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Elliptic CR-manifolds and shear invariant ordinary differential equations with additional symmetries

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Abstract. We classify the ordinary differential equations that correspond to elliptic CR-manifolds with maximal isotropy. It follows that the dimension of the isotropy group of an elliptic CR-manifold can only be 10 (for the quadric), 4 (for the listed examples) or less. This is in contrast with the situation of hyperbolic CR-manifolds, where the dimension can be 10 (for the quadric), 6 or 5 (for semi-quadrics) or less than 4. We also prove that, for all elliptic CR-manifolds with non-linearizable isotropy group, except for two special manifolds, the points with non-linearizable isotropy form exactly some complex curve on the manifold.

1. Introduction

In [3] the authors used a correspondence between so-called torsion-free elliptic CR-manifolds and complex second order ordinary differential equations (ODEs) to describe elliptic CR-manifolds with non-linearizable isotropy. This description was based on an investigation of ODEs with a shear symmetry $y\partial/\partial x$ on the xy-plane near the singularity $(0,0)$.

The major aim of this paper is to describe elliptic CR-manifolds with big isotropy. We will show that the maximal dimension of the isotropy for non-quadratic elliptic CR-manifolds is 4 and is attained exactly for manifolds that correspond to the ODEs

$$y''=y^k(y-xy')^3 \quad \text{and} \quad y''=y^l y'(y-xy')^2+Cy^{2l+2}(y-xy')^3$$

where k and l are non-negative integers and C is a complex constant. Thus, according to earlier results by the authors [2], the possible dimensions of the isotropy of elliptic CR-manifolds are $10, 4, 3, 2, 1$ and 0. This is somewhat unexpected, because

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the corresponding numbers for analogous hyperbolic manifolds are $10, 6, 5, 3, 2, 1$ and 0 (see [5]).

In Section 4 we represent open parts of elliptic CR-manifolds with non-linearizable isotropy as copies of $\mathrm{SL}(2,\mathbb{C})$ with the standard action of subgroups of $\mathrm{SL}(2,\mathbb{C})$.

In Section 5 we show that the duality of ODEs that results from swapping the rôles of the variables x and y and the parameters c_1 and c_2 corresponds simply to switching to the complex conjugate CR-manifold. We demonstrate this feature for the exceptional quartic.

Section 6 is devoted to shear invariant elliptic CR-manifolds with one additional non-isotropic symmetry. We show that these manifolds coincide with the manifolds obtained in Section 3 for a different choice of the reference point.

In Section 7 we conclude that the quartic is the only shear invariant elliptic CR-manifold with 6-dimensional automorphism group.

Finally, in Section 8 we show that the quadric and the quartic are characterized by the property that the points with non-linearizable isotropy fill more than a complex curve, whereas in all other cases they fill exactly a complex curve.

2. Preliminaries

Let M be a CR-manifold M of CR-dimension two and CR-codimension two, i.e., M is a 6-dimensional manifold with a rank-4 distribution $D\subset TM$ and a smooth field of endomorphisms $J_x\colon D_x\to D_x$ with $J_x^2=-\,\mathrm{id}$. The Levi form at $x\in M$ is a bilinear mapping

$$\mathcal{L}_x\colon D_x\times D_x\longrightarrow T_xM/D_x.$$

$\mathcal{L}_x(X,Y)$ is defined as the bracket of two sections $\widetilde{X}$ and $\widetilde{Y}$ of D which extend X and Y, followed by the natural projection $\pi\colon T_x\to T_x/D_x$.

M is called *elliptic* if any *real* linear combination of the two scalar components of $\mathcal{L}$ is a non-degenerate bilinear form. It follows that there exist two mutually conjugate *complex* degenerate combinations. Their null vectors define a canonical splitting $D_x=D_x^+\oplus D_x^-$. For a pair of sections $\widetilde{X}$ in D^+ and $\widetilde{Y}$ in D^- the vectors $(\widetilde{X}_x,\overline{\widetilde{Y}}_x,[\widetilde{X},\overline{\widetilde{Y}}]_x)$ define a complex structure on T_xM.

We assume that $\mathcal{L}_x(J_xX,J_xY)=\mathcal{L}_x(X,Y)$ for all $x\in M$, i.e. M is partially integrable.

For partially integrable elliptic CR-manifolds a Cartan connection was constructed in [1] (see also [6] and [8]). In the present paper we consider only elliptic manifolds, whose so-called torsion-part of the Cartan curvature vanishes. This algebraic condition is equivalent to the following geometric properties:

(1) M is embeddable;

(2) the line bundles D^+ and D^- are integrable;

(3) the canonical almost complex structure is integrable.

