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Maximal regularity via reverse Hölder inequalities for elliptic systems of n-Laplace type involving measures

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Abstract. In this note, we consider the regularity of solutions of the nonlinear elliptic systems of n-Laplacian type involving measures, and prove that the gradients of the solutions are in the weak Lebesgue space $L^{n,\infty}$. We also obtain the *a priori* global and local estimates for the $L^{n,\infty}$-norm of the gradients of the solutions without using BMO-estimates. The proofs are based on a new lemma on the higher integrability of functions.

1. Introduction

In this note, we consider the regularity of solutions $u\colon \Omega\to\mathbf{R}^m$ of the nonlinear elliptic system

$$\begin{cases} -\operatorname{div}\sigma(x,u,Du)=\mu & \text{in } \mathcal{D}'(\Omega), \\ u=0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Here Ω is an open set in $\mathbf{R}^n$ and μ a Radon measure on Ω with finite mass. The prototypical problem is the n-Laplace system

$$-\operatorname{div}(|Du|^{n-2}Du)=\mu.$$

The precise assumptions on the function σ in system (1.1) are listed in Section 2; throughout this note, we assume that σ satisfies the assumptions (H0)–(H3), and one of the conditions (i)–(iii).

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It was proven in [8] that the solutions u of system (1.1) enjoys the maximal regularity: the derivative is in the weak L^n space, $Du \in L^{n,\infty}(\Omega)$. Moreover, in [8] the authors establish an *a priori* global estimate for the $L^{n,\infty}$-norm of Du. In this note (Theorem 1.2) we give a new proof for this global estimate; we also obtain local(1) and boundary estimates.

For the global and boundary estimates, we need to impose a condition on the complement of Ω. We assume that $\mathbf{R}^n \setminus \Omega$ is *geometrically dense*, i.e. there is a constant K so that

$$|B(x,r) \setminus \Omega| \geq K r^n$$

for all $x \in \mathbf{R}^n \setminus \Omega$ and $r > 0$. Actually, we can relax this geometrical density condition on the domain in Theorem 1.2 below; we can assume that the complement of Ω is *uniformly n-fat*, see Section 2 for the definition.

Theorem 1.2. *Let $\Omega \subset \mathbf{R}^n$ be a bounded open set such that its complement is geometrically dense. Let μ be an $\mathbf{R}^m$-valued Radon measure on Ω with finite mass. Then system* (1.1) *has a solution $u \in W_0^{1,q}(\Omega; \mathbf{R}^m)$ for all $q < n$. Moreover, Du belongs to the weak Lebesgue space $L^{n,\infty}(\Omega; \mathbf{R}^{m \times n})$, and obeys both the global estimate*

$$\|Du\|_{L^{n,\infty}(\Omega)} \leq C\big(\|\mu\|_{\mathcal{M}(\Omega)}^{1/(n-1)} + \|g\|_{L^1(\Omega)}^{1/n}\big), \tag{1.3}$$

and the local estimate

$$\|Du\|_{L^{n,\infty}(B(a,r))} \leq C\left(\|\mu\|_{\mathcal{M}(B(a,2r))}^{1/(n-1)} + \|g\|_{L^1(B(a,2r))}^{1/n} + \frac{1}{r}\left(\int_{B(a,2r)} |\nabla u|^{n/2}\, dx\right)^{2/n}\right) \tag{1.4}$$

for all balls $B(a,r) \subset \mathbf{R}^n$, where

$$g = |\gamma_3| + |\gamma_4|^{n/(n-1)} + \gamma_5 |Du|^s + |\gamma_6|;$$

here the constant C depends only on the dimension n, the geometric density constant K, and the operator structure constants γ_1 and γ_2.

The function g in Theorem 1.2 depends on the assumptions on the function σ, see Section 2. We denote by $\|\mu\|_{\mathcal{M}(\Omega)}$ the total mass of a Radon measure μ in Ω.

In Theorem 1.2, we assume that u is defined on the whole space $\mathbf{R}^n$ by setting $u = 0$ outside Ω. The local estimate (1.4) is true for all balls in $\mathbf{R}^n$. Thus (1.4) gives not only the interior estimate, but also the boundary estimate. These interior and boundary estimates are both new.

(1) Giuseppe Mingione informed the authors that he found a different, new proof for the local estimate, see Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **6** (2007), 195–261.

The outline of our proof of Theorem 1.2 is the following. We first show in Lemma 4.2 that Du satisfies a kind of reverse Hölder inequality. Our Lemma 4.2 resembles [8, Lemma 3.3], but the main new feature here is that our estimate does not involve the BMO-norm of u. Consequently, we first derive the weak L^n estimates (1.3) and (1.4) for Du, and the BMO-norm estimate for u then follows by the Poincaré inequality. The argument in [8] is in reverse order: they first establish the BMO-estimate for u via a delicate blow up argument, and their weak L^n estimate for Du relies heavily on the BMO-estimate.

Our key observation is that the above mentioned reverse Hölder type inequality for Du enables us to prove higher integrability. In other words, the weak L^n-estimate of Du follows from two lemmas on the higher integrability of functions: a global version and a local version.

