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Boundary decay estimates for solutions of fourth-order elliptic equations

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Abstract. We obtain integral boundary decay estimates for solutions of fourth-order elliptic equations on a bounded domain with regular boundary. We apply these estimates to obtain stability bounds for the corresponding eigenvalues under small perturbations of the boundary.

1. Introduction

Let Ω be a bounded region in $\mathbf{R}^N$ and let H be a fourth-order, self-adjoint, uniformly elliptic operator on $L^2(\Omega)$ subject to Dirichlet boundary conditions on $\partial\Omega$,

$$Hu(x) = \sum_{\substack{|\eta|\le 2\\ |\zeta|\le 2}} D^\eta(a_{\eta\zeta}(x) D^\zeta u(x)), \quad x\in\Omega.$$

The scope of this paper is to obtain integral boundary decay estimates for solutions of the equation

$$Hu = f, \quad f\in L^2(\Omega). \tag{1}$$

More precisely, we want to establish ranges of $\beta>0$ for which the integrals

$$\int_\Omega d^{-2-\beta} u^2\,dx \quad \text{and} \quad \int_\Omega d^{-\beta}|\nabla u|^2\,dx$$

are finite (where $d(x)=\operatorname{dist}(x,\partial\Omega)$). If Ω is regular in the sense that the Hardy–Rellich inequality

$$\int_\Omega (\Delta u)^2\,dx \ge c\int_\Omega \left(\frac{|\nabla u|^2}{d^2} + \frac{u^2}{d^4}\right)dx, \quad u\in H_0^2(\Omega),$$

is valid, we then immediately have such an estimate since $H_0^2(\Omega)=\operatorname{Dom}(H^{1/2})$. Our aim is to establish better decay estimates that exploit the fact that the solution u of (1) belongs not only to $H_0^2(\Omega)$ but also to $\operatorname{Dom}(H)$.

This problem is well studied in the case of second-order operators. In [D2] Davies obtained boundary decay estimates of the form

$$(2)\qquad \int_\Omega \left(\frac{|\nabla u|^2}{d^{2\alpha}} + \frac{u^2}{d^{2+2\alpha}} \right) dx \le c(\|Hu\|_2 \|H^{1/2}u\|_2 + \|u\|_2^2), \quad u \in \mathrm{Dom}(H),$$

for $\alpha > 0$ in some interval $(0, \alpha_0)$. Here α_0 is an explicitly given constant which depends on the boundary regularity and the ellipticity constants of H. As an application of (2) stability estimates were obtained on the eigenvalues $\{\lambda_n\}_{n=1}^\infty$ of H under small perturbations of the boundary $\partial\Omega$. More precisely, if $\widetilde{\Omega} \subset \Omega$ is a domain such that $\partial\widetilde{\Omega} \subset \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) < \varepsilon\}$ and if $\{\tilde{\lambda}_n\}_{n=1}^\infty$ are the corresponding Dirichlet eigenvalues (the operator $\widetilde{H}$ being defined by form restriction), then it was shown that (2) implies

$$(3)\qquad 0 \le \tilde{\lambda}_n - \lambda_n \le c_n \varepsilon^{2\alpha}$$

for all $n \in \mathbf{N}$ and all $\varepsilon > 0$ small enough. This estimate has obvious applications in the numerical computation of eigenvalues; see [D2] for more on specific examples.

Inequality (3) was subsequently improved in [D3], where $\varepsilon^{2\alpha}$, $\alpha < \alpha_0$, was replaced by $\varepsilon^{2\alpha_0}$, for the same α_0; this was done by estimating the integrals $\int_{d(x)<\varepsilon} |\nabla u|^2 \, dx$ and $\int_{d(x)<\varepsilon} u^2 \, dx$ for small $\varepsilon > 0$. For results analogous to those of [D3] for the p-Laplacian together with applications we refer to Fleckinger et al. [FHT]. See also [EHK] where estimates of this type were first obtained for eigenfunctions of second-order operators. For relevant results in the case of singular operators see [M].

In our main theorem we obtain integral decay estimates analogous to (2) for fourth-order operators. More precisely, for a fourth-order operator H with L^∞ coefficients we establish boundary decay estimates of the form

$$(4)\quad \int_\Omega \left(\frac{|\nabla^2 u|^2}{d^{2\alpha}} + \frac{|\nabla u|^2}{d^{2+2\alpha}} + \frac{u^2}{d^{4+2\alpha}} \right) dx \le c(\|Hu\|_2 \|H^{\alpha/2}u\|_2 + \|u\|_2^2), \quad u \in \mathrm{Dom}(H),$$

for α in an interval $(0, \alpha_0)$. Under additional assumptions we obtain $\alpha_0 = \frac{1}{2}$, which is optimal. To prove (4) we first use some general inequalities, which lead to a property $(\mathbf{P}_\alpha)$ being identified as sufficient for the validity of (4). We then study property $(\mathbf{P}_\alpha)$, and find sufficient conditions under which it is valid; the distance function used here is taken to be the Finsler distance induced by the operator.

Technical reasons oblige us to make a regularity assumption that is not needed in the second-order case and requires $d(x)$ to be C^2 near $\partial\Omega$. This relates to a recurrent and largely unsolved issue in higher-order problems: a distance function is normally only once differentiable, but is required for technical reasons to be differentiated more than once; see for example [B], where such an issue has arisen in the context of heat kernel estimates.

Finally, as an application of (4) we obtain stability bounds analogous to (3) on the eigenvalues of H under small boundary perturbations.

The structure of the paper is as follows: in Section 2 we provide a sufficient condition ($\mathbf{P}_\alpha$) for the validity of (4); in Section 3 we establish a range of α for which ($\mathbf{P}_\alpha$) is valid for different classes of operators; and in Section 4 we present the application to eigenvalue stability.

Setting

We fix some notation. Given a multi-index $\eta=(\eta_1,...,\eta_N)$ we write $\eta!=\eta_1!...\eta_N!$ and $|\eta|=\eta_1+...+\eta_N$. We write $\gamma\leq\eta$ if $\gamma_i\leq\eta_i$ for all i, in which case we also set $c^\eta_\gamma=\eta!/\gamma!(\eta-\gamma)!$. We use the standard notation $D^\eta u=(\partial/\partial x_1)^{\eta_1}...(\partial/\partial x_N)^{\eta_N}u$ and $(\nabla u)^\eta=u^{\eta_1}_{x_1}...u^{\eta_N}_{x_N}$. By $\nabla^2 u$ we denote the vector $(u_{x_ix_j})^N_{i,j=1}$. The letter c will denote a constant whose value may change from line to line; the constants c_1, c_2 and c_3 however are the same throughout the paper.

We now describe our setting. We assume that Ω is a bounded domain in $\mathbf{R}^N$ with boundary $\partial\Omega$. We consider a distance function $d(\,\cdot\,,\cdot\,)$ on Ω, and denote by $d(\,\cdot\,)$ the corresponding distance to the boundary $\partial\Omega$. We say that $d(\,\cdot\,)$ belongs to the class $\mathcal{D}$ if it satisfies:

(D1) There exist $c_1, c_2>0$ such that for any $x,y\in\Omega$,

$$\text{(a)}\qquad c_1\leq|\nabla_z d(z,y)|\leq c_2, \qquad z\in\Omega,$$

$$\text{(b)}\qquad c_1 d_{\mathrm{Euc}}(x,y)\leq d(x,y)\leq c_2 d_{\mathrm{Euc}}(x,y).$$

(D2) There exist $\theta,\tau>0$ such that

$$\text{(a)}\qquad d(x) \text{ is } C^2 \qquad \text{on } \{x\in\Omega: 0<d(x)<\theta\},$$

$$\text{(b)}\qquad |\nabla^2 d(x)|\leq c d(x)^{-1+\tau} \qquad \text{on } \{x\in\Omega: 0<d(x)<\theta\}.$$

(D3) The following Hardy–Rellich inequalities are valid for all $v\in C^2_c(\Omega)$:

$$\text{(a)}\qquad \int_\Omega (\Delta v)^2\,dx\geq c_3\int_\Omega \frac{v^2}{d^4}\,dx,$$

$$\text{(b)}\qquad \int_\Omega (\Delta v)^2\,dx\geq c_3\int_\Omega \frac{|\nabla v|^2}{d^2}\,dx.$$

We note that a sufficient condition for (D3)(b) is the Hardy inequality

$$\int_\Omega |\nabla v|^2\,dx\geq c_3\int_\Omega \frac{v^2}{d^2}\,dx.$$

The distance $d(\cdot,\cdot)$ will typically be a Finsler distance, in which case (a) and (b) of (D1) are equivalent. Condition (D2) is essentially a strong regularity assumption on $\partial\Omega$, as will be seen below. Its validity in examples will always involve the specific value $\tau=1$; we choose however this more general and somewhat axiomatic setting because, we believe, it shows more clearly what the essential ingredients are. Finally, for more on Hardy–Rellich inequalities, optimal constants as well as improved versions of such inequalities we refer to [BFT] and [BT] and references therein.