It follows that M must be real-analytic. For a smooth embedded elliptic CR-manifold, vanishing of the torsion at a point $x \in M$ can also be expressed by the equivalent condition that M has contact of third order with its osculating quadric at x (see [7]).

We have the following result.

Proposition 1. *There is a local one-to-one correspondence between*

(1) *torsionfree elliptic CR-manifolds of CR-dimension two and CR-codimension two*

(2) *complex* 3*-folds with two holomorphic direction fields that span a non-involutive distribution*

(3) *complex second order ODEs.*

Proof. If $\widetilde{M}$ is a complex 3-fold with a pair of non-involutive direction fields one can introduce local coordinates x, y and p such that $\mathfrak{Z}_1 = \partial/\partial p$ and $\mathfrak{Z}_2 = \partial/\partial x + p\partial/\partial y + B(x,y,p)\partial/\partial p$ (see [3]). This allows us to interpret a local part of $\widetilde{M}$ as a chart of the projectivized tangent bundle over $\mathbb{C}^2$ with coordinates x and y in the base and $p = dy/dx$ in the fibre. The projections of the integral curves of $\mathfrak{Z}_2$ are then nothing but the integral curves of $y'' = B(x,y,y')$ in $\mathbb{C}^2$. Vice versa, the lifts of integral curves of a second order ODE to the projectivized tangent bundle define a direction field $\mathfrak{Z}_2$ that does not commute with $\mathfrak{Z}_1 = \partial/\partial p$.

If M is a torsionfree elliptic CR-manifold then M has an integrable almost complex structure and two holomorphic direction fields that generate $D^+, \overline{D}^-$. Vice versa, if $\widetilde{M}$ is a complex 3-fold with holomorphic direction fields $\mathfrak{Z}_1$ and $\mathfrak{Z}_2$ then D_x can be defined as the span of these direction fields. J_x is defined by $J_x \mathfrak{Z}_{1,x} = i\mathfrak{Z}_{1,x}$ and $J_x \mathfrak{Z}_{2,x} = -i\mathfrak{Z}_{2,x}$. Non-involutivity of the two direction fields is equivalent to the ellipticity of the Levi form.

It remains to show that the obtained CR-manifold M is torsionfree. It is convenient to represent M as an embedded CR-submanifold of $\mathbb{C}^4$. We look for four independent coordinate functions that are annihilated by

$$\bar{\mathfrak{Z}}_1 = \frac{\partial}{\partial \bar{p}} \quad \text{and} \quad \mathfrak{Z}_2 = \frac{\partial}{\partial x} + p\frac{\partial}{\partial y} + B(x,y,p)\frac{\partial}{\partial p}.$$

Two obvious solutions are $z_2 = \bar{x}$ and $w_2 = \bar{y}$. We need two additional coordinate functions of the form $f(x,y,p)$. Thus, we have to solve

$$\frac{\partial}{\partial x} f + p\frac{\partial}{\partial y} f + B(x,y,p)\frac{\partial}{\partial p} f = 0.$$

The characteristic equation of this partial different equation is

$$\dot{x}=1, \quad \dot{y}=p \quad \text{and} \quad \dot{p}=B(x,y,p).$$

It is equivalent to $\ddot{y}=B(t,y,\dot{y})$. Let the characteristic curves be

$$x=t, \quad y=\phi(t,C_1,C_2) \quad \text{and} \quad p=\dot{\phi}(t,C_1,C_2).$$

Then the desired coordinate functions are $z_1=C_1(x,y,p)$ and $z_2=C_2(x,y,p)$. In $\mathbb{C}^4$ with coordinates z_1, z_2, w_1 and w_2 the equation of the manifold M takes the form

$$\overline{w}_2=\phi(\bar{z}_2, z_1, w_1).$$

M has two foliations: into holomorphic curves (for $\bar{z}_2$ and $\overline{w}_2$ fixed) and into anti-holomorphic curves (for z_1 and w_1 fixed). The tangent spaces to the curves which pass through a given point span the maximal complex subspace of the tangent space of M at this point. The corresponding directions annihilate the degenerate complex linear combinations of the components of the Levi form. Thus, they provide the canonical splitting. By construction, the corresponding line bundles are integrable. The induced almost complex structure is the one that is obtained by adopting $z_1, \bar{z}_2, w_1$ and $\overline{w}_2$ as holomorphic coordinates in the ambient space. Therefore, it is clearly integrable. □

Remark 1. The embedding constructed in the proof of Proposition 1 has the property that the two canonical foliations coincide with the foliations into the fibres of the projections to the z_1w_1-plane and the z_2w_2-plane, respectively.