Lemma 1.5. *Let $f\in L^p(\mathbf{R}^n)$ be nonnegative, $1<p<\infty$, and μ be a nonnegative measure with finite mass in $\mathbf{R}^n$. Suppose that there are constants $\gamma>0$ and $p<q<\infty$ such that the inequality*

$$\left(\fint_B f^p\,dx\right)^{1/p}\le\gamma\fint_{2B} f\,dx+\left(\frac{\mu(2B)}{|2B|}\right)^{1/q} \tag{1.6}$$

holds for all balls B in $\mathbf{R}^n$. Then there is a constant $\delta=\delta(n,p,\gamma)>0$ such that $f\in L^{q,\infty}(\mathbf{R}^n)$ whenever $q-p<\delta$. Moreover,

$$\|f\|^q_{L^{q,\infty}(\mathbf{R}^n)}\le c\|\mu\|_{\mathcal{M}(\mathbf{R}^n)}, \tag{1.7}$$

where $c=c(n,p,q)>0$.

The following local version of the above lemma will be proved using Lemma 1.5.

Lemma 1.8. *Let $f\in L^p_{\mathrm{loc}}(\Omega)$ be nonnegative, $1<p<\infty$, and μ be a nonnegative Radon measure in $\Omega\subset\mathbf{R}^n$. Suppose that there are constants $\gamma>0$ and $p<q<\infty$ such that the inequality*

$$\left(\fint_B f^p\,dx\right)^{1/p}\le\gamma\fint_{2B} f\,dx+\left(\frac{\mu(2B)}{|2B|}\right)^{1/q} \tag{1.9}$$

holds for all balls B with $2B\Subset\Omega$. Then there is a constant $\delta=\delta(n,p,\gamma)>0$ such that $f\in L^{q,\infty}_{\mathrm{loc}}(\Omega)$ whenever $q-p<\delta$. Moreover,

$$\|f\|_{L^{q,\infty}(B)}\le c|B|^{1/q-1/p}\|f\|_{L^p(2B)}+c\|\mu\|^{1/q}_{\mathcal{M}(2B)} \tag{1.10}$$

for any balls B with $2B\subset\Omega$; here $c=c(n,p,q)>0$.

Gehring's pioneering work [11] initiated intensive research on higher integrability of functions satisfying reverse Hölder inequalities. We refer to the monographs [12] and [16] for detailed discussions of the Gehring lemma and its applications in analysis. The original Gehring lemma is in the setting of Lebesgue spaces L^p. The paper [15] significantly extends Gehring's lemma to the framework of Orlicz spaces. Recently, a new higher integrability lemma of this type was proven in [3] and can be considered as a limiting case of the Gehring lemma.

Lemmas 1.5 and 1.8 above are in the setting of Lorentz spaces. They are new and, we believe, of independent interest and might have other applications. One of our motivations to write this note is to publish those lemmas. The proof is based on a modification of Gehring's original idea.

The paper is organized as follows. In Section 2, we fix our notation and list the assumptions on the operator. The proofs of the higher integrability Lemmas 1.5 and 1.8 are in Section 3. A proof for Theorem 1.2 is given in Section 4. There we also comment on the proof of Theorem 1.2 under the assumption that the complement of Ω is uniformly n-fat.

The p-Laplacian equations and systems with measure-valued right-hand side have been intensively studied, see e.g. [2], [4], [5], [6], [10], [13], [17], [18], [20], [21] and [22]. The existence of solutions is well known, but the problem of uniqueness of solutions is largely open except for the case $p=n$ ([8], [13] and [25]). In [8] the uniqueness was reached through the regularity result Theorem 1.2; in [13] there is another approach based on the nonlinear Hodge decomposition.

2. Notation and the assumptions on the operator

Let $\Omega\subset\mathbf{R}^n$ be measurable and $1<p<\infty$. Let $|E|$ denote the Lebesgue measure of E. The integral average of a measurable function u over E is written as

$$u_E = \fint_E u\,dx = \frac{1}{|E|}\int_E u\,dx.$$

We say that a measurable function u belongs to the *weak Lebesgue space* $L^{p,\infty}(\Omega)$ if

$$\|u\|_{L^{p,\infty}(\Omega)} := \sup_E |E|^{-(p-1)/p}\int_E |u|\,dx < \infty,$$

where the supremum is taken over all measurable subsets E of Ω of positive and finite measure. The space $L^{p,\infty}(\Omega)$ is a Banach space under this norm. The above defined norm is equivalent to the quasinorm

$$|\!|\!| u |\!|\!|_{L^{p,\infty}(\Omega)} := \sup_{t>0} t|\{x\in\Omega : |u(x)|>t\}|^{1/p}.$$

An integrable function u is said to be in $\mathrm{BMO}(\Omega)$, the space of functions of *bounded mean oscillation*, if

$$[u]_{\mathrm{BMO}(\Omega)} := \sup \fint_{B(a,r)} |u - u_{B(a,r)}|\, dx < \infty,$$

where the supremum is taken over all balls $B(a,r) \subset \Omega$. It follows from the Poincaré inequality that

$$[u]_{\mathrm{BMO}(\Omega)} \le c(n) \|\nabla u\|_{L^{n,\infty}(\Omega)}.$$

The p-*capacity* of a compact set E in an open set D is the number

$$\mathrm{cap}_p(E, D) = \inf_{\varphi} \int_D |\nabla \varphi|^p\, dx,$$

where the infimum is taken over all $\varphi \in W_0^{1,p}(D)$ with $\varphi = 1$ on an open neighborhood of E. It is known that

$$\mathrm{cap}_p(B(a,r), B(a,2r)) \approx r^{n-p} \quad \text{for } 1 < p \le n.$$

See [1] or [14] for more information on capacities.

