{D}(\mathbb{S})$, $R_\psi(\varphi)=\varphi\circ\psi$ be the right multiplication map. Its tangent map is

$$TR_\psi\colon V\in T_\varphi\mathcal{D}(\mathbb{S})\longmapsto V\circ\psi\in T_{\varphi\circ\psi}\mathcal{D}(\mathbb{S}).$$

Thus the right-invariant vector fields X and Y on $\mathcal{D}(\mathbb{S})$ corresponding to two functions $u,v\in T_{\mathrm{id}}\mathcal{D}(\mathbb{S})\simeq C^\infty(\mathbb{S})$ are $X(\varphi)=u\circ\varphi$ and $Y(\varphi)=v\circ\varphi$ for $\varphi\in\mathcal{D}(\mathbb{S})$, respectively. Therefore, for a right-invariant vector field Y, we have

$$DY(\varphi)\cdot X(\varphi)=\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0}Y(\varphi+\varepsilon(u\circ\varphi))=\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0}v\circ(\varphi+\varepsilon(u\circ\varphi))=v_x\circ\varphi\cdot u\circ\varphi. \tag{8.4}$$

Note that, as $(Y-Y_\varphi^R)(\varphi)=0$, locally

$$[X,Y-Y_\varphi^R](\varphi)=DY(\varphi)\cdot X(\varphi)-DY_\varphi^R(\varphi)\cdot X(\varphi).$$

Now fix $\varphi\in\mathcal{D}(\mathbb{S})$ and let $u=X(\varphi)\circ\varphi^{-1}\in C^\infty(\mathbb{S})$ and $v=Y(\varphi)\circ\varphi^{-1}\in C^\infty(\mathbb{S})$. By (8.4) we have

$$DY_\varphi^R(\varphi)\cdot X(\varphi)=(uv_x)\circ\varphi.$$

We infer that locally ∇ as defined in Theorem 8.1 is given by

$$(\nabla_X Y)(\varphi) = DY(\varphi)\cdot X(\varphi) - (uv_x)\circ\varphi + \tfrac{1}{2}([u,v] - B(u,v) - B(v,u))\circ\varphi. \tag{8.5}$$

By (8.2) we have

$$B(u,v)+B(v,u) = -A^{-1}(2v_xAu + vAu_x + 2u_xAv + uAv_x).$$

Using that

$$A(uv)_x = (uv)_x - 3(u_xv_x)_x - uv_{xxx} - u_{xxx}v,$$

a computation shows that

$$B(u,v)+B(v,u) = -(uv)_x - 2A^{-1}\big(uv + \tfrac{1}{2}u_xv_x\big)_x.$$

Since $[u,v] = uv_x - u_xv$ we get

$$\begin{aligned}(\nabla_X Y)(\varphi) &= DY(\varphi)\cdot X(\varphi) - (uv_x)\circ\varphi \\ &\quad + \tfrac{1}{2}\big(uv_x - u_xv + (uv)_x + 2A^{-1}\big(uv + \tfrac{1}{2}u_xv_x\big)_x\big)\circ\varphi.\end{aligned}$$

Thus, recalling that

$$\Gamma_{\mathrm{id}}(u,v) = -A^{-1}\big(uv + \tfrac{1}{2}u_xv_x\big)_x,$$

we arrive by right invariance of Γ at

$$(\nabla_X Y)(\varphi) = DY(\varphi)\cdot X(\varphi) - \Gamma_\varphi(Y(\varphi), X(\varphi)). \quad \square$$

A. Differential calculus in Banach spaces

For Banach spaces $\mathbf{E}$ and $\mathbf{F}$ we let $L(\mathbf{E},\mathbf{F})$ be the Banach space of continuous linear maps $\mathbf{E}\to\mathbf{F}$. For any $k\geq 1$, $L^k(\mathbf{E};\mathbf{F})$ denotes the Banach space of continuous k-multilinear maps $\mathbf{E}\to\mathbf{F}$, and $L^k_{\mathrm{sym}}(\mathbf{E};\mathbf{F})\subset L^k(\mathbf{E};\mathbf{F})$ is the subset of symmetric maps. For a Banach manifold $\mathcal{M}$, $L^k_{\mathrm{sym}}(T\mathcal{M};\mathbb{R})$ denotes the vector bundle over $\mathcal{M}$ with fiber $L^k_{\mathrm{sym}}(T_m\mathcal{M};\mathbb{R})$ over $m\in\mathcal{M}$.