It was proved in [3] that elliptic CR-manifolds with non-linearizable isotropy group are in one-to-one correspondence with shear invariant second order ODEs. Such ODEs can be represented by

$$y''=B(x,y,y')=f_0(y)(y-xy')^3+f_1(y)y'(y-xy')^2, \tag{1}$$

where two ODEs are equivalent if and only if there is a mapping

$$(x,y)\longmapsto\left(\frac{c_1x}{1-cy}, \frac{c_2y}{1-cy}\right)$$

which takes one to the other. A finer classification can be obtained if we take into account possible additional symmetries. Sophus Lie [4] classified second order ODEs with one-, two-, and three-dimensional symmetry groups. The difference with our approach is that we are interested in fixed points of the automorphisms, whereas Lie always chooses a point where one of the symmetries is the translation $\partial/\partial y$. In our situation one of the symmetries is the shear $y\partial/\partial x$. Our choice of the canonical symmetries and regularity of the ODEs at the reference point imply that B is a third order polynomial with respect to y' and x.

3. Classification of shear invariant ODEs with 4-dimensional isotropy

If there is only one (up to scale) shear symmetry of a shear invariant ODE then it can be used as an invariant. On the other hand, as is known from [3], all ODEs with more than one shear can be written as

$$y''=\frac{K(y-xy')^3}{(1-cy)^3}.$$

If we exclude these ODEs, then any additional isotropic symmetry of the ODE $y''=B(x,y,y')$ must preserve the single shear symmetry and, consequently, must have the form

$$((\phi(y)+a)x+\psi(y))\frac{\partial}{\partial x}+\phi(y)y\frac{\partial}{\partial y}. \tag{2}$$

The general equation for infinitesimal symmetries $\xi\partial/\partial x+\eta\partial/\partial y$ is

$$\begin{aligned}\xi\frac{\partial B}{\partial x}+\eta\frac{\partial B}{\partial y}+\phi\frac{\partial B}{\partial p}+\left(2\frac{\partial \xi}{\partial x}+3p\frac{\partial \xi}{\partial y}-\frac{\partial \eta}{\partial y}\right)B\\ -\frac{\partial^2\eta}{\partial x^2}+p\left(\frac{\partial^2\xi}{\partial x^2}-2\frac{\partial^2\eta}{\partial x\partial y}\right)+p^2\left(2\frac{\partial^2\xi}{\partial x\partial y}-\frac{\partial^2\eta}{\partial y^2}\right)+p^3\frac{\partial^2\xi}{\partial y^2}=0.\end{aligned} \tag{3}$$

We plug in B from (1) and the infinitesimal automorphism (2). The component of degree 3 in p and degree 0 in x in equation (3) immediately implies that $\psi''=0$. Since we are here interested only in isotropic automorphisms and since we know that the shear $y\partial/\partial x$ is an automorphism we may assume that $\psi=0$.

The component of degree 3 in p and degree 1 in x in equation (3) yields $\phi''=0$, and thus $\phi=\beta_1+\alpha_3 y$.

From the components of degree 3 in p and degree 2 and 3 in x we get

$$\begin{aligned}af_1+3\beta_1 f_1+3\alpha_3 y f_1+\beta_1 y f_1'+\alpha_3 y^2 f_1'&=0,\\ 2af_0+4\beta_1 f_0+3y\alpha_3 f_0+y\beta_1 f_0'+y^2\alpha_3 f_0'&=0.\end{aligned}$$

If $f_0=\sum_{n=k}^{\infty} b_n y^n$ and $f_1=\sum_{n=l}^{\infty} c_n y^n$ then

$$\begin{aligned}(a+(n+3)\beta_1)c_n+(n+2)\alpha_3 c_{n-1}&=0,\\ (2a+(n+4)\beta_1)b_n+(n+2)\alpha_3 b_{n-1}&=0.\end{aligned}$$

The first equation for $n=l$ and the second equation for $n=k$ give rise to a linear system that implies $\beta_1=a=0$ and, consequently, $\alpha_3=0$, unless $k=2l+2$, or either $f_0=0$ or $f_1=0$.