In the sequel we shall often need to twice differentiate $d(x)$ near $\partial\Omega$. In order to avoid repeatedly splitting integrals in two, we redefine $d(x)$ on $\{x\in\Omega:d(x)>\theta\}$ so that now $d(x)$ is a positive C^2 function on Ω which equals $\inf\{d(x,y):y\in\partial\Omega\}$ for $x\in\{x\in\Omega:\mathrm{dist}(x,\partial\Omega)<\theta\}$ but not necessarily for all $x\in\Omega$ (of course $d(x,\cdot)$ extends to $\partial\Omega$ by uniform continuity). In relation to this we emphasize that throughout the paper what really matters is what happens near the boundary $\partial\Omega$. We also note that, while the validity of estimate (4) and assumption (D3) is independent of the specific distance $d(\cdot)\in\mathcal{D}$ chosen, we shall need to consider non-Euclidean distances since some of the intermediate calculations do depend on the specific choice of the distance.

We will consider operators of the form

$$Hu(x)=\sum_{\substack{|\eta|=2\\|\zeta|=2}} D^\eta(a_{\eta\zeta}(x)D^\zeta u(x)), \quad x\in\Omega, \tag{5}$$

subject to Dirichlet boundary conditions on $\partial\Omega$; lower-order terms can be easily accomodated. More precisely, we start with a matrix-valued function $a(x)=\{a_{\eta\zeta}(x)\}$ which is assumed to be have entries in $L^\infty(\Omega)$ and to take its values in the set of all real, $\frac{1}{2}N(N+1)\times\frac{1}{2}N(N+1)$ matrices ($\frac{1}{2}N(N+1)$ is the number of multi-indices η of length $|\eta|=2$). We assume that $\{a_{\eta\zeta}(x)\}$ is symmetric for all $x\in\Omega$ and define a quadratic form $Q(\cdot)$ on the Sobolev space $H^2_0(\Omega)$ by

$$Q(u)=\int_\Omega \sum_{\substack{|\eta|=2\\|\zeta|=2}} a_{\eta\zeta}(x)D^\eta u(x)D^\zeta u(x)\,dx, \quad u\in H^2_0(\Omega).$$

We make the ellipticity assumption that there exist $\lambda,\Lambda>0$ such that

$$\lambda Q_0(u)\le Q(u)\le \Lambda Q_0(u), \quad u\in H^2_0(\Omega),$$

where $Q_0(u)=\int_\Omega(\Delta u)^2\,dx$ denotes the quadratic form corresponding to the bilaplacian Δ^2. We then define H to be the associated self-adjoint operator on $L^2(\Omega)$, so that $\langle Hu,u\rangle=Q(u)$ for all $u\in\mathrm{Dom}(H)$.

2. Boundary decay

Let $d(\cdot)\in\mathcal{D}$. Let $\alpha>0$ be fixed and let us define

$$\omega(x)=d(x)^{-\alpha},\quad x\in\Omega.$$

We regularize ω defining

$$d_n(x)=d(x)+\frac{1}{n},\ \omega_n(x)=d_n(x)^{-\alpha},\quad n=1,2,.... \tag{6}$$

We note that $u\in H_0^2(\Omega)$ implies $\omega_n u\in H_0^2(\Omega)$, $n\in\mathbf{N}$. It is crucial for the estimates which follow that, while they contain the functions d_n and ω_n, they involve constants that are independent of $n\in\mathbf{N}$.

Lemma 1. *Let $\alpha>0$. There exists a constant c which is independent of $n\in\mathbf{N}$ such that*

$$\int_\Omega\left(\frac{|\nabla^2 u|^2}{d_n^{2\alpha}}+\frac{|\nabla u|^2}{d_n^{2+2\alpha}}+\frac{u^2}{d_n^{4+2\alpha}}\right)dx\le cQ(\omega_n u),\quad u\in H_0^2(\Omega). \tag{7}$$

Proof. It suffices to prove (7) for all $u\in C_c^2(\Omega)$. So let $u\in C_c^2(\Omega)$ be given and let $v=\omega_n u$, a function also in $C_c^2(\Omega)$. Using (D3) we have

$$\int_\Omega\frac{u^2}{d_n^{4+2\alpha}}\,dx=\int_\Omega\frac{v^2}{d_n^4}\,dx\le\int_\Omega\frac{v^2}{d^4}\,dx\le c\int_\Omega(\Delta v)^2\,dx.$$

Similarly,

$$\begin{aligned}\int_\Omega\frac{|\nabla u|^2}{d_n^{2+2\alpha}}\,dx&=\int_\Omega\frac{1}{d_n^{2+2\alpha}}\left|\alpha d_n^{\alpha-1}v\nabla d_n+d_n^\alpha\nabla v\right|^2dx\\&\le c\int_\Omega\frac{v^2}{d_n^4}\,dx+c\int_\Omega\frac{|\nabla v|^2}{d_n^2}\,dx\le c\int_\Omega(\Delta v)^2\,dx,\end{aligned}$$

where we have used the fact that

$$\int_\Omega|\nabla^2 v|^2\,dx=\int_\Omega(\Delta v)^2\,dx. \tag{8}$$

Finally, since d and d_n differ by a constant,

$$u_{x_ix_j}=d_n^\alpha v_{x_ix_j}+\alpha d_n^{\alpha-1}(d_{x_i}v_{x_j}+d_{x_j}v_{x_i})+\alpha d_n^{\alpha-1}d_{x_ix_j}v+\alpha(\alpha-1)d_n^{\alpha-2}d_{x_i}d_{x_j}v,$$

and therefore

$$\int_\Omega\frac{|\nabla^2 u|^2}{d_n^{2\alpha}}\,dx\le c\left(\int_\Omega|\nabla^2 v|^2\,dx+\int_\Omega\frac{|\nabla v|^2}{d_n^2}\,dx+\int_\Omega\frac{v^2}{d_n^4}\,dx+\int_\Omega\frac{|\nabla^2 d|^2}{d_n^2}v^2\,dx\right).$$

Since $d_n\ge d$, the second and third terms in the brackets are smaller than $c\int_\Omega(\Delta v)^2\,dx$ by the Hardy–Rellich inequalities (D3). The same is true for the last term by (D2). Thus, one more application of (8) concludes the proof. $\square$

Lemma 2. *Let $\alpha\in(0,1)$ and $\omega_n=d_n^{-\alpha}$. Then there exists a constant $c>0$, independent of $n\in\mathbf{N}$, such that*

$$Q(u,\omega_n^2 u)\le c\|Hu\|_2\|H^{\alpha/2}u\|_2, \quad u\in \mathrm{Dom}(H).$$

Proof. For any $n\in\mathbf{N}$ and $u\in C_c^2(\Omega)$ we have

$$\int_\Omega \omega_n^{4/\alpha}u^2\,dx\le\int_\Omega \omega^{4/\alpha}u^2\,dx=\int_\Omega \frac{u^2}{d^4}\,dx\le cQ(u).$$

Hence $\omega_n^{4/\alpha}\le cH$ in the quadratic form sense, which by [D1, Lemma 4.20] implies that $\omega_n^4\le cH^\alpha$ (since $\alpha\in(0,1)$). Hence given $u\in\mathrm{Dom}(H)$ we have

$$Q\big(u,\omega_n^2u\big)\le\|Hu\|_2\big\|\omega_n^2u\big\|_2\le c\|Hu\|_2\|H^{\alpha/2}u\|_2,$$

which is the stated inequality. □

We can now establish a sufficient condition for the boundary decay estimates.

Theorem 3. *Let $\alpha\in(0,1)$ be fixed and let $\omega_n=d_n^{-\alpha}$. Assume that there exist $k,k'>0$ independent of $n\in\mathbf{N}$ such that*

$$Q(\omega_n u)\le kQ\big(u,\omega_n^2u\big)+k'\|u\|_2^2, \quad u\in C_c^2(\Omega), \tag{9}$$

for all $n\in\mathbf{N}$. Then there exists $c>0$ such that

$$\int_\Omega\left(\frac{|\nabla^2u|^2}{d^{2\alpha}}+\frac{|\nabla u|^2}{d^{2+2\alpha}}+\frac{u^2}{d^{4+2\alpha}}\right)dx\le c\|Hu\|_2\|H^{\alpha/2}u\|_2, \quad u\in\mathrm{Dom}(H). \tag{10}$$

Proof. The validity of (9) for all $u\in C_c^2(\Omega)$ implies its validity for all $u\in H_0^2(\Omega)$ and in particular for all $u\in\mathrm{Dom}(H)$. Hence given $u\in\mathrm{Dom}(H)$ and applying Lemmas 1 and 2 we conclude that there exists a constant c such that for any $n\in\mathbf{N}$ there holds

$$\int_\Omega\left(\frac{|\nabla^2u|^2}{d_n^{2\alpha}}+\frac{|\nabla u|^2}{d_n^{2+2\alpha}}+\frac{u^2}{d_n^{4+2\alpha}}\right)dx\le c\big(\|Hu\|_2\|H^{\alpha/2}u\|_2+\|u\|_2^2\big).$$

Letting $n\to+\infty$, applying the dominated convergence theorem and using the fact that the spectrum of H is bounded away from zero we obtain (10). □

3. The property $(\mathbf{P}_\alpha)$

The validity of assumption (9) of Theorem 3 will be our main interest in this section. For the sake of simplicity, for any $\alpha\in(0,1)$ we define the property $(\mathbf{P}_\alpha)$ (relative to the distance function $d\in\mathcal{D}$) as

$$(\mathbf{P}_\alpha)\qquad \begin{cases} \text{There exists constants } k,k'>0 \text{ such that} \\ Q\big(d_n^{-\alpha}u\big)\le kQ\big(u,d_n^{-2\alpha}u\big)+k'\|u\|_2^2 \\ \text{for all } n\in\mathbf{N} \text{ and } u\in C_c^2(\Omega). \end{cases}$$

This is precisely assumption (9) of Theorem 3. Our aim in this section is to obtain sufficient conditions under which property $(\mathbf{P}_\alpha)$ is valid. In the following three subsections we present three theorems that provide such conditions. The first applies to all operators in the class under consideration; the second applies to operators of a specific type but gives a better range of $\alpha>0$; and the third applies to small perturbations of operators in the second class.