We say that a (closed) set $E \subset \mathbf{R}^n$ is *uniformly* p*-fat*, if there is a constant $K > 0$ such that

$$\mathrm{cap}_p(\overline{B}(x,r) \cap E, B(x,2r)) \ge K r^{n-p}$$

for all $x \in E$ and $0 < r < \mathrm{diam}(E)$. It follows from the well known capacity density estimates (see [14]) that a set E is uniformly p-fat for all $p > 1$ if it is geometrically dense.

It follows from Hölder's inequality that a uniformly p-fat set is also uniformly q-fat for all $q > p$. A fundamental property of uniformly p-fat sets is the following deep result [23]: a closed uniformly p-fat set E with constant K is also uniformly p_0-fat for some $1 \le p_0 = p_0(n, p, K) < p$.

We shall need the following form of the Sobolev–Poincaré inequality.

Lemma 2.1. *Suppose that $\mathbf{R}^n \setminus \Omega$ is uniformly q-fat and $p \ge q$. Let $B = B(a,r)$ be a ball such that $B(a,r/2) \setminus \Omega \ne \varnothing$. If $q \le p < n$, then*

$$\left(\int_B |u|^{np/(n-p)}\, dx \right)^{(n-p)/np} \le c \left(\int_B |\nabla u|^p\, dx \right)^{1/p}$$

for every $u \in W_0^{1,p}(\Omega)$; here $c = c(n, K) > 0$.

Proof. Using a capacitary version of the Sobolev–Poincaré inequality

$$\biggl(\fint_B |u|^{np/(n-p)}\,dx\biggr)^{(n-p)/np} \le c\biggl(\frac{1}{\operatorname{cap}_p(\{x\in\overline{B}:u(x)=0\},2B)}\int_B |\nabla u|^p\,dx\biggr)^{1/p}$$

(see [1] or [19, Lemma 3.1]), it suffices to estimate the capacity of the zero set of u in B. That estimate is obtained by the uniform fatness condition, since $u=0$ in the complement of Ω; hence by the Poincaré inequality

$$\biggl(\frac{\operatorname{cap}_p(\overline{B}\setminus\Omega,2B)}{r^{n-p}}\biggr)^{1/p} \ge \biggl(\frac{\operatorname{cap}_q(\overline{B}\setminus\Omega,2B)}{r^{n-q}}\biggr)^{1/q} \ge c\biggl(\frac{\operatorname{cap}_q(\overline{B}_1\setminus\Omega,2B_1)}{r^{n-q}}\biggr)^{1/q} \ge c_0>0,$$

where the constant c_0 depends only on n and K, and B_1 is a ball of radius $r/2$ with center in $B\setminus\Omega$. For the equivalence of the capacities above see [14, Lemma 2.16]. □

Finally, we assume that the function σ satisfies the following hypotheses as in [8]:

(H0) (Continuity) $\sigma\colon \Omega\times\mathbf{R}^m\times\mathbf{R}^{m\times n}\to\mathbf{R}^{m\times n}$ is a Carathéodory function, that is, the mapping $x\mapsto\sigma(x,u,p)$ is measurable for every (u,p), and the mapping $(u,p)\mapsto\sigma(x,u,p)$ is continuous for almost every $x\in\Omega$.

(H1) (Monotonicity) For all $x\in\Omega$, $u\in\mathbf{R}^m$ and all $F,G\in\mathbf{R}^{m\times n}$,

$$(\sigma(x,u,F)-\sigma(x,u,G)):(F-G)\ge 0.$$

Here we used the notation $F{:}G=\sum_{j=1}^m\sum_{k=1}^n a_{jk}b_{jk}$ if $F=(a_{jk})$ and $G=(b_{jk})$.

(H2) (Coercivity and growth) There exist constants $\gamma_1>0$ and $\gamma_2\ge 0$, and functions $\gamma_3\in L^1(\Omega)$ and $\gamma_4\in L^{n/(n-1)}(\Omega)$ such that for all $x\in\Omega$, $u\in\mathbf{R}^m$ and $F\in\mathbf{R}^{m\times n}$,

$$\begin{aligned}\sigma(x,u,F):F &\ge \gamma_1|F|^n-\gamma_3(x),\\ |\sigma(x,u,F)| &\le \gamma_2|F|^{n-1}+\gamma_4(x).\end{aligned}$$

(H3) (Structure condition) There exist constants $1\le s<n$ and $\gamma_5\ge 0$, and a function $\gamma_6\in L^1(\Omega)$ such that for all $x\in\Omega$, $u\in\mathbf{R}^m$ and $F\in\mathbf{R}^{m\times n}$ the inequality

$$\sigma(x,u,F):MF\ge -\gamma_5|F|^s-\gamma_6(x)$$

holds for all matrices $M\in\mathbf{R}^{m\times m}$ of the form $M=\mathrm{Id}-a\otimes a$ with $|a|\le 1$.

Moreover, to prove the existence of solutions of system (1.1), we need one of the following conditions on σ:

(i) $F\mapsto\sigma(x,u,F)$ is a C^1 function.