From the recursive formulae we find

$$f_0 = C_1\, y^k (1-cy)^{-k-3},$$
$$f_1 = C_2\, y^l (1-cy)^{-l-3}.$$

By applying a transformation $x_1=c_1x/(1-cy)$, $y_1=c_2y/(1-cy)$ this can be reduced to one of the following two series of ODEs

$$y'' = y^k (y-xy')^3, \tag{4}$$
$$y'' = y^l y'(y-xy')^2 + Cy^{2l+2}(y-xy')^3, \tag{5}$$

where k and l are non-negative integers and C is a complex constant. According to [3, Theorem 3] these ODEs are pairwise non-equivalent.

The additional symmetry is

$$(k+2)x\frac{\partial}{\partial x} - 2y\frac{\partial}{\partial y}, \quad \text{resp.} \quad (l+2)x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}.$$

The corresponding CR-manifolds are exactly the CR-manifolds with an isotropy group of real dimension 4.

We conclude the following result.

Theorem 1. *The isotropy group of an elliptic CR-manifold has*
(1) *dimension* 10 *if and only if it is equivalent to the quadric;*
(2) *dimension* 4 *if and only if it corresponds to one of the ODEs* (4) *and* (5)*;*
(3) *dimension* ≤ 3 *in all other cases.*

Proof. Statements (1) and (2) follow from the obtained classification. Statement (3) was proved in [2]. □

4. SL(2, ℂ) representation of the shear invariant manifolds

Any shear invariant ODE

$$y'' = f_0(y)(y-xy')^3 + f_1(y)y'(y-xy')^2$$

obviously admits the solutions $y=cx$ for any constant c. Thus, the solution passing through $(x_0,y_0)\in\mathbb{C}^2_*=\mathbb{C}^2\setminus\{(0,0)\}$ with slope $p_0=y_0/x_0$ is $y=p_0x$.

Notice that the equation $y=px$ describes a canonical section in the trivial fibre bundle $\mathbb{C}^2_*\times\mathbb{CP}^1$, which is induced by the tautological mapping

$$\begin{aligned}\tau\colon \mathbb{C}^2_* &\longrightarrow \mathbb{CP}^1,\\ (x,y) &\longmapsto [x:y].\end{aligned}$$

By M^* we denote the bundle with deleted section τ.

Here we will give a representation of the part of the solution manifold that corresponds to initial conditions $(x_0,y_0,p_0)\in M^*$. M^* can be identified with $\mathrm{SL}(2,\mathbb{C})$ using the map

$$(x,y,p)\longmapsto \begin{pmatrix} \dfrac{1}{y-xp} & x \\ \dfrac{p}{y-xp} & y\end{pmatrix} = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}.$$

The two distinguished direction fields now take the form

$$\begin{aligned}\mathfrak{z}_1 &= \beta\frac{\partial}{\partial\alpha}+\delta\frac{\partial}{\partial\gamma},\\ \mathfrak{z}_2 &= \alpha\frac{\partial}{\partial\beta}+\gamma\frac{\partial}{\partial\delta}+(f_0(\delta)+\gamma f_1(\delta))\,\mathfrak{z}_1.\end{aligned}$$

The one-parametric action produced by the field $\mathfrak{z}_1$ is right multiplication with

$$\begin{pmatrix}1 & 0\\ t & 1\end{pmatrix}.$$

The second field generates a linear action only if $f_1\equiv 0$ and f_0 is constant, i.e., in the cases of a quadric ($f_0\equiv 0$) or a quartic ($f_0\equiv 1$).

The shear symmetry is represented by

$$\theta=\gamma\frac{\partial}{\partial\alpha}+\delta\frac{\partial}{\partial\beta}$$

and produces left multiplication by

$$\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix}.$$

In the quadric case $\mathfrak{z}_2=\mathfrak{z}_Q=\alpha\partial/\partial\beta+\gamma\partial/\partial\delta$ corresponds to right multiplication with

$$\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix}$$

and in the quartic case $\mathfrak{Z}_2 = \mathfrak{Z}_Q + \mathfrak{Z}_1$ to right multiplication with

$$\begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}.$$

It is clear that in both cases these actions commute with the complete left multiplication by $\mathrm{SL}(2, \mathbb{C})$.