Remark. If $\partial\Omega$ is smooth then the ground state ϕ of Δ^2 decays as $d(x)^2$ as $x\to\partial\Omega$. Hence the integral in the left-hand side of (10) is not finite for $\alpha\ge\frac{1}{2}$. For this reason and throughout the rest of the paper we restrict our attention to $\alpha\in(0,\frac{1}{2})$.

3.1. General operators

We always work in the context described at the beginning of Section 2. We recall that for $\alpha\in(0,\frac{1}{2})$ we have $\omega_n=d_n^{-\alpha}=(d+1/n)^{-\alpha}$; we also recall that λ and Λ are the ellipticity constants of the operator H.

Theorem 4. *There exists a computable constant $c>0$ such that property $(\mathbf{P}_\alpha)$ relative to the Euclidean distance is valid for H for all $\alpha\in(0,c^{-1}\Lambda^{-1}\lambda)$.*

Proof. Let $u\in C_c^2(\Omega)$ be fixed. Setting $v=d_n^{-\alpha}u$ and using Leibniz' rule we have

$$\begin{aligned} Q(d_n^{-\alpha}u)-Q(u,d_n^{-2\alpha}u) &= Q(v)-Q(d_n^{\alpha}v,d_n^{-\alpha}v) \\ &= \int_\Omega \sum_{\substack{|\eta|=2\\|\zeta|=2}} a_{\eta\zeta}\big(D^\eta v D^\zeta v-D^\eta\big(d_n^\alpha v\big)D^\zeta\big(d_n^{-\alpha}v\big)\big)\,dx \end{aligned}$$

$$= -\int_\Omega \sum_{\substack{|\eta|=2\\|\zeta|=2}} \sum_{\substack{\gamma\le\eta\\\delta\le\zeta\\\gamma+\delta>0}} c_\gamma^\eta c_\delta^\zeta a_{\eta\zeta} \big(D^\gamma d_n^\alpha\big)\big(D^\delta d_n^{-\alpha}\big)(D^{\eta-\gamma}v)(D^{\zeta-\delta}v)\,dx$$

$$\le c\Lambda \int \sum_{\substack{0\le i,j\le 2\\i+j>0}} \big|\nabla^i d_n^\alpha\big|\,\big|\nabla^j d_n^{-\alpha}\big|\,|\nabla^{2-i}v|\,|\nabla^{2-j}v|\,dx.$$

But, by (D1) and (D2),

$$\big|\nabla d_n^{\pm\alpha}\big| = \alpha d_n^{\pm\alpha-1} \quad \text{and} \quad \big|\nabla^2 d_n^{\pm\alpha}\big| \le c\alpha d_n^{\pm\alpha-2},$$

and we thus obtain

$$Q(v) - Q\big(d_n^\alpha v, d_n^{-\alpha} v\big) \le c\Lambda\alpha \int_\Omega \left(|\nabla^2 v|^2 + \frac{|\nabla v|^2}{d_n^2} + \frac{v^2}{d_n^4}\right) dx \le c\Lambda\lambda^{-1}\alpha Q(v).$$

Hence, if α is such that $c\Lambda\lambda^{-1}\alpha<1$, then property $(\mathbf{P}_\alpha)$ is valid for H. □

3.2. Regular coefficients

The weak point of Theorem 4 is the poor information it provides on the range of α for which $(\mathbf{P}_\alpha)$ is valid. In this subsection we shall consider operators of a more specific type and for which we shall see that $(\mathbf{P}_\alpha)$ is valid for all $\alpha\in(0,\frac{1}{2})$.

It will be useful in this subsection to drop the multi-index notation and write the quadratic form as

$$Q(u) = \int_\Omega \sum_{i,j,k,l=1}^N a_{ijkl} u_{x_i x_j} u_{x_k x_l}\,dx, \quad u\in H_0^2(\Omega).$$

We may clearly assume that the functions a_{ijkl} have the following symmetries:

$$a_{ijkl} = a_{jikl}, a_{ijlk}, a_{klij}. \tag{11}$$

We make the following additional assumptions on the coefficients $\{a_{ijkl}\}$:

(i) There exist $\theta, \tau>0$ such that

(12.a) each a_{ijkl} is differentiable in $\{x\in\Omega : d(x)<\theta\}$

(12.b) $|\nabla a_{ijkl}| \le c d^{-1+\tau}$ on $\{x\in\Omega : d(x)<\theta\}$

(ii)

$$\sum_{i,j,k,l=1}^N a_{ijkl}(x)\xi_i\xi_k\eta_j\eta_l \le \sum_{i,j,k,l=1}^N a_{ijkl}(x)\xi_i\xi_j\eta_k\eta_l, \quad \xi, \eta\in\mathbf{R}^N,\ x\in\Omega. \tag{12.c}$$

Without any loss of generality we assume that τ in (i) is the same as in (D2). Condition (ii) is a technical one, whose necessity is not clear. We present two examples in which it is valid.

Example 1. Suppose that $a_{ijkl}=b_{ij}b_{kl}$ for some non-negative $N\times N$ matrix $\{b_{ij}\}_{i,j}$. Then (ii) is valid by the Cauchy-Schwarz inequality for the non-negative form $(\xi,\eta)\mapsto b_{ij}\xi_i\eta_j$. This for example includes operators of the form $\Delta a(x)\Delta$, for which we have $a_{ijkl}=a(x)\delta_{ij}\delta_{kl}$.

Example 2. Suppose that $a_{ijkl}=\delta_{ij}\delta_{kl}a_{ik}$, where $a_{ik}=a_{ki}\geq 0$ for $i,k=1,...,N$. Then it is easily seen that (ii) is again valid.

We choose the distance function $d(\,\cdot\,)$ to be the one naturally associated with H, that is the one induced by the Finsler metric $p(x,\eta)$ whose dual metric (cf. (15) below) is

$$p_*(x,\xi)=\Bigg(\sum_{i,j,k,l=1}^{N} a_{ijkl}(x)\xi_i\xi_j\xi_k\xi_l\Bigg)^{1/4}. \tag{13}$$

This implies in particular that the function $d(\,\cdot\,)$ satisfies

$$\sum_{i,j,k,l=1}^{N} a_{ijkl}(x)d_{x_i}d_{x_j}d_{x_k}d_{x_l}=1, \quad \text{a.e. } x\in\Omega. \tag{14}$$

Indeed, the inequality $\sum_{i,j,k,l=1}^{N} a_{ijkl}(x)d_{x_i}d_{x_j}d_{x_k}d_{x_l}\leq 1$ is shown in [A, Lemma 1.3]. To prove the converse inequality let y denote a point of differentiability of d. Then y has a unique nearest point $y_0\in\partial\Omega$; so $d(y)=d(y,y_0)=:s$. Let y_t, $t\in[0,s]$, be the geodesic joining y_0 and y parametrised by arc length so that $y_s=y$. Then for small $\varepsilon>0$ we have on the one hand

$$d(y_{s-\varepsilon})-d(y)=d(y,y_{s-\varepsilon})=p(y,y-y_{s-\varepsilon})+o(\varepsilon),$$

and on the other hand, by differentiability,

$$d(y_{s-\varepsilon})-d(y)=\nabla d(y)\cdot(y_{s-\varepsilon}-y)+o(\varepsilon).$$

Hence

$$p_*(y,\nabla d(y))=\sup_{\xi\in\mathbf{R}^N}\frac{\nabla d(y)\cdot\xi}{p(y,\xi)}\geq\lim_{\varepsilon\searrow 0}\frac{\nabla d(y)\cdot(y_{s-\varepsilon}-y)}{p(y,y-y_{s-\varepsilon})}=1. \tag{15}$$

We note that the metric is Riemannian if the symbol of the operator H is the square of a polynomial of degree two.