(ii) There exists $W\colon \Omega\times\mathbf{R}^m\times\mathbf{R}^{m\times n}\to\mathbf{R}$ with $\sigma(x,u,F)=\partial W/\partial F(x,u,F)$ such that $F\mapsto W(x,u,F)$ is convex and C^1.

(iii) σ is strictly monotone, i.e., σ is monotone and

$$(\sigma(x,u,F)-\sigma(x,u,G)):(F-G)=0$$

if and only if $F=G$.

3. Proofs for Lemmas 1.5 and 1.8

Proof of Lemma 1.5. Taking the supremum over all balls $B\subset\mathbf{R}^n$, we have by (1.6),

$$M(f^p)(x)^{1/p}\le\gamma M(f)(x)+M_\mu(x)^{1/q}$$

for every $x\in\mathbf{R}^n$, where

$$M(h)(x)=\sup_{r>0}\fint_{B(x,r)}|h|\,dy\quad\text{and}\quad M_\mu(x)=\sup_{r>0}\frac{\mu(B(x,r))}{|B(x,r)|}$$

are the Hardy-Littlewood maximal functions of the integrable function h and the measure μ, respectively. Hence

$$\begin{aligned}|\{x\in\mathbf{R}^n:M(f^p)(x)^{1/p}>\lambda\}|&\le|\{x\in\mathbf{R}^n:\gamma M(f)(x)>\lambda/2\}|\\&\quad+|\{x\in\mathbf{R}^n:M_\mu(x)^{1/q}>\lambda/2\}|\end{aligned}$$

for $\lambda>0$. By [9, Proposition 2.1] or [24, Chapter 1, Section 5.2(b) and Section 1.5], we obtain that

$$\frac{c(n)}{\lambda^p}\int_{\{f>\lambda\}}f^p\,dx\le\frac{c(n)\gamma}{\lambda}\int_{\{4\gamma f>\lambda\}}f\,dx+\frac{c(n)}{\lambda^q}\|\mu\|_{\mathcal{M}(\mathbf{R}^n)}\quad\text{for all }\lambda>0.$$

Let $\delta>0$; we will specify the choice of $\delta=\delta(n,p,\gamma)>0$ later. Assume that $q>p$ with $q-p<\delta$. Fixing $t>0$ for now, we multiply both sides of the inequality above by $\lambda^{p-1+2\delta}$, and integrate with respect to λ over $(0,t)$. Changing the order of the integration, we arrive at

$$\begin{aligned}\int_{\mathbf{R}^n}f^p\int_0^{\min(f,t)}\lambda^{-1+2\delta}\,d\lambda\,dx&\le c(n)\gamma\int_{\mathbf{R}^n}f\int_0^{\min(4\gamma f,t)}\lambda^{p-2+2\delta}\,d\lambda\,dx\\&\quad+c(n)\|\mu\|_{\mathcal{M}(\mathbf{R}^n)}\int_0^t\lambda^{p-q-1+2\delta}\,d\lambda,\end{aligned}$$

that is, for all $t>0$,

$$\begin{aligned}\frac{1}{2\delta}\int_{\mathbf{R}^n}f^p\min(f,t)^{2\delta}\,dx&\le\frac{c(n)\gamma}{p-1+2\delta}\int_{\mathbf{R}^n}f\min(4\gamma f,t)^{p-1+2\delta}\,dx\\&\quad+\frac{c(n)}{p-q+2\delta}t^{p-q+2\delta}\|\mu\|_{\mathcal{M}(\mathbf{R}^n)}.\end{aligned}\tag{3.1}$$

We will show that if δ is small enough, the first integral on the right-hand side can be absorbed into the term on the left-hand side. Indeed, we assume, as we clearly may, that $4\gamma \geq 1$. Then

$$f\min(4\gamma f,t)^{p-1+2\delta} \leq (4\gamma)^{p-1+2\delta} f\min(f,t)^{p-1+2\delta} \leq (4\gamma)^{p-1+2\delta} f^p \min(f,t)^{2\delta},$$

and if $0<\delta<1$ is so small that

$$\frac{2c(n)\delta\gamma}{p-1+2\delta}(4\gamma)^{p-1+2\delta} \leq \frac{2c(n)\delta\gamma}{p-1}(4\gamma)^{p+1} \leq \frac{1}{2}, \tag{3.2}$$

(3.1) yields that

$$\int_{\mathbf{R}^n} f^p \min(f,t)^{2\delta}\,dx \leq \frac{c(n)\delta}{p-q+2\delta}\|\mu\|_{\mathcal{M}(\mathbf{R}^n)} t^{p-q+2\delta}.$$

