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Power weighted L^p-inequalities for Laguerre–Riesz transforms

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Abstract. In this paper we give a complete description of the power weighted inequalities, of strong, weak and restricted weak type for the pair of Riesz transforms associated with the Laguerre function system $\{\mathcal{L}_k^\alpha\}$, for any given $\alpha > -1$. We achieve these results by a careful estimate of the kernels: near the diagonal we show that they are local Calderón–Zygmund operators while in the complement they are majorized by Hardy type operators and the maximal heat-diffusion operator. We also show that in all the cases our results are sharp.

1. Introduction

The purpose of this article is to study boundedness properties in $L^p((0,\infty), x^\delta\,dx)$ for the Riesz transforms associated with the orthogonal systems of Laguerre functions $\mathcal{L}_k^\alpha$, $\alpha > -1$, started in [4], see also [17]. There, following the ideas developed in [12], the appropriate Riesz transforms are introduced and, using a transference method from Hermite systems along with Kanjin's transplantation theorem for Laguerre expansions, continuity results in $L^p((0,\infty), x^\delta\,dx)$ were obtained for δ ranging in a certain interval depending on p and α. Recently (see [6] and [3]), sharper results of this kind have been given for the maximal heat operator related to such expansions. There, the authors not only obtain strong type inequalities but they analyse the behaviour at the extreme points, proving weak type or restricted weak type inequalities that, as they show, are sharp.

[†] During the revision process of this article, Carlos Segovia unexpectedly died. The other authors are deeply touched by the loss of such an outstanding colleague and friend. We are also indebted for all we have learnt and enjoyed working in his company. Undoubtedly, his clearness of thinking and his ability for computations are vividly present throughout this manuscript.

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Our aim in this paper is to perform an analogous analysis for the case of the Riesz operators. Our approach this time is through a careful study of their kernels. We make use of a technique, that nowadays might be called classical, of splitting the kernel into its "global" and "local" parts (see (3.1) and (3.2) for their precise definitions). The local parts of the Riesz transforms are shown to be local Calderón–Zygmund operators in the sense defined in [10] (see Theorem 3), which gives no restriction on the exponent δ. On the other hand, the global parts are estimated by a sum of positive operators, namely a variation of the maximal heat operator and some Hardy type operators. We combine both estimates to give our main result stated as Theorem 1.

We remark at this point that our boundedness results for the Riesz transforms are a little worse than those for the maximal heat operator. Not only do we not get boundedness on L^∞, as expected, but we also get a weaker result at one of the end points, when the range of p is a finite interval. Nevertheless it turns to be sharp.

As we just mentioned, the main key to achieve our results is a careful analysis of the kernels. Since they involve modified Bessel functions and their derivatives, part of the proof becomes highly technical. However, such precise knowledge of the behaviour of the kernels allows us to get really sharp inequalities. In fact, in Section 6, we show that our boundedness results on $L^p\big((0,\infty),x^\delta\,dx\big)$ cannot be improved any further (see Theorem 2).

Let us remark that this technique of splitting the kernel according to the local and global regions, goes back to Muckenhoupt and Stein (see [7] and [9]) in connection with problems concerning Poisson summation and conjugacy for different systems of orthogonal polynomials, like those of Jacobi, Hermite or Laguerre.

Later, even though it is not pointed out explicitly, K. Stempak in [13] did also make use of such technique in order to bound the heat and Poisson maximal operators for various systems of Laguerre expansions. Finally, very recently, A. Nowak and K. Stempak in [11], have obtained boundedness results for Riesz transforms associated to other Laguerre orthonormal functions, using again this type of partition of the kernel. We include, as a final section, a brief discussion on how their results are related to ours.

Given a real number $\alpha>-1$, we consider the Laguerre second order differential operator L_α defined by

$$L_\alpha = -y\,\frac{d^2}{dy^2} - \frac{d}{dy} + \frac{y}{4} + \frac{\alpha^2}{4y}, \quad y>0.$$

It is well known that L_α is non-negative and selfadjoint with respect to the Lebesgue measure on $(0,\infty)$, furthermore its eigenfunctions are the Laguerre functions $\mathcal{L}_k^\alpha$

defined by

$$\mathcal{L}_k^\alpha(y) = \left(\frac{\Gamma(k+1)}{\Gamma(k+\alpha+1)}\right)^{1/2} e^{-y/2} y^{\alpha/2} L_k^\alpha(y),$$

where L_k^α are the Laguerre polynomials of type α see [15, p. 100] and [16, p. 7]. The orthogonality of Laguerre polynomials with respect to the measure $e^{-y}y^\alpha$ leads to the orthonormality of the family $\{\mathcal{L}_k^\alpha\}_{k=0}^\infty$ in $L^2((0,\infty), dy)$.

Given the second order non negative and selfadjoint differential operator L_α and its heat semigroup $T_t = e^{-tL_\alpha}$, following [12], we shall consider the Riesz potentials

$$L^{-\sigma} f(x) = \frac{1}{\Gamma(\sigma)} \int_0^\infty t^{\sigma-1} T_t f(x)\, dt, \quad \sigma > 0,$$

which can be derived from the identity $s^{-\sigma} = \Gamma(\sigma)^{-1} \int_0^\infty t^{\sigma-1} e^{-ts}\, dt$.