We assume that our basic hypotheses (D1)–(D3) of the introduction are valid; Concerning in particular the validity of condition (D2), we note that it is satisfied if enough regularity is imposed on the boundary and the coefficients. If for example the boundary is C^3 and the coefficients a_{ijkl} lie in $C^3(\{x\in\overline{\Omega}:0\le d(x)<\theta\})$, then $d\in C^2(\{x\in\overline{\Omega}:0\le d(x)<\theta\})$; see [LN, Section 1.3]. On the other hand, for the Euclidean distance a C^2 boundary is enough [GT, p. 354].

It is useful to introduce at this point a class $\mathcal{A}$ of integrals that are in a sense negligible.

Definition. A family of quadratic integral forms $T_n(v)$, $v\in C_c^2(\Omega)$, $n\in\mathbf{N}$, belongs to the class $\mathcal{A}$ if for any $\varepsilon>0$ there exists $c_\varepsilon>0$ (independent of $n\in\mathbf{N}$) such that

$$|T_n(v)|\le\varepsilon Q(v)+c_\varepsilon\int_\Omega v^2\,dx,\quad n\in\mathbf{N},\ v\in C_c^2(\Omega). \tag{16}$$

Lemma 5. *Let $I_n(v)=\int_\Omega b_n(D^\gamma v)(D^\delta v)\,dx$, $|\gamma|,|\delta|\le 2$, be a term that results after expanding $Q(d_n^\alpha v,d_n^{-\alpha}v)$ and integrating by parts a number of times. If b_n contains as a factor either a derivative of $a_{\eta\zeta}$ or a second-order derivative of d_n, then $\{I_n\}_{n=1}^\infty\in\mathcal{A}$.*

Proof. After expanding $Q(d_n^\alpha v,d_n^{-\alpha}v)$ (cf. (19) below) we obtain a linear combination of integrals, and direct observation shows that each one of them has one of the following three forms (we switch temporarily to multi-index notation):

(a) $\displaystyle\int_\Omega a_{\eta\zeta}d_n^{-4+|\gamma+\delta|}(\nabla d_n)^{\eta+\zeta-\gamma-\delta}(D^\gamma v)(D^\delta v)\,dx,\qquad |\eta|=|\zeta|=2,\ \gamma\le\eta,\ \delta\le\zeta,$

(b) $\displaystyle\int_\Omega a_{\eta\zeta}d_n^{-3+|\gamma|}(\nabla d_n)^{\eta-\gamma}(D^\zeta d_n)v(D^\gamma v)\,dx,\qquad |\eta|=|\zeta|=2,\ \gamma\le\eta,$

(c) $\displaystyle\int_\Omega a_{\eta\zeta}d_n^{-2}(D^\eta d_n)(D^\zeta d_n)v^2\,dx,\qquad |\eta|=|\zeta|=2.$

(These are distinguished by the number of second-order derivatives of d_n that they contain – none, one and two, respectively.) Hence all resulting integrals have the form

$$\int_\Omega b_n(x)(D^\gamma v)(D^\delta v)\,dx,\quad 0\le|\gamma|,|\delta|\le 2,$$

where $b_n(x)$ is a product of $a_{\eta\zeta}$ with powers and/or derivatives of d_n and, since ∇d_n is bounded,

$$|b_n(x)|\le c d_n(x)^{-4+|\gamma+\delta|},\quad x\in\Omega. \tag{17}$$

In cases (b) and (c) however, where $b_n(x)$ contains as a factor at least one second-order derivative of d_n, it follows from condition (D2) of the introduction that we have something more, namely

$$|b_n(x)| \le c d_n(x)^{-4+|\gamma+\delta|+\tau}, \quad x \in \Omega.$$

This easily implies that the integral lies in $\mathcal{A}$ in this case.

Suppose now that we integrate by parts in the integral (a) above, transferring one derivative from, say, $D^\gamma v$, ($|\gamma| \ge 1$), to the remaining functions. If the derivative being transfered is $\partial/\partial x_i$, we obtain – in an obvious notation – the integral

$$\int_\Omega [a_{\eta\zeta} d_n^{-4+|\gamma+\delta|} (\nabla d_n)^{\eta+\zeta-\gamma-\delta} (D^\delta v)]_{x_i} (D^{\gamma-e_i} v)\, dx.$$

If the derivative $\partial/\partial x_i$ "hits" either $a_{\eta\zeta}$ or one of the factors that make up $(\nabla d_n)^{\eta+\zeta-\gamma-\delta}$ we obtain an integral of the form

$$\int_\Omega b_n(x) (D^{\gamma-e_i} v)(D^\delta v)\, dx,$$

where $|b_n| \le c d_n^{-1+\tau} d_n^{-4+|\gamma+\delta|}$; hence this integral belongs to $\mathcal{A}$. $\square$

Example. We illustrate the last lemma with an example: in (21) below there appears the integral

$$I_n(v) = \int_\Omega a_{ijkl} d_n^{-2} d_{x_i} d_{x_j} v v_{x_k x_l}\, dx.$$

Letting $\{T_n\}_{n=1}^\infty, \{T'_n\}_{n=1}^\infty$ denote elements in $\mathcal{A}$ we compute

$$\begin{aligned}
&\int_\Omega a_{ijkl} d_n^{-2} d_{x_i} d_{x_j} v v_{x_k x_l}\, dx \\
&\quad = -\int_\Omega (a_{ijkl})_{x_k} d_n^{-2} d_{x_i} d_{x_j} v v_{x_l}\, dx + 2\int_\Omega a_{ijkl} d_n^{-3} d_{x_i} d_{x_j} d_{x_k} v v_{x_l}\, dx \\
&\qquad - \int_\Omega a_{ijkl} d_n^{-2} d_{x_i x_k} d_{x_j} v v_{x_l}\, dx - \int_\Omega a_{ijkl} d_n^{-2} d_{x_i} d_{x_j x_k} v v_{x_k}\, dx \\
&\qquad - \int_\Omega a_{ijkl} d_n^{-2} d_{x_i} d_{x_j} v_{x_k} v_{x_l}\, dx \\
&\quad = \int_\Omega a_{ijkl} d_n^{-3} d_{x_i} d_{x_j} d_{x_k} (v^2)_{x_l}\, dx - \int_\Omega a_{ijkl} d_n^{-2} d_{x_i} d_{x_j} v_{x_k} v_{x_l}\, dx + T_n(v) \\
&\quad = 3\int_\Omega a_{ijkl} d_n^{-4} d_{x_i} d_{x_j} d_{x_k} d_{x_l} v^2\, dx - \int_\Omega a_{ijkl} d_n^{-2} d_{x_i} d_{x_j} v_{x_k} v_{x_l}\, dx + T'_n(v).
\end{aligned}$$

Note. The summation convention over repeated indices will be used from now on.

Lemma 6. *There exists* $\{T_n\}_{n=1}^{\infty}\in\mathcal{A}$ *such that*

$$Q(v)-Q\big(d_n^{\alpha}v,d_n^{-\alpha}v\big)=2\alpha^2\int_{\Omega}a_{ijkl}d_n^{-2}d_{x_i}d_{x_j}v_{x_k}v_{x_l}\,dx +4\alpha^2\int_{\Omega}a_{ijkl}d_n^{-2}d_{x_i}d_{x_k}v_{x_j}v_{x_l}\,dx -(\alpha^4+11\alpha^2)\int_{\Omega}d_n^{-4}v^2\,dx+T_n(v) \tag{18}$$

for all $v\in C_c^2(\Omega)$.

Proof. For $\beta=\alpha$ and $\beta=-\alpha$ we have

$$(d_n^{\beta}v)_{x_ix_j}=d_n^{\beta}v_{x_ix_j}+\beta d_n^{\beta-1}d_{x_i}v_{x_j}+\beta d_n^{\beta-1}d_{x_j}v_{x_i} +\beta(\beta-1)d_n^{\beta-2}d_{x_i}d_{x_j}v+\beta d_n^{\beta-1}d_{x_ix_j}v. \tag{19}$$

We substitute in $Q(d_n^{\alpha}v,d_n^{-\alpha}v)$ and expand. Now, by Lemma 5 all terms containing second-order derivatives of d_n belong to $\mathcal{A}$. Further, the symmetries (11) of a_{ijkl} give

$$\begin{aligned} a_{ijkl}d_{x_i}d_{x_j}d_{x_k}v_{x_l}&=a_{ijkl}d_{x_i}d_{x_k}d_{x_l}v_{x_j}=\ldots,\\ a_{ijkl}d_{x_i}d_{x_j}v_{x_kx_l}&=a_{ijkl}d_{x_k}d_{x_l}v_{x_ix_j}=\ldots,\\ a_{ijkl}d_{x_i}d_{x_l}v_{x_j}v_{x_k}&=a_{ijkl}d_{x_i}d_{x_k}v_{x_j}v_{x_l}=\ldots. \end{aligned} \tag{20}$$