This implies that

$$t^q|\{x\in\mathbf{R}^n : f(x)>t\}| \leq \frac{c(n)\delta}{p-q+2\delta}\|\mu\|_{\mathcal{M}(\mathbf{R}^n)} \leq c(n)\|\mu\|_{\mathcal{M}(\mathbf{R}^n)},$$

and the lemma is proven. □

Proof of Lemma 1.8. We fix $B_0=B(x_0,r_0)\Subset\Omega$. Let $d(x)=\operatorname{dist}(x,\mathbf{R}^n\setminus B_0)$ and χ_E be the characteristic function of the set $E\subset\mathbf{R}^n$. We define an auxiliary function

$$\tilde{f}(x)=d(x)^{n/p}f(x),$$

and a measure

$$d\tilde{\mu}(x)=c(n,p,q)d(x)^{nq/p}\,d\mu+c(n,p,q)\biggl(\int_{B_0} f(y)^p\,dy\biggr)^{q/p}\chi_{B_0}(x)\,dx, \tag{3.3}$$

that is,

$$\tilde{\mu}(E)=c(n,p,q)\int_E d(x)^{nq/p}\,d\mu+c(n,p,q)|E\cap B_0|\biggl(\int_{B_0} f(y)^p\,dy\biggr)^{q/p}$$

for all Borel sets $E\subset\mathbf{R}^n$, where $c(n,p,q)>0$ is a constant to be suitably chosen. We claim that

$$\biggl(\fint_B \tilde{f}^p\,dx\biggr)^{1/p} \leq 4^{n/p}\gamma\fint_{2B}\tilde{f}\,dx+\biggl(\frac{\tilde{\mu}(2B)}{|2B|}\biggr)^{1/q} \tag{3.4}$$

for all balls $B\subset\mathbf{R}^n$. To this end, we may assume that B meets B_0; otherwise (3.4) is trivial. We treat two cases separately.

Case 1. Assume that $3B\subset B_0$. Then an elementary geometric consideration yields

$$\max_{x\in B} d(x)\le 4\min_{x\in 2B} d(x).$$

Thus it follows from the assumption (1.9) that

$$\begin{aligned}\left(\fint_B \tilde{f}^p\,dx\right)^{1/p} &\le \max_{x\in B} d(x)^{n/p}\left(\fint_B f^p\,dx\right)^{1/p}\\ &\le 4^{n/p}\min_{x\in 2B} d(x)^{n/p}\left(\gamma\fint_B f\,dx+\left(\frac{\mu(2B)}{|2B|}\right)^{1/q}\right)\\ &\le 4^{n/p}\gamma\fint_{2B}\tilde{f}\,dx+\left(\frac{\tilde{\mu}(2B)}{|2B|}\right)^{1/q},\end{aligned}$$

provided we choose the constant $c(n,p,q)\ge 4^{nq/p}$ in (3.3).

Case 2. Suppose that $3B$ is not contained in B_0; recall that B intersects B_0. Therefore

$$\max_{x\in B} d(x)\le \max_{x\in 2B} d(x)\le c(n)|2B\cap B_0|^{1/n}.$$

We conclude that

$$\begin{aligned}\left(\fint_B \tilde{f}^p\,dx\right)^{1/p} &\le \max_{x\in B} d(x)^{n/p}\left(\frac{1}{|B|}\int_{B\cap B_0} f^p\,dx\right)^{1/p}\\ &\le 2^n c(n)^{n/p}\left(\frac{|2B\cap B_0|}{|2B|}\int_{B_0} f^p\,dx\right)^{1/p}\\ &\le 2^n c(n)^{n/p}\left(\frac{|2B\cap B_0|}{|2B|}\right)^{1/q}\left(\int_{B_0} f^p\,dx\right)^{1/p}\\ &\le \left(\frac{\tilde{\mu}(2B)}{|2B|}\right)^{1/q},\end{aligned}$$

provided that $c(n,p,q)\ge 2^{nq}c(n)^{nq/p}$. Combining these two cases proves inequality (3.4), with the choice of $c(n,p,q)=\max(4^{nq/p},2^{nq}c(n)^{nq/p})$.

Now we may apply Lemma 1.5 and obtain

$$\|\tilde{f}\|^q_{L^{q,\infty}(\mathbf{R}^n)}\le c(n,p,q)\|\tilde{\mu}\|_{\mathcal{M}(\mathbf{R}^n)},$$

if $q-p<\delta=\delta(n,p,\gamma)>0$. Using the definitions of $\tilde{f}$ and $\tilde{\mu}$, we arrive at the desired estimate

$$\|f\|_{L^{q,\infty}((1/2)B_0)}\le c|B_0|^{1/q-1/p}\|f\|_{L^p(B_0)}+c\|\mu\|^{1/q}_{\mathcal{M}(B_0)}.$$

The proof is now complete. $\square$

Remark 3.5. From (3.2) it is clear that we only need the constant $\delta>0$ of Lemmas 1.5 and 1.8 to satisfy

$$\delta \leq \frac{p-1}{c(n)(4\gamma)^{p+2}}.$$

If $1<p_1<p<p_2$, it follows that δ can be chosen to depend only on n, γ, p_1 and p_2;

$$\delta \leq \frac{p_1-1}{c(n)(4\gamma)^{p_2+2}};$$

in our application, we will have $p_1=\frac{3}{2}$ and $p_2=2$.

4. Proof of Theorem 1.2

For the proof of Theorem 1.2, we need the following two lemmas.

The first one, a Caccioppoli type estimate, is Lemma 2.2 in [8]. Here we use the following notation: $\mathcal{D}^{1,n}(\Omega;\mathbf{R}^m)$ consists of functions in $W^{1,n}_{\mathrm{loc}}(\Omega;\mathbf{R}^m)$ that can be approximated in the seminorm $\|Du\|_{L^n(\Omega)}$ by functions from $C_0^\infty(\Omega;\mathbf{R}^m)$. We always tacitly assume that all functions in $\mathcal{D}^{1,n}(\Omega;\mathbf{R}^m)$ are defined as zero in $\mathbf{R}^n\setminus\Omega$; the same assumption applies for f and g below. The proof of the lemma is standard; see [8].