For manifolds with two isotropic symmetries the second (linear) symmetry has the form

$$\mathfrak{L} = 2\alpha \frac{\partial}{\partial \alpha} + (k+2)\beta \frac{\partial}{\partial \beta} - (k+2)\gamma \frac{\partial}{\partial \gamma} - 2\delta \frac{\partial}{\partial \delta},$$

and respectively,

$$\mathfrak{L} = \alpha \frac{\partial}{\partial \alpha} + (l+2)\beta \frac{\partial}{\partial \beta} - (l+2)\gamma \frac{\partial}{\partial \gamma} - \delta \frac{\partial}{\partial \delta}.$$

It generates the one-parametric action

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \longmapsto \begin{pmatrix} t^{k+4} & 0 \\ 0 & t^{-k-4} \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} t^{-k} & 0 \\ 0 & t^{k} \end{pmatrix},$$

and, respectively,

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \longmapsto \begin{pmatrix} t^{l+3} & 0 \\ 0 & t^{-l-3} \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} t^{-l-1} & 0 \\ 0 & t^{l+1} \end{pmatrix}.$$

Since $\mathfrak{Z}_1$ commutes with the left part of the action and is mapped to $k\mathfrak{Z}_1$ (resp. $(l+1)\mathfrak{Z}_1$) by the right part of the action we find

$$[\mathfrak{L}, \mathfrak{Z}_1] = k\mathfrak{Z}_1 \quad \text{resp.} \quad [\mathfrak{L}, \mathfrak{Z}_1] = (l+1)\mathfrak{Z}_1.$$

For the second field

$$\mathfrak{Z}_2 = \mathfrak{Z}_Q + F\mathfrak{Z}_1$$

we have

$$[\mathfrak{L}, \mathfrak{Z}_Q] = -k\mathfrak{Z}_Q \quad \text{resp.} \quad [\mathfrak{L}, \mathfrak{Z}_Q] = (-l-1)\mathfrak{Z}_Q$$

and

$$[\mathfrak{L}, F\mathfrak{Z}_1] = kF\mathfrak{Z}_1 + (\mathfrak{L}F)\mathfrak{Z}_1 \quad \text{resp.} \quad [\mathfrak{L}, F\mathfrak{Z}_1] = (l+1)F\mathfrak{Z}_1 + (\mathfrak{L}F)\mathfrak{Z}_1.$$

In the first case this requires $\mathfrak{L}F = -2kF$, which is satisfied for $F = \delta^k$. In the second case this requires $\mathfrak{L}F = -2(l+1)F$, which is satisfied for combinations of $\gamma\delta^l$ and δ^{2l+2}.

5. Dual ODEs

A duality of ODEs appears from the symmetry of interchanging the distinguished direction fields $\mathfrak{Z}_1$ and $\mathfrak{Z}_2$. This corresponds to interchanging the rôles of the variables x and y and the parameters c_1 and c_2 of the solutions. In terms of the embedded CR-manifold this will be achieved by complex conjugation. The symmetry group of the dual ODE clearly will be isomorphic to the symmetry group of the initial ODE, though the action is different. It follows that ODEs corresponding to elliptic CR-manifolds that are complexifications of real hypersurfaces in $\mathbb{C}^2$ are self-dual. The non-quadratic CR-manifolds with non-linearizable automorphisms are never self-dual.

If the complete solution of an ODE is known then the dual ODE can be easily obtained by differentiating with respect to the parameters and eliminating the variables x and y.

In the case of $y''=(y-xy')^3$ the complete solution is the quartic

$$(y-c_1x)^2-c_2^2x^2-c_2=0.$$

We find the dual ODE

$$y''=\frac{1-(y')^2}{x(y'+\sqrt{(y')^2-1})}. \tag{6}$$

(Here we adopted c_2 as the new independent variable x and c_1 as the new dependent variable y.)

The family of solutions can be written in the form

$$(x-c_1)^2-(y-c_2)^2=c_1^2.$$

The symmetries are generated by

$$\frac{\partial}{\partial y},\quad x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y},\quad \text{and}\quad 2xy\frac{\partial}{\partial x}+(x^2+y^2)\frac{\partial}{\partial y}.$$

The ODE (6) is equivalent to

$$\eta''+\frac{2\eta'(1-\sqrt{\eta'})^2}{\xi-\eta}=0$$

from Lie's list of ODEs with three symmetries. The equivalence is established by $\xi=y+x$ and $\eta=y-x$. In this notation the infinitesimal automorphisms become

$$\frac{\partial}{\partial \xi}+\frac{\partial}{\partial \eta},\quad \xi\frac{\partial}{\partial \xi}+\eta\frac{\partial}{\partial \eta},\quad \text{and}\quad \xi^2\frac{\partial}{\partial \xi}+\eta^2\frac{\partial}{\partial \eta}.$$

The corresponding group is PSL$(2,\mathbb{C})$ acting by "coupled" Möbius transformations on the complex ξ and η planes

$$(\xi,\eta)\longmapsto\left(\frac{\alpha\xi+\beta}{\gamma\xi+\delta},\frac{\alpha\eta+\beta}{\gamma\eta+\delta}\right).$$

6. Shear invariant ODE with one non-isotropic symmetry

If a shear invariant ODE admits a non-isotropic symmetry we may assume that, after a coordinate change $x=f(x^*,y^*)$, $y=g(x^*,y^*)$, it takes the form $\partial/\partial x^*$. Then the shear becomes $\theta=\xi\partial/\partial x^*+\eta\partial/\partial y^*$, where $\xi=g\partial f^*/\partial x$, $\eta=g\partial g^*/\partial x$, and (f^*,g^*) is the inverse coordinate change. We prove the following result.