Denoting by $\{T_n\}_{n=1}^{\infty}$ an element of $\mathcal{A}$ which may change within the proof we thus arrive at

$$Q\big(d_n^{\alpha}v,d_n^{-\alpha}v\big)=\int_{\Omega}a_{ijkl}[v_{x_ix_j}v_{x_kx_l}+2\alpha^2d_n^{-2}d_{x_i}d_{x_j}vv_{x_kx_l}-4\alpha^2d_n^{-2}d_{x_i}d_{x_k}v_{x_j}v_{x_l} +4\alpha^2d_n^{-3}d_{x_i}d_{x_j}d_{x_k}vv_{x_l} +\alpha^2(\alpha^2-1)d_n^{-4}d_{x_i}d_{x_j}d_{x_k}d_{x_l}v^2]\,dx+T_n(v). \tag{21}$$

We integrate by parts the second and fourth terms in the last integral. By Lemma 5, all terms that contain either derivatives of a_{ijkl} or second-order derivatives of d_n, belong to $\mathcal{A}$. Hence, denoting always by $\{T_n\}_{n=1}^{\infty}$ a generic element of $\mathcal{A}$ we obtain (cf. the example above)

$$\int_{\Omega}a_{ijkl}d_n^{-2}d_{x_i}d_{x_j}vv_{x_kx_l}\,dx=3\int_{\Omega}a_{ijkl}d_n^{-4}d_{x_i}d_{x_j}d_{x_k}d_{x_l}v^2\,dx -\int_{\Omega}a_{ijkl}d_n^{-2}d_{x_i}d_{x_j}v_{x_k}v_{x_l}\,dx+T_n(v)$$

and, similarly,

$$\int_\Omega a_{ijkl} d_n^{-3} d_{x_i} d_{x_j} d_{x_k} v v_{x_l}\, dx = \frac{3}{2} \int_\Omega a_{ijkl} d_n^{-4} d_{x_i} d_{x_j} d_{x_k} d_{x_l} v^2\, dx + T_n(v).$$

Substituting in (21) yields

$$\begin{aligned} Q\big(d_n^\alpha v, d_n^{-\alpha} v\big) = \int_\Omega a_{ijkl} [& v_{x_i x_j} v_{x_k x_l} - 4\alpha^2 d_n^{-2} d_{x_i} d_{x_k} v_{x_j} v_{x_l} \\ & - 2\alpha^2 d_n^{-2} d_{x_i} d_{x_j} v_{x_k} v_{x_l} \\ & + (\alpha^4 + 11\alpha^2) d_n^{-4} d_{x_i} d_{x_j} d_{x_k} d_{x_l} v^2]\, dx + T_n(v). \end{aligned}$$

Recalling that (14) holds, relation (18) follows. □

Lemma 7. *Let $v \in C_c^2(\Omega)$ and $w = d_n^{-3/2} v$. Then*

$$\begin{aligned} Q(v) - Q(d_n^\alpha v, d_n^{-\alpha} v) = 2\alpha^2 & \int_\Omega a_{ijkl} d_n d_{x_i} d_{x_j} w_{x_k} w_{x_l}\, dx \\ & + 4\alpha^2 \int_\Omega a_{ijkl} d_n d_{x_i} d_{x_k} w_{x_j} w_{x_l}\, dx \\ & + \left(-\alpha^4 + \frac{5\alpha^2}{2}\right) \int_\Omega d_n^{-1} w^2\, dx + T_n(v), \end{aligned}$$

where $\{T_n\}_{n=1}^\infty \in \mathcal{A}$.

Remark 1. When working with the function w, all integrals have the form

$$\int_\Omega b_n(x) (D^\gamma w)(D^\delta w)\, dx$$

where the function b_n satisfies

$$(22) \qquad |b_n(x)| \le c d_n(x)^{-1+|\gamma+\delta|}, \quad x \in \Omega.$$

Such an integral lies in $\mathcal{A}$ if in addition

$$|b_n(x)| \le c d_n(x)^{-1+|\gamma+\delta|+\tau}, \quad x \in \Omega,$$

for some $\tau > 0$; as before, these are precisely the integrals that contain either second-order derivatives of d_n or (first-order) derivatives of a_{ijkl}.

Remark 2. Since d and d_n differ by a constant, for the sake of simplicity we shall write d_{x_i} instead of $(d_n)_{x_i}$, etc.

Proof of Lemma 7. We substitute $v_{x_i}=d_n^{3/2}w_{x_i}+\frac{3}{2}d_n^{1/2}d_{x_i}w$ in (18). Recalling the symmetry relations (20) (with w in the place of v) and using the fact that $a_{ijkl}d_{x_i}d_{x_j}d_{x_k}d_{x_l}=1$ we obtain

$$
\begin{aligned}
&Q(v)-Q\big(d_n^{\alpha}v, d_n^{-\alpha}v\big)\\
&\quad=4\alpha^2\int_\Omega a_{ijkl}d_n^{-2}d_{x_i}d_{x_k}\left[d_n^{3/2}w_{x_j}+\frac{3}{2}d_n^{1/2}d_{x_j}w\right]\left[d_n^{3/2}w_{x_l}+\frac{3}{2}d_n^{1/2}d_{x_l}w\right]dx\\
&\qquad+2\alpha^2\int_\Omega a_{ijkl}d_n^{-2}d_{x_i}d_{x_j}\left[d_n^{3/2}w_{x_k}+\frac{3}{2}d_n^{1/2}d_{x_k}w\right]\left[d_n^{3/2}w_{x_l}+\frac{3}{2}d_n^{1/2}d_{x_l}w\right]dx\\
&\qquad-(\alpha^4+11\alpha^2)\int_\Omega d_n^{-1}w^2\,dx+T_n(v)\\
&\quad=4\alpha^2\int_\Omega a_{ijkl}d_n^{-2}d_{x_i}d_{x_k}\left[d_n^3w_{x_j}w_{x_l}+3d_n^2d_{x_j}w_{x_l}w+\frac{9}{4}d_nd_{x_j}d_{x_l}w^2\right]dx\\
&\qquad+2\alpha^2\int_\Omega a_{ijkl}d_n^{-2}d_{x_i}d_{x_j}\left[d_n^3w_{x_k}w_{x_l}+3d_n^2d_{x_k}w_{x_l}w+\frac{9}{4}d_nd_{x_k}d_{x_l}w^2\right]dx\\
&\qquad-(\alpha^4+11\alpha^2)\int_\Omega d_n^{-1}w^2\,dx+T_n(v)\\
&\quad=4\alpha^2\int_\Omega a_{ijkl}d_nd_{x_i}d_{x_k}w_{x_j}w_{x_l}\,dx+2\alpha^2\int_\Omega a_{ijkl}d_nd_{x_i}d_{x_j}w_{x_k}w_{x_l}\,dx\\
&\qquad+\left(-\alpha^4+\frac{5\alpha^2}{2}\right)\int_\Omega d_n^{-1}w^2\,dx+18\alpha^2\int_\Omega a_{ijkl}d_{x_i}d_{x_j}d_{x_k}w_{x_l}w\,dx+T_n(v).
\end{aligned}
$$

But the last integral belongs to $\mathcal{A}$ by an integration by parts; hence the proof is complete. □

Lemma 8. *Let* $v\in C_c^2(\Omega)$ *and* $w=d_n^{-3/2}v$. *Then*

$$
\begin{aligned}
Q(v)=&\int_\Omega a_{ijkl}d_n^3w_{x_ix_j}w_{x_kx_l}\,dx+\frac{9}{2}\int_\Omega a_{ijkl}d_nd_{x_i}d_{x_j}w_{x_k}w_{x_l}\,dx\\
&-3\int_\Omega a_{ijkl}d_nd_{x_i}d_{x_k}w_{x_j}w_{x_l}\,dx+\frac{9}{16}\int_\Omega d_n^{-1}w^2\,dx+T_n(v),
\end{aligned}
$$

where $\{T_n\}_{n=1}^\infty$ *is an element of* $\mathcal{A}$.