Lemma 4.1. *Let u be a solution of* (1.1) *with $f\in L^1(\Omega;\mathbf{R}^m)\cap C^\infty(\Omega;\mathbf{R}^m)$ in place of μ. Let $g=|\gamma_3|+|\gamma_4|^{n/(n-1)}+\gamma_5|Du|+|\gamma_6|$, and $0<\rho<r$. There exists a constant C_1, depending only on γ_1,γ_2 and n, such that the following inequalities hold*:

(i) (*Interior estimate*) *For all balls $B(a,r)\subset\Omega$, $\beta\in\mathbf{R}^m$ and $\alpha>0$,*

$$\int_{\{|u-\beta|<\alpha\}\cap B(a,\rho)} |Du|^n\,dx \leq \frac{C_1}{(r-\rho)^n}\int_{B(a,r)}|u-\beta|^n\,dx + C_1\int_{B(a,r)}(\alpha|f|+g)\,dx.$$

(ii) (*Boundary estimate*) *For all balls $B(a,r)\subset\mathbf{R}^n$ and $\alpha>0$,*

$$\int_{\{|u|<\alpha\}\cap B(a,\rho)} |Du|^n\,dx \leq \frac{C_1}{(r-\rho)^n}\int_{B(a,r)}|u|^n\,dx + C_1\int_{B(a,r)}(\alpha|f|+g)\,dx.$$

The second lemma gives a quantitative estimate for the L^p-norm of Du for all $p<n$. While it is similar to Lemma 3.3 of [8], the novelty here is that in contrast to [8], the BMO-norm of u is not involved in the proof of the following lemma. As mentioned in the introduction, we will directly obtain an estimate for the weak L^n-norm of Du, and as a consequence via the Poincaré inequality, an estimate for the BMO-norm of u.

Lemma 4.2. *Assume that* $u\in\mathcal{D}^{1,n}(\Omega;\mathbf{R}^m)$ *satisfies the Caccioppoli inequalities* (i) *and* (ii) *of Lemma* 4.1 *with* $f\in L^1(\Omega;\mathbf{R}^m)$ *and* $g\in L^1(\Omega)$. *If* $p\in[n/2,n)$ *and* $r>0$, *then*

$$\left(\fint_{B(a,r)}|Du|^p\,dx\right)^{1/p}\le C_2\left(\fint_{B(a,8r)}|Du|^{n/2}\,dx\right)^{2/n}+C_2\left(\fint_{B(a,8r)}|g|\,dx\right)^{1/n}+\frac{C_3}{r}\left(\int_{B(a,8r)}|f|\,dx\right)^{1/(n-1)}; \tag{4.3}$$

here $C_2=C_2(n,K,C_1)>0$ *and* $C_3=C_3(n,K,C_1,p)>0$.

Remark 4.4. For the proof of Theorem 1.2 it is important to observe that the constant C_2 is independent of p.

Proof. Case 1. Assume that $B(a,2r)\subset\Omega$. We fix a nonnegative constant T, to be chosen soon. Define

$$S_0=\{x\in B(a,r):|u(x)-u_{B(a,r)}|\le T\},$$

and for $k=1,2,...$,

$$S_k=\{x\in B(a,r):2^{k-1}T<|u(x)-u_{B(a,r)}|\le 2^kT\}.$$

By the Sobolev–Poincaré inequality, we have for each $k=1,2,...$,

$$\begin{aligned}\int_{B(a,r)}|Du|^p\,dx&\ge A_p\left(\int_{B(a,r)}|u-u_{B(a,r)}|^{np/(n-p)}\,dx\right)^{(n-p)/n}\\&\ge A_p2^{p(k-1)}T^p|S_k|^{(n-p)/n}.\end{aligned} \tag{4.5}$$

Here $A_p>0$ is a constant, depending solely on n and p. Now we choose T. Let

$$T=\left(\frac{1}{A_pr^{n-p}}\int_{B(a,r)}|\nabla u|^p\,dx\right)^{1/p}. \tag{4.6}$$

Hence, (4.5) yields

$$|S_k|^{(n-p)/n}\le 2^{-p(k-1)}r^{n-p},$$

and we have by the Hölder inequality for $k=1,2,...$,

$$\int_{S_k}|Du|^p\,dx\le|S_k|^{(n-p)/n}\left(\int_{S_k}|Du|^n\,dx\right)^{p/n}\le 2^{-p(k-1)}r^{n-p}\left(\int_{S_k}|Du|^n\,dx\right)^{p/n}.$$

Now we combine this with the estimate (given by Lemma 4.1)

$$\int_{S_k} |Du|^n \, dx \le \frac{C_1}{r^n} \int_{B(a,2r)} |u - u_{B(a,r)}|^n \, dx + C_1 \int_{B(a,2r)} (2^k T|f| + |g|) \, dx$$

to obtain for $k=1,2,...,$

$$\int_{S_k} |Du|^p \, dx \le 2^{-p(k-1)} r^{n-p} \left(\frac{C_1}{r^n} \int_{B(a,2r)} |u - u_{B(a,r)}|^n \, dx + C_1 \int_{B(a,2r)} (2^k T|f| + |g|) \, dx \right)^{p/n}. \tag{4.7}$$