Lemma 1. *If $\partial/\partial x^*$ and the shear θ are the only symmetries of the ODE then*

$$\left[\frac{\partial}{\partial x^*},\theta\right]=\mu\theta.$$

Proof. From

$$\left[\frac{\partial}{\partial x^*},\theta\right]=\mu\theta+\nu\frac{\partial}{\partial x^*}$$

we conclude that

$$\frac{\partial}{\partial x^*}\xi=\nu+\mu\xi\quad\text{and}\quad\frac{\partial}{\partial x^*}\eta=\mu\eta.$$

Now, if $\mu=0$, then

$$\xi=\nu x^*+K_1(y^*)\quad\text{and}\quad\eta=K_2(y^*).$$

We distinguish two subcases: If $K_2\equiv0$ then $\partial g^*/\partial x\equiv0$, and therefore $g=g(y^*)$ with $g(0)=0$. It follows that $\xi=0$ for $y^*=0$. Since θ vanishes exactly at one curve, this curve must coincide with $y^*=0$. Hence $\nu=0$.

In the second subcase $y^*=0$ is an isolated zero of K_2. Thus again y^* is the only curve on which θ can vanish and therefore $\nu=0$.

Suppose now that $\mu\neq0$. Then

$$\xi=\frac{-\nu}{\mu}+K_1(y^*)\,\mathrm{e}^{\mu x^*}\quad\text{and}\quad\eta=K_2(y^*)\,\mathrm{e}^{\mu x^*}.$$

Again, either $K_2\equiv0$ or 0 is an isolated zero of K_2. Analogous arguments to the ones used above show that $\nu=0$ in this case as well. □

As in Section 3 we consider the equation (3). We conclude that $\psi(y)=\alpha_0$ but now we assume that $\alpha_0\neq 0$. Then we can rescale the additional infinitesimal automorphism in such a way that $\alpha_0=1$. Thus we look for an infinitesimal automorphism of the form

$$(1+(\phi(y)+a)x)\frac{\partial}{\partial x}+\phi(y)y\frac{\partial}{\partial y}.$$

From the component of degree 3 in p and 1 in x we find that

$$f_1=-\frac{\phi''}{2\alpha_0}=-\frac{-\phi''}{2}.$$

The components of degree 3 in p and 2 respectively 3 in x yield now the system

$$(7)\qquad \begin{cases} -\frac{1}{6}(y\phi'''+3\phi'')\phi-\frac{a}{6}\phi''=f_0, \\ (4\phi-y\phi'+2a)f_0+y\phi f_0'=0, \end{cases}$$

which is equivalent to the ODE

$$(8)\qquad (y^2\phi^{\mathrm{IV}}+8y\phi'''+12\phi'')\phi^2+a(3y\phi'''\phi+10\phi''\phi-y\phi''\phi')+2a^2\phi''=0$$

on ϕ. We see immediately that $a=0$ implies $\phi''=0$ and therefore $f_0=f_1=0$. Assume that $a\neq 0$. Then the ODE (8) yields the following equations on the coefficients of an analytic solution $\phi(y)=\sum_{n=0}^{\infty}\phi_n y^n$:

$$(9)\qquad \sum_{\beta=0}^{j-2}\sum_{\alpha=0}^{\beta}\frac{(j-\beta+2)!}{(j-\beta-2)!}\phi_\alpha\phi_{\beta-\alpha}\phi_{j+2-\beta}+8\sum_{\beta=0}^{j-1}\sum_{\alpha=0}^{\beta}\frac{(j-\beta+2)!}{(j-\beta-1)!}\phi_\alpha\phi_{\beta-\alpha}\phi_{j+2-\beta}$$
$$+12\sum_{\beta=0}^{j}\sum_{\alpha=0}^{\beta}\frac{(j-\beta+2)!}{(j-\beta)!}\phi_\alpha\phi_{\beta-\alpha}\phi_{j+2-\beta}+3a\sum_{\beta=0}^{j-1}\frac{(j-\beta+2)!}{(j-\beta-1)!}\phi_\beta\phi_{j+2-\beta}$$
$$+10a\sum_{\beta=0}^{j}\frac{(j-\beta+2)!}{(j-\beta)!}\phi_\beta\phi_{j+2-\beta}-a\sum_{\beta=0}^{j}\frac{(j-\beta+1)!}{(j-\beta-1)!}(\beta+1)\phi_{\beta+1}\phi_{j+1-\beta}$$
$$+2a^2(j+2)(j+1)\phi_{j+2}=0.$$