Proof. We have

$$
v_{x_ix_j}=d^{3/2}w_{x_ix_j}+\tfrac{3}{2}d_n^{1/2}d_{x_i}w_{x_j}+\tfrac{3}{2}d_n^{1/2}d_{x_j}w_{x_i}+\tfrac{3}{4}d_n^{-1/2}d_{x_i}d_{x_j}w+\tfrac{3}{2}d_n^{1/2}d_{x_ix_j}w.
$$

As already mentioned, all terms involving second-order derivatives of d_n belong to $\mathcal{A}$. Hence, using the symmetry relations (20) once more we compute,

$$
\begin{aligned}
Q(v) &= \int_\Omega a_{ijkl}\Big[d_n^{3/2}w_{x_ix_j}+3d_n^{1/2}d_{x_i}w_{x_j}+\frac{3}{4}d_n^{-1/2}d_{x_i}d_{x_j}w\Big] \\
&\qquad\times\Big[d_n^{3/2}w_{x_kx_l}+3d_n^{1/2}d_{x_k}w_{x_l}+\frac{3}{4}d_n^{-1/2}d_{x_k}d_{x_l}w\Big]\,dx+T_n(v) \\
&= \int_\Omega a_{ijkl}\Big[d_n^3 w_{x_ix_j}w_{x_kx_l}+6d_n^2 d_{x_i}w_{x_j}w_{x_kx_l}+\frac{3}{2}d_n d_{x_i}d_{x_j}w_{x_kx_l}w \\
&\qquad +9d_n d_{x_i}d_{x_k}w_{x_j}w_{x_l}+\frac{9}{2}d_{x_i}d_{x_j}d_{x_k}w_{x_l}w \\
&\qquad +\frac{9}{16}d_n^{-1}d_{x_i}d_{x_j}d_{x_k}d_{x_l}w^2\Big]\,dx+T_n(v).
\end{aligned} \tag{23}
$$

The fifth term belongs to $\mathcal{A}$ be a simple integration by parts. We also integrate the second and third terms by parts, obtaining, respectively,

$$
\begin{aligned}
\int_\Omega a_{ijkl}d_n^2 d_{x_i}w_{x_j}w_{x_kx_l}\,dx &= -2\int_\Omega a_{ijkl}d_n d_{x_i}d_{x_k}w_{x_j}w_{x_l}\,dx \\
&\qquad +\int_\Omega a_{ijkl}d_n d_{x_i}d_{x_j}w_{x_k}w_{x_l}\,dx+T_n(v), \\
\int_\Omega a_{ijkl}d_n d_{x_i}d_{x_j}w_{x_kx_l}w\,dx &= -\int_\Omega a_{ijkl}d_n d_{x_i}d_{x_j}w_{x_k}w_{x_l}\,dx+T_n(v).
\end{aligned}
$$

Substituting in (23) and recalling (14) we obtain the stated relation. □

We can now prove the main theorem of this subsection. For any $\alpha\in(0,\frac{1}{2})$ we define

$$k_\alpha=\frac{9}{(1-4\alpha^2)(9-4\alpha^2)}.$$

Theorem 9. *For the operator H and relative to the metric* (13), *property* $(\mathbf{P}_\alpha)$ *is valid for all $\alpha\in(0,\frac{1}{2})$. More precisely, for any $\alpha\in(0,\frac{1}{2})$ and any $k>k_\alpha$ there exists $k'<+\infty$ such that*

$$Q\big(d_n^{-\alpha}u\big)\le kQ\big(u,d_n^{-2\alpha}u\big)+k'\|u\|_2^2,\quad u\in C_c^2(\Omega). \tag{24}$$

Proof. Let $u\in C_c^2(\Omega)$ be given and let v and w be defined by $v=d_n^{-\alpha}u$ and $w=d_n^{-3/2}v$, respectively. Define $\gamma_\alpha=(40\alpha^2-16\alpha^4)/9$ and observe that $\gamma_\alpha\in(0,1)$.

Applying Lemmas 7 and 8 and assumption (12.c) we obtain

$$
\begin{aligned}
(25)\quad & \gamma_\alpha Q\big(d_n^{-\alpha}u\big)-[Q(d_n^{-\alpha}u)-Q(u,d_n^{-2\alpha}u)] \\
&=\gamma_\alpha Q(v)-[Q(v)-Q(d_n^{\alpha}v,d_n^{-\alpha}v)] \\
&=\gamma_\alpha\int_\Omega a_{ijkl}d_n^3 w_{x_ix_j}w_{x_kx_l}\,dx+\left(\frac{9\gamma_\alpha}{2}-2\alpha^2\right)\int_\Omega a_{ijkl}d_n d_{x_i}d_{x_j}w_{x_k}w_{x_l}\,dx \\
&\quad-(3\gamma_\alpha+4\alpha^2)\int_\Omega a_{ijkl}d_n d_{x_i}d_{x_k}w_{x_j}w_{x_l}\,dx \\
&\quad+\left(\alpha^4-\frac{5\alpha^2}{2}+\frac{9\gamma_\alpha}{16}\right)\int_\Omega a_{ijkl}d_n^{-1}d_{x_i}d_{x_j}d_{x_k}d_{x_l}w^2\,dx+T_n(v) \\
&\geq\gamma_\alpha\int_\Omega a_{ijkl}d_n^3 w_{x_ix_j}w_{x_kx_l}\,dx+\left(\frac{3\gamma_\alpha}{2}-6\alpha^2\right)\int_\Omega a_{ijkl}d_n d_{x_i}d_{x_j}w_{x_k}w_{x_l}\,dx \\
&\quad+\left(\alpha^4-\frac{5\alpha^2}{2}+\frac{9\gamma_\alpha}{16}\right)\int_\Omega a_{ijkl}d_n^{-1}d_{x_i}d_{x_j}d_{x_k}d_{x_l}w^2\,dx+T_n(v).
\end{aligned}
$$

Therefore

$$
\gamma_\alpha Q(d_n^{-\alpha}u)-[Q(d_n^{-\alpha}u)-Q(u,d_n^{-2\alpha}u)]\geq T_n(v), \tag{26}
$$

since the coefficient of the last integral is zero and those of the other two integrals are non-negative. Now, for any $\varepsilon_1,\varepsilon_2>0$ we have from (16),

$$
\begin{aligned}
|T_n(v)| &\leq \varepsilon_1 Q(v)+c_{\varepsilon_1}\|v\|_2^2 \\
&=\varepsilon_1 Q(d_n^{-\alpha}u)+c_{\varepsilon_1}\|d_n^{-\alpha}u\|_2^2 \\
&\leq \varepsilon_1 Q(d_n^{-\alpha}u)+c_{\varepsilon_1}(\varepsilon_2\|d_n^{-\alpha-2}u\|_2^2+c_{\varepsilon_2}\|u\|_2^2) \\
&\leq \varepsilon_1 Q(d_n^{-\alpha}u)+c_{\varepsilon_1}(c\varepsilon_2 Q(d_n^{-\alpha}u)+c_{\varepsilon_2}\|u\|_2^2),
\end{aligned}
$$

and therefore

$$
|T_n(v)|\leq \varepsilon Q(d_n^{-\alpha}u)+c_\varepsilon\|u\|_2^2 \tag{27}
$$

for any $\varepsilon>0$ small. Choosing $\varepsilon>0$ so that $\gamma_\alpha+\varepsilon<1$ we obtain from (26) and (27),

$$
Q(d_n^{-\alpha}u)\leq\frac{1}{1-\gamma_\alpha-\varepsilon}Q(u,d_n^{-2\alpha}u)+\frac{c_\varepsilon}{1-\gamma_\alpha-\varepsilon}\|u\|_2^2.
$$

Hence (24) is valid for any $k>1/(1-\gamma_\alpha)=k_\alpha$. $\square$

3.3. Small perturbations

In this subsection we prove a stability theorem on the validity of $(\mathbf{P}_\alpha)$. We denote by M_+ the cone of all coefficient matrices for the operators under consideration, that is

$$M_+ = \big\{a = \{a_{\eta\zeta}\}_{|\eta|=|\zeta|=2} : a_{\eta\zeta} \text{ is symmetric, real-valued and measurable} \\ \text{with } \lambda Q_0(u) \le Q(u) \le \Lambda Q_0(u),\ u \in C_c^2(\Omega)\ (\lambda, \Lambda > 0)\big\},$$

equipped with the uniform norm

$$\|a\|_\infty := \operatorname{ess\,sup} |a(x)|_\infty\,; \tag{28}$$

here $|a(x)|$ is the norm of the matrix $a(x) = \{a_{\eta\zeta}(x)\}_{\eta,\zeta}$ considered as an operator on $\mathbf{R}^{N(N+1)/2}$. We recall that λ, $\tilde\lambda$, etc, denote the lower ellipticity constants for the operators induced by the matrices a, $\tilde a$, etc. We have the following result.

Lemma 10. *There exists a computable constant $c>0$ such that for all $\alpha \in (0, \frac{1}{2})$,*

$$\int_\Omega |\nabla^2(d_n^\alpha v)|\,|\nabla^2(d_n^{-\alpha} v)|\,dx \le cQ_0(v), \qquad v \in C_c^2(\Omega).$$

Note. For an estimate on the constant c see the remark at the end of this subsection.