For $k=0$ we employ Hölder's inequality and (i) of Lemma 4.1 together with the fact that $|S_0| \le |B(a,r)| = c(n) r^n$ to obtain the estimate

$$\begin{aligned} \int_{S_0} |Du|^p \, dx &\le |S_0|^{n-p/n} \left(\int_{S_0} |Du|^n \, dx \right)^{p/n} \\ &\le c(n) r^{n-p} \left(\frac{C_1}{r^n} \int_{B(r,2r)} |u - u_{B(a,r)}|^n \, dx + C_1 \int_{B(a,2r)} (T|f| + |g|) \, dx \right)^{p/n}. \end{aligned}$$

We used the fact that $c(n)^{1-p/n} \le c(n)$. Now by taking the sum over $k=0,1,2,...,$ we obtain

$$\begin{aligned} \int_{B(a,r)} |Du|^p \, dx \le & \left[c(n) + \sum_{k=1}^{\infty} 2^{-p(k-1)} \right] r^{n-p} \left(\frac{C_1}{r^n} \int_{B(a,2r)} |u - u_{B(a,r)}|^n \, dx \right)^{p/n} \\ & + \left[c(n) + \sum_{k=1}^{\infty} 2^{-p(k-1)} \right] r^{n-p} \left(C_1 \int_{B(a,2r)} |g| \, dx \right)^{p/n} \\ & + \left[c(n) + \sum_{k=1}^{\infty} 2^{-p(k-1)+pk/n} \right] r^{n-p} \left(C_1 T \int_{B(a,2r)} |f| \, dx \right)^{p/n}. \end{aligned}$$

The sums in the above estimate are bounded by a constant depending only on n since $1<p<n$. Thus,

$$\begin{aligned} \left(\fint_{B(a,r)} |Du|^p \, dx \right)^{1/p} \le & \frac{c(n) C_1^{1/n}}{r^2} \left(\int_{B(a,r)} |u - u_{B(a,r)}|^n \, dx \right)^{1/n} \\ & + c(n) C_1^{1/n} \left(\fint_{B(a,2r)} |g| \, dx \right)^{1/n} \\ & + \frac{c(n) C_1^{1/n} T^{1/n}}{r} \left(\int_{B(a,2r)} |f| \, dx \right)^{1/n}. \end{aligned} \tag{4.8}$$

We estimate the first integral in the right-hand side with the help of the Sobolev–Poincaré inequality

$$\left(\int_{B(a,2r)} |u-u_{B(a,r)}|^n\,dx\right)^{1/n} \le c(n)\left(\int_{B(a,2r)} |Du|^{n/2}\,dx\right)^{2/n}, \tag{4.9}$$

and the third integral is estimated with the aid of Young's inequality:

$$\begin{aligned}\frac{1}{r}&\left[(A_p^{1/np}T^{1/n})\left(A^{-1/np}c(n)C_1^{1/n}\left(\int_{B(a,2r)}|f|\,dx\right)^{1/n}\right)\right]\\ &\le \frac{\delta A_p^{1/p}}{r}T+c(\delta)\frac{C_1^{1/(n-1)}}{rA_p^{1/p(n-1)}}\left(\int_{B(a,2r)}|f|\,dx\right)^{1/(n-1)}\\ &\le \frac{1}{2}\left(\fint_{B(a,r)}|Du|^p\,dx\right)^{1/p}+\frac{C_3}{r}\left(\int_{B(a,2r)}|f|\,dx\right)^{1/(n-1)};\end{aligned}$$

here the last inequality holds if δ is small enough. Recall that T was defined in (4.6). Thus we arrive at the inequality

$$\begin{aligned}\left(\fint_{B(a,r)}|Du|^p\,dx\right)^{1/p} &\le C_2\left(\fint_{B(a,2r)}|Du|^{n/2}\,dx\right)^{2/n}+C_2\left(\fint_{B(a,2r)}|g|\,dx\right)^{1/n}\\ &\quad+\frac{C_3}{r}\left(\int_{B(a,2r)}|f|\,dx\right)^{1/(n-1)},\end{aligned} \tag{4.10}$$

and the lemma is proved in this case.

Case 2. Assume that $B(a,2r)\setminus\Omega\neq\varnothing$. We proceed as in Case 1; we list only the necessary modifications here. Define

$$S_0=\{x\in B(a,4r):|u(x)|\le T\},$$

and

$$S_k=\{x\in B(a,4r):2^{k-1}T<|u(x)|\le 2^kT\},\quad k=1,2,\ldots.$$

Recall that $u=0$ in $\mathbf{R}^n\setminus\Omega$. As $\mathbf{R}^n\setminus\Omega$ is geometrically dense and hence is uniformly p-fat for every $p>1$, we can apply the Sobolev–Poincaré inequality (2.1) to obtain for each $k=1,2,\ldots$,

$$\begin{aligned}\int_{B(a,4r)}|Du|^p\,dx &\ge A_p\left(\int_{B(a,4r)}|u|^{np/(n-p)}\,dx\right)^{(n-p)/n}\\ &\ge A_p2^{p(k-1)}T^p|S_k|^{(n-p)/n}.\end{aligned} \tag{4.11}$$