It follows for $j\geq 0$ that

$$(j+2)(j+1)(a+(j+3)\phi_0)(2a+(j+4)\phi_0)\phi_{j+2}$$
$$+(j+2)(j+1)j(3a+2(j+3)\phi_0)\phi_1\phi_{j+1}=...,$$

where the dots indicate a sum whose summands contain only factors ϕ_n with $n\leq j$ and at least one factor ϕ_n with $n\geq 2$.

Let ϕ_k with $k\geq 2$ be the first non-vanishing coefficient. Then either

$$\phi_0 = -\frac{a}{1+k} \quad \text{and} \quad \phi_0 = -\frac{a}{1+k/2},$$

and, consequently,

$$\phi_1 = \frac{a\phi_{k+1}}{(-1+k)(1+k)\phi_k} \quad \text{and} \quad \phi_1 = \frac{a\phi_{k+1}}{(-1+k)(1+k/2)\phi_k},$$

respectively.

For any parameter $a\neq 0$ related to the automorphism, we obtain two series of solutions:

If $2a+(k+2)\phi_0=0$ (the second option) then $(a+(j+1)\phi_0)(2a+(j+2)\phi_0)\neq 0$ for all $j\geq k+2$ and therefore all ϕ_j with $j\geq k+2$ can be determined recursively for given parameters $k, \phi_k\neq 0$ and ϕ_{k+1}.

If $a+(k+1)\phi_0=0$ (the first option) then again all ϕ_j with $j\geq k+2$ can be obtained recursively, except for $j=2k+2$. Here an additional parameter ϕ_{2k+4} appears.

All other components in ϕ (and, thus, in f_0 and f_1) can be computed recursively from (9), which can be rewritten as

$$\begin{aligned}(a+(j+k+3)\phi_0)&(2a+(j+k+4)\phi_0)\phi_{j+k+2}\\ &+(j+k)(3a+2(j+k+3)\phi_0)\phi_1\phi_{j+k+1}\\ &+\sum_{\beta=0}^{j-k+2}\sum_{\alpha=0}^{\beta}\frac{(j-\beta-k+4)!}{(j-\beta-k)!}\frac{\phi_{k+\alpha}\phi_{k+\beta-\alpha}\phi_{j-\beta-k+2}}{(j+k+2)(j+k+1)}\\ &+a\sum_{\alpha=0}^{j}\frac{(3j-k-4\alpha+10)(j+2-\alpha)(j+1-\alpha)\phi_{k+\alpha}\phi_{j+2-\alpha}}{(j+k+2)(j+k+1)}=0.\end{aligned}$$

The convergence of the formal solutions follows from an induction argument.

By applying a map of the form

$$x_1 = \frac{c_1 x}{1-cy} \text{ and } y_1 = \frac{c_2 y}{1-cy}$$

we renormalize a solution in such a way that we get $-a=\alpha_0=1$, $f_{1,k-2}=-(k(k-1)\phi_k)/2\alpha_0=1$ and $f_{1,k-1}=-(k(k+1)\phi_{k+1})/2\alpha_0=0$. Thus, up to equivalence, we obtain exactly two series of solutions, such that a solution of the first series is determined by a non-negative integer k and a solution of the second series is determined by a non-negative integer k and a complex number C which is related to ϕ_{2k+4}. In all cases we have

$$f_0(y) = -\frac{1}{6}(y\phi'''+3\phi'')\phi+\frac{1}{6}\phi'' \quad \text{and} \quad f_1(y) = \frac{-\phi''}{2}$$

with the additional symmetry

$$(1+(\phi-1)x)\frac{\partial}{\partial x}+\phi y\frac{\partial}{\partial y},$$

where ϕ satisfies (8) with initial conditions

$$\phi_0=\frac{1}{j+1},\ \phi_1=0 \quad \text{or} \quad \phi_0=\frac{2}{j+2},\ \phi_1=0.$$

The first option corresponds to

$$y''=y^j(y-(x-c)y')^3,$$

which is obtained by shifting the ODE (4) in the x-direction by c. The parameter c can be rescaled by applying the additional isotropic automorphism.