Proof. For any $\beta \in \mathbf{R}$ we have

$$(d_n^\beta v)_{x_i x_j} = d^\beta v_{x_i x_j} + \beta d_n^{\beta-1} d_{x_i} v_{x_j} + \beta d_n^{\beta-1} d_{x_j} v_{x_i} \\ + \beta(\beta-1) d_n^{\beta-2} d_{x_i} d_{x_j} v + \beta d_n^{\beta-1} d_{x_i x_j} v. \tag{29}$$

We write this for $\beta=\alpha$ and for $\beta=-\alpha$, and we multiply the two relations; d_n^α cancels with $d_n^{-\alpha}$ and we obtain

$$|\nabla^2(\omega_n^{-1} v)|\,|\nabla^2(\omega_n v)| \le c\bigg(|\nabla^2 v|^2 + \frac{|\nabla v|^2}{d_n^2} + \frac{v^2}{d_n^4} + \frac{|\nabla^2 d_n|^2}{d_n^2} v^2\bigg).$$

The proof is concluded by using assumption (D2) on $\nabla^2 d$ and the Hardy–Rellich inequalities (D3); here we have also used the fact that $\int_\Omega |\nabla^2 v|^2\,dx = \int_\Omega (\Delta v)^2\,dx$. $\square$

Proposition 11. *Let $\alpha\in(0,\frac{1}{2})$ be fixed. Assume that* $(\mathbf{P}_\alpha)$ *is valid for the matrix $a\in M_+$ relative to some distance $d(\,\cdot\,)\in\mathcal{D}$ and let $k,k'>0$ be such that*

$$Q(\omega_n u)\le kQ\big(u,\omega_n^2u\big)+k'\|u\|_2^2,\quad n\in\mathbf{N},\ u\in C_c^2(\Omega).$$

Then there is a constant $c>0$ such that if $\tilde{a}\in M_+$ satisfies $\|\tilde{a}-a\|_\infty<\tilde{\lambda}[(1+ck)]^{-1}$, then $(\mathbf{P}_\alpha)$ *is also satisfied for $\tilde{a}$ relative to $d(\,\cdot\,)$; more precisely, there exists $\tilde{k}'<+\infty$ so that*

$$\widetilde{Q}(\omega_n u)\le \frac{k}{1-\tilde{\lambda}^{-1}(1+ck)\|a-\tilde{a}\|_\infty}\widetilde{Q}\big(u,\omega_n^2u\big)+\tilde{k}'\|u\|_2^2,\quad n\in\mathbf{N},\quad u\in C_c^2(\Omega).$$

Proof. We first note that

$$(30)\qquad |\widetilde{Q}(v)-Q(v)|\le\int_\Omega|\tilde{a}-a|\,|\nabla^2v|^2\,dx\le\|\tilde{a}-a\|_\infty Q_0(v),\quad v\in C_c^2(\Omega).$$

Moreover, setting $v=\omega_n u$ we have from Lemma 10,

$$(31)\qquad \begin{aligned}|\widetilde{Q}(u,\omega_n^2u)-Q(u,\omega_n^2u)| &\le\|\tilde{a}-a\|_\infty\int_\Omega|\nabla^2u|\,|\nabla^2(\omega_n^2u)|\,dx\\ &=\|\tilde{a}-a\|_\infty\int_\Omega|\nabla^2(\omega_n^{-1}v)|\,|\nabla^2(\omega_n v)|\,dx\\ &\le c\|\tilde{a}-a\|_\infty Q_0(v).\end{aligned}$$

From (30) and (31) we conclude that for any $n\in\mathbf{N}$ and $u\in C_c^2(\Omega)$ we have

$$\begin{aligned}\widetilde{Q}(\omega_n u)&\le Q(\omega_n u)+\|\tilde{a}-a\|_\infty Q_0(\omega_n u)\\ &\le kQ(u,\omega_n^2u)+k'\|u\|_2^2+\|\tilde{a}-a\|_\infty Q_0(\omega_n u)\\ &\le k(\widetilde{Q}(u,\omega_n^2u)+c\|\tilde{a}-a\|_\infty Q_0(\omega_n u))+k'\|u\|_2^2+\|\tilde{a}-a\|_\infty Q_0(\omega_n u)\\ &=k\widetilde{Q}(u,\omega_n^2u)+(1+ck)\|\tilde{a}-a\|_\infty Q_0(\omega_n u)+k'\|u\|_2^2\\ &\le k\widetilde{Q}(u,\omega_n^2u)+\tilde{\lambda}^{-1}(1+ck)\|\tilde{a}-a\|_\infty\widetilde{Q}(\omega_n u)+k'\|u\|_2^2,\end{aligned}$$

from which the statement of the lemma follows. □

Let $\mathcal{G}$ denote the cone of all coefficient matrices that satisfy assumptions (i) and (ii) of Section 3.2. Let also k_α be as in Theorem 9. Combining Proposition 11 and Theorem 9 we obtain immediately the following result.

Theorem 12. *There exists a computable constant $c>0$ such that if for some $\alpha\in(0,\frac{1}{2})$ the coefficient matrix a of the operator H satisfies*

$$\operatorname{dist}_{L^\infty}(a,\mathcal{G})<\frac{\lambda}{1+ck_\alpha},$$

then $(\mathbf{P}_\alpha)$ *is satisfied for H.*

Proof. Let $\tilde{a}\in\mathcal{G}$ be such that

$$\|a-\tilde{a}\|_\infty < \frac{\lambda}{1+ck_\alpha}$$

By Theorem 9, $(\mathbf{P}_\alpha)$ is satisfied for $\tilde{a}$ and (24) is valid for any $k>k_\alpha$. If in addition k satisfies

$$\|a-\tilde{a}\|_\infty < \frac{\lambda}{1+ck},$$

then $(\mathbf{P}_\alpha)$ is also valid for a by Proposition 11. $\square$

Example. Suppose that the coefficients $a_{\eta\zeta}$ are uniformly continuous and satisfy (12.c). Then property $(\mathbf{P}_\alpha)$ is valid for H for all $\alpha\in(0,\frac{1}{2})$. This is seen by approximating $a_{\eta\zeta}$ with smooth functions using an approximate identity; note that the approximating functions also satisfy (ii).

Remark. The constant c of the above proposition is precisely the constant c of Lemma 10. Precise estimates for this constant can be easily obtained. Indeed, it follows from (29) that for $\alpha\in\left(0,\frac{1}{2}\right)$ there holds modulo $\mathcal{A}$

$$\begin{aligned}
&\int_\Omega |\nabla^2(d_n^\alpha v)|\,|\nabla^2(d_n^{-\alpha} v)|\,dx\\
&\le 3\int_\Omega\left(|\nabla^2 v|^2+4\alpha^2|\nabla d|^2\frac{|\nabla v|^2}{d_n^2}+\alpha^2(\alpha+1)^2|\nabla d|^4\frac{v^2}{d_n^4}\right)dx\\
&\le 3\int_\Omega\left(|\nabla^2 v|^2+|\nabla d|^2\frac{|\nabla v|^2}{d_n^2}+\frac{9}{16}|\nabla d|^4\frac{v^2}{d_n^4}\right)dx.
\end{aligned}$$

Hence, letting c_2 and c_3 be as in (D1) and (D3), we obtain (modulo $\mathcal{A}$)

$$\int_\Omega |\nabla^2(d_n^\alpha v)|\,|\nabla^2(d_n^{-\alpha} v)|\,dx \le 3\left(1+c_2^2c_3^{-1}+\frac{9}{16}c_2^4c_3^{-1}\right)\int_\Omega(\Delta v)^2\,dx.$$

In fact, since we work modulo $\mathcal{A}$, the last constant can be improved to become $3\left(1+c_2^2A^{-1}+\frac{9}{16}c_2^4B^{-1}\right)$, where A and B are the *weak* Hardy constants, that is they satisfy

$$\begin{aligned}
\int_\Omega|\nabla v|^2\,dx &\ge A\int_\Omega\frac{v^2}{d^2}\,dx-c'\int_\Omega v^2\,dx,\\
\int_\Omega(\Delta v)^2\,dx &\ge B\int_\Omega\frac{v^2}{d^4}\,dx-c''\int_\Omega v^2\,dx.
\end{aligned}$$

For smooth boundaries with a smooth Riemannian metric this amounts to $A=\frac{1}{4}$ and $B=\frac{9}{16}$.

4. An application: eigenvalue stability

In this final section we demonstrate how the boundary decay estimate of Theorem 3 yield stability bounds on the eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$ of H under small perturbations of the boundary $\partial\Omega$. The proof follows closely the corresponding proof in [D2] for the second-order case, however we include it here for the sake of completeness. So we consider a distance function $d(\,\cdot\,)\in\mathcal{D}$, an operator H as above and assume that the boundary decay estimates (10) are valid for some fixed $\alpha\in(0,\frac{1}{2})$. For $\varepsilon>0$ we define $\Omega_\varepsilon=\{x\in\Omega: d(x)>\varepsilon\}$. We assume that $\varepsilon<\theta/2$ so that $d(x)$ is C^2 on $\Omega\setminus\Omega_{2\varepsilon}$. We define $d_\varepsilon(x)=\mathrm{dist}(x,\partial\Omega_\varepsilon)$, $x\in\Omega$, and make the additional assumption that there exists $c>0$ such that for small enough $\varepsilon>0$,

$$|\nabla^2 d_\varepsilon|\leq c \quad \text{on } \{x\in\Omega: d(x)<2\varepsilon\}. \tag{32}$$

Now, let $\widetilde{\Omega}$ be a domain such that

$$\Omega_\varepsilon\subset\widetilde{\Omega}\subset\Omega;$$

we do not make any regularity assumptions on $\partial\widetilde{\Omega}$. We denote by $\{\tilde{\lambda}_n\}_{n=1}^{\infty}$ the eigenvalues of the operator $\widetilde{H}$ on $L^2(\widetilde{\Omega})$, which is defined by restricting the quadratic form $Q(\,\cdot\,)$ on $H_0^2(\widetilde{\Omega})$.