Here $A_p>0$ is a constant that depends only on n,p, and K. Again we proceed as in Case 1. In this case we use part (ii) of Lemma 4.1 instead of part (i). Also, we need to replace the Sobolev–Poincaré inequality (4.9) with the inequality

$$\left(\int_{B(a,8r)} |u|^n\,dx\right)^{1/n} \le c(n,K)\left(\int_{B(a,8r)} |Du|^{n/2}\,dx\right)^{2/n}, \tag{4.12}$$

which follows from Lemma 2.1 with $p=n/2$. Eventually, we arrive at the following estimate similar to (4.10)

$$\begin{aligned}\left(\fint_{B(a,4r)} |Du|^p\,dx\right)^{1/p} &\le C_2\left(\fint_{B(a,8r)} |Du|^{n/2}\,dx\right)^{2/n} + C_2\left(\fint_{B(a,8r)} |g|\,dx\right)^{1/n}\\ &\quad + \frac{C_3}{r}\left(\int_{B(a,8r)} |f|\,dx\right)^{1/(n-1)}.\end{aligned} \tag{4.13}$$

This completes the proof of case 2, and hence, that of Lemma 4.2. □

Proof of Theorem 1.2. As shown in [8], under the assumptions (H0)–(H3) and one of the conditions (i)–(iii) on the operator (see Section 2), we can construct approximating solutions $u_i\in W_0^{1,n}(\Omega)$ of the regularized system

$$-\operatorname{div}\sigma(x,u_i,Du_i)=f_i$$

for smooth functions f_i, bounded in $L^1(\Omega)$, with $f_i\to\mu$ weakly in $\mathcal{M}(\Omega)$. It was shown in [7] (see also [8]) that we may pass to the limit $i\to\infty$ to prove the existence of a solution u of system (1.1). By Lemma 4.1, the approximating solutions u_i, together with g_i and f_i, satisfy the hypotheses of Lemma 4.2, and hence satisfy the inequality (4.3) for all balls $B(a,r)\subset\mathbf{R}^n$. An easy covering argument shows that for all balls $B(a,r)\subset\mathbf{R}^n$,

$$\begin{aligned}\left(\fint_{B(a,r)} |Du_i|^p\,dx\right)^{1/p} &\le C_2\left(\fint_{B(a,2r)} |Du_i|^{n/2}\,dx\right)^{2/n} + C_2\left(\fint_{B(a,2r)} |g_i|\,dx\right)^{1/n}\\ &\quad + \frac{C_3}{r}\left(\int_{B(a,2r)} |f_i|\,dx\right)^{1/(n-1)}.\end{aligned} \tag{4.14}$$

Now we fix a ball $B(y,2r_0)\subset\mathbf{R}^n$. Then for each ball $B(a,r)$ with $B(a,2r)\subset B(y,2r_0)$, we rewrite (4.14) as

$$\begin{aligned}\left(\fint_{B(a,r)} |Du_i|^p\,dx\right)^{1/p} &\le C_2\left(\fint_{B(a,2r)} |Du_i|^{n/2}\,dx\right)^{2/n}\\ &\quad + \left(\fint_{B(a,2r)} (C_2'|g_i|+C_3'\nu_i|f_i|)\,dx\right)^{1/n},\end{aligned} \tag{4.15}$$

where

$$\nu_i = \left(\int_{B(y,2r_0)} |f_i| \, dx \right)^{1/(n-1)}.$$

Hence with $F_i = |Du_i|^{n/2}$,

(4.16)

$$\left(\fint_{B(a,r)} F_i^{2p/n} \, dx \right)^{n/2p} \le C_4 \fint_{B(a,2r)} F_i \, dx + \left(\fint_{B(a,2r)} (C_5 |g_i| + C_6 \nu_i |f_i|) \, dx \right)^{1/2},$$

and we are now in a position to apply Lemma 1.8 with $q=2$ and with $2p/n$ instead of p, provided $n>p\ge 3n/4$ and $2(1-p/n)<\delta$. By Remark 3.5, we see that it is possible to choose such a p. We conclude that

$$\begin{aligned} |Du_i|_{L^{n,\infty}(B(y,r_0))} \le{}& C\|f_i\|^{1/(n-1)}_{\mathcal{M}(B(y,2r_0))} + C\|g_i\|^{1/n}_{L^1(B(y,2r_0))} \\ &+ Cr_0 \left(\fint_{B(y,2r_0)} |Du_i|^p \, dx \right)^{1/p}. \end{aligned}$$

Again we use Lemma 4.2 to estimate the last term in the above inequality to obtain the inequality (1.4) for the approximating solutions u_i. In view of the weak lower semicontinuity of the $L^{n,\infty}$-norm, we conclude the validity of (1.4) for u as well.

The proof of (1.3) is similarly obtained by applying Lemma 1.5; we leave the details to the reader. □

In Theorem 1.2 if we only assume that the complement of Ω is uniformly n-fat, we can prove the theorem as follows. By the result in [23] mentioned in Section 2, the complement of Ω is uniformly n_0-fat for some $n_0 = n_0(n, K) < n$. We may assume that $n_0 > n/2$. In this case, the inequality (4.3) is true if we replace the $L^{n/2}$-norm of Du in the right-hand side with the L^{n_0}-norm. The proof is the same as that of (4.3), except that in the Sobolev–Poincaré inequality (4.12), we replace the $L^{n/2}$-norm by the L^{n_0}-norm. The rest of the proof of the theorem in this case requires only minor changes. We omit the details.

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