The second option corresponds to shifts of (5),

$$y''=y^jy'(y-(x-c)y')^2+Cy^{2j+2}(y-(x-c)y')^3.$$

In the special case $C=0$ we deduce $f_0\equiv 0$ and

$$(y\phi'''+3\phi'')\phi-\phi''=0.$$

In terms of f_1 the latter equation becomes

$$\left(\frac{f_1}{3f_1+yf_1'}\right)''=-2f_1.$$

7. Shear invariant ODEs with two additional symmetries

If a shear invariant ODE has two additional symmetries then either one of them can be chosen to be isotropic or both give rise to a transitive sub-semigroup on $\mathbb{C}^2$. According to the results of Section 3 the first case leads to three particular series of ODEs, which have only isotropic symmetries. The only ODE (up to equivalence) with two additional isotropic symmetries is

$$y''=(y-xy')^3.$$

Consider the second case. We may assume that there is an infinitesimal non-isotropic automorphism σ in the direction of the line of fixed points of the shear θ. Without loss of generality we have then

$$\sigma=\frac{\partial}{\partial x} \quad \text{and} \quad \theta=(y+a)\frac{\partial}{\partial x}+b\frac{\partial}{\partial y},$$

where $a(x,y)$ and $b(x,y)$ are of at least second order. But then

$$[\sigma,\theta]=\lambda\theta$$

because θ is the only isotropic symmetry. According to the results of Section 6 we conclude that the ODE must be a shift of (4) or (5). Again, only

$$y''=(y-(x-c)y')^3$$

has three-dimensional symmetry.

8. Non-linearizable automorphisms of elliptic CR-manifolds

In [3] we proved that the phenomenon of non-linearizable isotropy takes place on a whole complex curve. As a consequence of the classification results from above we prove here the following converse statement for an elliptic CR manifold M with the additional property that all infinitesimal automorphisms are globally defined.

Theorem 2. *Let M be an elliptic CR-manifold with non-linearizable isotropy group at $p\in M$. If M is neither equivalent to the quadric*

$$w_1-\overline{w}_2-z_1\bar{z}_2=0$$

nor to the quartic

$$w_1+w_1^2\bar{z}_2^2-(\overline{w}_2-z_1\bar{z}_2)^2=0,$$

then there exists a neighbourhood U of 0 *such that* $\operatorname{Aut}_q M$ *is linearizable for $q\in U$ outside a complex curve γ.*

Proof. Let $q\in M$ be a point with non-linearizable isotropy. If M is not equivalent neither to the quadric nor to the quartic then there is a single shear at q, which either coincides with the single shear in 0 or it provides an additional symmetry at 0. In the first case q is a fixed point of $y\partial/\partial x$ and therefore belongs to $\gamma=\{(x,y,p):y=p=0\}$.

In the second case M corresponds to one of the ODEs listed above. But then only the shear has non-isolated fixed points outside 0. All these fixed points belong to $\{(x,y,p):y=p=0\}$. □

The quartic can be characterized by the following property.

Proposition 2. *The set of points at the quartic with non-linearizable isotropy is the complex hypersurface $\Gamma=\{(x,y,p):y=xp\}$.*

Proof. The mapping

$$(x,y)=(a(x_1+1)+cy_1, b(x_1+1)+dy_1)$$

takes the the point $(a,b,b/a)$ to $(0,0,0)$ and the ODE $y''=(y-xy')^3$ again to an ODE that admits a shear, namely to

$$y''=(ad-bc)^2(y-(x+1)y')^3.$$

Since the orbit of 0 under these mappings is the hypersurface Γ, at all points of Γ the isotropy group is non-linearizable.

We show that the isotropy of the quartic is linearizable (even trivial) at any point outside Γ. Any infinitesimal automorphism of the quartic at 0 has the form

$$(\alpha x+\beta y)\frac{\partial}{\partial x}+(\delta x-\alpha y)\frac{\partial}{\partial y}+(\delta-2\alpha p-\beta p^2)\frac{\partial}{\partial p}.$$

If the discriminant $\Delta=\alpha^2+\beta\delta$ is different from 0 then fixed points occur only for $x=y=0$. If the discriminant vanishes we distinguish the two subcases $\beta\neq 0$ and $\beta=0$. In the first subcase we find fixed points for $\alpha x+\beta y=0$ and $p=-\alpha/\beta$. This implies that $y-xp=0$. If $\beta=0$ we conclude that $\alpha=\delta=0$. Then only the identical automorphism has fixed points other than 0. □

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