Let ϕ be a non-negative, smooth, increasing function on $\mathbf{R}$ such that

$$\phi(t)=\begin{cases} 0, & t\leq 0,\\ 1, & t\geq 1.\end{cases}$$

We define a C^2 cut-off function τ on Ω by

$$\tau(x)=\begin{cases} 0, & x\in\Omega\setminus\Omega_\varepsilon,\\ \phi(d_\varepsilon(x)/\varepsilon), & x\in\Omega_\varepsilon.\end{cases}$$

Note that $\tau(x)=1$ when $d(x)>2\varepsilon$; moreover (32) yields

$$|\tau(x)|\leq 1,\quad |\nabla\tau(x)|\leq c\varepsilon^{-1}\quad\text{and}\quad |\nabla^2\tau(x)|\leq c\varepsilon^{-2}.$$

Let us now denote by $\{\phi_n\}_{n=1}^{\infty}$ the normalized eigenfunctions of H. For $n\geq 1$ we set

$$L_n=\mathrm{span}\{\phi_1,...,\phi_n\}\quad\text{and}\quad \tilde{L}_n=\mathrm{span}\{\tau\phi_1,...,\tau\phi_n\},$$

and observe that $\tilde{L}_n\subset H_0^2(\widetilde{\Omega})$.

Lemma 13. *There exists a constant $c>0$ such that for all small $\varepsilon>0$ and all $u\in\mathrm{Dom}(H)$,*

$$\text{(i)}\qquad |Q(\tau u)-Q(u)|\le c\varepsilon^{2\alpha}\|Hu\|_2\|H^{\alpha/2}u\|_2,$$

$$\text{(ii)}\qquad \big|\|\tau u\|_2-\|u\|_2\big|\le c\varepsilon^{2+\alpha}\|Hu\|_2^{1/2}\|H^{\alpha/2}u\|_2^{1/2}.$$

Proof. Let $u\in\mathrm{Dom}(H)$. On $\Omega_{2\varepsilon}$ we have $\tau(x)=1$, hence

$$\begin{aligned}
|Q(\tau u)-Q(u)| &= \left|\int_\Omega \sum_{\substack{|\eta|=2\\|\zeta|=2}} a_{\eta\zeta}((D^\eta(\tau u))(D^\zeta(\tau u))-(D^\eta u)(D^\zeta u))\,dx\right|\\
&\le c\int_{d(x)<2\varepsilon}(|\nabla^2(\tau u)|^2+|\nabla^2 u|^2)\,dx\\
&\le c\int_{d(x)<2\varepsilon}(|\nabla^2 u|^2+|\nabla\tau|^2|\nabla u|^2+|\nabla^2\tau|^2u^2)\,dx\\
&\le c\int_{d(x)<2\varepsilon}(|\nabla^2 u|^2+\varepsilon^{-2}|\nabla u|^2+\varepsilon^{-4}u^2)\,dx\\
&\le c\varepsilon^{2\alpha}\int_{d(x)<2\varepsilon}\left(\frac{|\nabla^2u|^2}{d^{2\alpha}}+\frac{|\nabla u|^2}{d^{2+2\alpha}}+\frac{|u|^2}{d^{4+2\alpha}}\right)dx,
\end{aligned}$$

from which (i) follows by means of Theorem 3. Similarly,

$$\begin{aligned}
\big|\|\tau u\|_2-\|u\|_2\big|^2 \le \|\tau u-u\|_2^2 &\le \int_{d(x)<2\varepsilon}|u|^2\,dx\\
&\le \varepsilon^{4+2\alpha}\int_{d(x)<2\varepsilon}\frac{u^2}{d^{4+2\alpha}}\,dx\le c\varepsilon^{4+2\alpha}\|Hu\|_2\|H^{\alpha/2}u\|_2,
\end{aligned}$$

from which (ii) follows. □

Theorem 14. *Assume that there exists a distance function $d\in\mathcal{D}$ and an $\alpha\in(0,\frac{1}{2})$ such that* (10) *is satisfied. Assume also that* (32) *is valid. Then there exists $c,c'>0$ such that for each $n\ge1$,*

$$0<\lambda_n\le\tilde\lambda_n\le\lambda_n+c\lambda_n^{5/4}\varepsilon^{2\alpha} \tag{33}$$

for all $\varepsilon>0$ satisfying $\varepsilon^{2\alpha}<c'\lambda_n^{-5/4}$.

Proof. We fix $n\ge1$. Since $\tilde L_n\subset H_0^2(\widetilde\Omega)$ we have by min-max

$$\tilde\lambda_n\le\sup\{Q(v)/\|v\|_2^2:v\in\tilde L_n\}=\sup\{Q(\tau u)/\|\tau u\|_2^2:u\in L_n\}. \tag{34}$$

Now, let $u\in L_n$ be given. It follows from Lemma 13 (i) that

$$Q(\tau u)\leq Q(u)+c\varepsilon^{2\alpha}\|Hu\|_2\|H^{\alpha/2}u\|_2\leq Q(u)+c\varepsilon^{2\alpha}\lambda_n^{5/4}\|u\|_2^2. \tag{35}$$

Similarly Lemma 13 (ii) gives

$$\|\tau u\|_2^2\geq \|u\|_2^2-c\varepsilon^{2+\alpha}\lambda_n^{5/8}\|u\|_2(\|u\|_2+\|\tau u\|_2)\geq \|u\|_2^2-c\varepsilon^{2+\alpha}\lambda_n^{5/8}\|u\|_2^2. \tag{36}$$

Assuming in addition that $\|u\|_2=1$ we thus obtain from (35) and (36) that

$$\frac{Q(\tau u)}{\|\tau u\|_2^2}\leq \frac{Q(u)+c\varepsilon^{2\alpha}\lambda_n^{5/4}}{1-c\varepsilon^{2+\alpha}\lambda_n^{5/8}}\leq Q(u)+c\lambda_n^{5/4}\varepsilon^{2\alpha}\leq \lambda_n+c\lambda_n^{5/4}\varepsilon^{2\alpha},$$

where for the second inequality we have used the fact that $\varepsilon^{2\alpha}<c'\lambda_n^{-5/4}$, with c' small enough but fixed (and independent of n and ε). Hence (34) implies

$$\lambda_n\leq\tilde\lambda_n\leq\lambda_n+c\lambda_n^{5/4}\varepsilon^{2\alpha},$$

which completes the proof of the theorem. □

Remark. In the case where $\Omega=B(1)$ and $\widetilde\Omega=B(1-\varepsilon)$ we have $\tilde\lambda_n=(1-\varepsilon)^{-4}\lambda_n$ and hence $\tilde\lambda_n-\lambda_n=4\lambda_n(\varepsilon+O(\varepsilon^2))$. Hence the value $\alpha=\frac12$ is the best possible for estimate (33).

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References

[A] Agmon, S., *Lectures on Exponential Decay of Solutions of Second-Order Elliptic Equations: Bounds on Eigenfunctions of N-body Schrödinger Operators*, Mathematical Notes **29**, Princeton University Press, Princeton, NJ, 1982.

[B] Barbatis, G., Explicit estimates on the fundamental solution of higher-order parabolic equations with measurable coefficients, *J. Differential Equations* **174** (2001), 442–463.

[BFT] Barbatis, G., Filippas, S. and Tertikas, A., Refined geometric L^p Hardy inequalities, *Commun. Contemp. Math.* **5** (2003), 869–881.

[BT] Barbatis, G. and Tertikas, A., On a class of Rellich inequalities, *J. Comput. Appl. Math.* **194** (2006), 156–172.

[D1] Davies, E. B., *One-parameter Semigroups*, London Mathematical Society Monographs **15**, Academic Press, London, 1980.

[D2] Davies, E. B., Eigenvalue stability bounds via weighted Sobolev spaces, *Math. Z.* **214** (1993), 357–371.

[D3] Davies, E. B., Sharp boundary estimates for elliptic operators, *Math. Proc. Cambridge Philos. Soc.* **129** (2000), 165–178.

[EHK] Evans, W. D., Harris, D. J. and Kauffman, R. M., Boundary behaviour of Dirichlet eigenfunctions of second order elliptic operators, *Math. Z.* **204** (1990), 85–115.

[FHT] Fleckinger, J., Harrell, E. M. II and de Thélin, F., Boundary behavior and estimates for solutions of equations containing the p-Laplacian, *Electron. J. Differential Equations* **38** (1999), 19pp.

[GT] Gilbarg, D. and Trudinger, N. S., *Elliptic Partial Differential Equations of Second Order*, 2nd edn., Grundlehren der Mathematischen Wissenschaften **224**, Springer, Berlin–Heidelberg, 1983.

[LN] Li, Y. and Nirenberg, L., The distance function to the boundary, Finsler geometry, and the singular set of viscosity solutions of some Hamilton–Jacobi equations, *Comm. Pure Appl. Math.* **58** (2005), 85–146.

[M] Mason, C., Perturbation of domain: singular Riemannian metrics, *Proc. London Math. Soc.* **84** (2002), 473–491.

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