Let $\mathcal{U}$ be an open subset of $\mathbf{E}$. As usual when dealing with Banach spaces a continuous map $f:\mathcal{U}\to\mathbf{F}$ is said to be C^1 if $Df:\mathcal{U}\to L(\mathbf{E},\mathbf{F})$ is continuous. Since $L(\mathbf{E},\mathbf{F})$ is a Banach space we may define D^pf recursively for any $p\geq 1$. If $\mathbf{G}$ is also a Banach spaces and f a map $(\mathcal{U}\times\mathcal{V}\subset\mathbf{E}\times\mathbf{F})\to\mathbf{G}$, we write $D_1f:\mathcal{U}\times\mathcal{V}\to L(\mathbf{E},\mathbf{G})$ for the partial derivative with respect to the first variable.

We will also need a different notion of differentiability. Recall that a Fréchet space is a complete Hausdorff metrizable locally convex topological vector space.

Definition A.1. Let E and F be Fréchet spaces, $\mathcal{U}$ be an open subset of E, and $f:\mathcal{U}\to F$ be a continuous mapping. We say that f is *Gateaux-*C^1 if for each point $x\in\mathcal{U}$ there exists a linear map $Df(x)\colon E\to F$ such that

$$Df(x)v=\lim_{t\to 0}\frac{f(x+tv)-f(x)}{t}$$

for all $v\in E$ and the map

$$(x,v)\longmapsto Df(x)v\colon \mathcal{U}\times E\longrightarrow F \tag{A.1}$$

is continuous (jointly as a function on a subset of the product).

We say that f is Gateaux-C^2 if both f and the map in (A.1) are Gateaux-C^1. The notion of Gateaux-C^p, $p\geq 3$, is defined inductively. See [15] for an extensive presentation of calculus for Gateaux differentiable functions in Fréchet spaces.

Clearly, if f is a map between Banach spaces so that both definitions of differentiability apply, then normal differentiability implies Gateaux differentiability, that is, if f is C^p, $p\geq 1$, then f is Gateaux-C^p. The basic result in the converse direction is the following.

Proposition A.2. *Let* $\mathbf{E}$ *and* $\mathbf{F}$ *be Banach spaces,* $\mathcal{U}$ *be an open subset of* $\mathbf{E}$*, and* $f\colon\mathcal{U}\to\mathbf{F}$ *be a continuous mapping. If* f *is Gateaux-*C^{p+1} *for some* $p\geq 0$*, then* f *is* C^p*. In particular, for smooth maps between Banach spaces the two definitions coincide.*

Proof. We refer to the book by Keller [19, p. 99 (see also the remark on p. 110)]. Note that our Gateaux-C^p maps correspond to the class C^p_c in [19] . □

We also have the following result.

Proposition A.3. ([26, Theorem 5.3]) *Let* $\mathbf{E}$*,* $\mathbf{F}$ *and* $\mathbf{G}$ *be Banach spaces, and let* $\mathcal{U}$ *be an open subset of* $\mathbf{E}$*. Let* f *be a* C^p*-mapping of* $\mathcal{U}\times\mathbf{F}$ *into* $\mathbf{G}$ *such that* $f(x,u)$ *is linear with respect to the second variable* u*. Set* $h(x)u=f(x,u)$ *and regard* h *as a mapping of* $\mathcal{U}$ *into* $L(\mathbf{F},\mathbf{G})$*. Then* h *is a* C^{p-1}*-mapping.*

The typical application in this paper of Propositions A.2 and A.3 is the following consequence. Let $\mathbf{E}$ and $\mathbf{F}$ be Banach spaces, $\mathcal{U}$ be an open subset of $\mathbf{E}$, and $(x,u,v)\mapsto P(x,u,v)\colon\mathcal{U}\times\mathbf{E}\times\mathbf{E}\to\mathbf{F}$ be a continuous mapping linear in u and v. Suppose we can show that P is Gateaux-C^{p+1}. Then P is C^p by Proposition A.2. Hence Proposition A.3 shows that the map

$$(x,u)\longmapsto(v\mapsto P(x,u,v))\colon\mathcal{U}\times\mathbf{E}\longrightarrow L(\mathbf{E},\mathbf{F}),$$

is C^{p-1}. Using that $L(\mathbf{E},L(\mathbf{E},\mathbf{F}))\simeq L^2(\mathbf{E};\mathbf{F})$, another application of Proposition A.3 yields that

$$x\longmapsto((u,v)\mapsto P(x,u,v))\colon\mathcal{U}\times\mathbf{E}\longrightarrow L^2(\mathbf{E};\mathbf{F})$$

is C^{p-2}. The upshot is that if P is Gateaux-smooth then $x \mapsto P_x = P(x,\cdot,\cdot)$ is a smooth map $\mathcal{U} \to L^2(\mathbf{E};\mathbf{F})$.

We also need the following corollary of the inverse function theorem.

Proposition A.4. ([21, Proposition 5.3]) *Let $\mathcal{U}$ and $\mathcal{V}$ be open subsets of Banach spaces and let $f:\mathcal{U}\to\mathcal{V}$ be a C^p-map which is also a C^1-diffeomorphism. Then f is a C^p-diffeomorphism.*

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