In order to define the corresponding Riesz transforms, the following first order derivatives can be introduced, see [4],

$$\delta_y^\alpha = \sqrt{y}\, \frac{d}{dy} + \frac{1}{2}\left(\sqrt{y} - \frac{\alpha}{\sqrt{y}}\right) \quad \text{and} \quad \partial_y^\beta = -\sqrt{y}\, \frac{d}{dy} + \frac{1}{2}\left(\sqrt{y} - \frac{\beta}{\sqrt{y}}\right) = (\delta_y^{\beta-1})^*.$$

The action for $\alpha > -1$ and $\beta > 0$ on the Laguerre functions is given by

$$\delta_y^\alpha(\mathcal{L}_k^\alpha) = -\sqrt{k}\,\mathcal{L}_{k-1}^{\alpha+1} \quad \text{and} \quad \partial_y^\beta(\mathcal{L}_k^\beta) = -\sqrt{k+1}\,\mathcal{L}_{k+1}^{\beta-1}.$$

Moreover

$$L_\alpha - \left(\frac{\alpha+1}{2}\right) = (\delta_y^\alpha)^* \delta_y^\alpha = \partial_y^{\alpha+1} \delta_y^\alpha.$$

Hence the Riesz transforms for the Laguerre function expansions are defined by

$$R_+^\alpha = \delta_y^\alpha (L_\alpha)^{-1/2},\ \alpha > -1, \quad \text{and} \quad R_-^\beta = \partial_y^\beta (L_\beta)^{-1/2},\ \beta > 0,$$

that is

$$R_+^\alpha(\mathcal{L}_k^\alpha) = -\frac{\sqrt{k}}{\sqrt{k+(\alpha+1)/2}}\mathcal{L}_{k-1}^{\alpha+1} \quad \text{and} \quad R_-^\beta(\mathcal{L}_k^\beta) = -\frac{\sqrt{k+1}}{\sqrt{k+(\beta+1)/2}}\mathcal{L}_{k+1}^{\beta-1}.$$

One of the main interests in studying Riesz transforms lies in their intimate connection with potential and Sobolev spaces. In our case it is easy to check that $R_-^{\alpha+1} \circ R_+^\alpha = T_m$, the multiplier operator associated with the sequence

$$m_k = \frac{k}{\sqrt{(k+(\alpha+1)/2)(k+\alpha/2)}}.$$

The boundedness of T_m and $T_{m^{-1}}$ in $L^2((0,\infty),dx)$ is obvious, but moreover [16, Theorem 6.3.4] gives the continuity in $L^p((0,\infty),dx)$. From this observation and the boundedness of the above Riesz transforms in $L^p((0,\infty),dx)$ given in Theorem 1 below, we may write

$$\|f\|_p = \|T_{m^{-1}} \circ R_-^{\alpha+1} \circ R_+^\alpha f\|_p \le C\|R_+^\alpha f\|_p = C\|\delta_y^\alpha (L_\alpha)^{-1/2} f\|_p \le C\|f\|_p,$$

for f satisfying $\int_0^\infty f(y)\mathcal{L}_0^\alpha(y)\,dy=0$. Therefore for good enough functions, we get

$$\|\delta_y^\alpha f\|_p \sim \|(L_\alpha)^{1/2} f\|_p.$$

This equivalence is the key to define the Sobolev space associated to this Laplacian in terms of the δ_y^α derivative.

The above observation leads us to give the following definition.

Definition 1. In what follows we denote by $\mathcal{R}^\alpha$ the *Riesz transform vector* associated to L_α, that is

$$\mathcal{R}^\alpha = (R_+^\alpha, R_-^{\alpha+1}).$$

Since we are interested in L^p-boundedness of both Riesz transforms simultaneously we shall state our results in terms of the operator

$$\|\mathcal{R}^\alpha\|(f) = (|R_+^\alpha f|^2 + |R_-^{\alpha+1} f|^2)^{1/2}.$$

Our main result is the following.

Theorem 1. *Let $\alpha>-1$ and δ be a real number. Then the operator $\|\mathcal{R}^\alpha\|$ satisfies.*

(a) $\|\mathcal{R}^\alpha\|$ *is of strong type (p,p) with respect to $x^\delta\,dx$ as long as $-\alpha p/2<\delta+1<(\alpha+2)p/2$ and $1<p<\infty$.*

(b) $\|\mathcal{R}^\alpha\|$ *is of weak type $(1,1)$ with respect to $x^\delta\,dx$ as long as $-\alpha/2\le\delta+1\le(\alpha+2)/2$ if $\alpha\ne0$ and $0<\delta+1\le1$ when $\alpha=0$.*

(c) *If $\alpha\ne0$ and $-\alpha p/2=\delta+1$ for some $1<p<\infty$, then $\|\mathcal{R}^\alpha\|$ is of restricted weak type (p,p) with respect to the measure $x^\delta\,dx$.*

(d) *If $(\alpha+2)p/2=\delta+1$ for some $1<p<\infty$, then $\|\mathcal{R}^\alpha\|$ is of restricted weak type (p,p) with respect to $x^\delta\,dx$.*

We notice that our results include negative values of $\delta+1$ when $\alpha>0$. In fact if we fix α and δ the range of p may be described as follows.

For $\alpha>0$:

If $\delta\le-1-\alpha/2$ it is of strong type for $-2(\delta+1)/\alpha<p<\infty$, and if $p=-2(\delta+1)/\alpha$ it is of restricted weak type if $p>1$ and weak type when $p=1$.

If $-1-\alpha/2<\delta\leq\alpha/2$ it is of strong type for $1<p<\infty$ and weak type $(1,1)$.

If $\delta>\alpha/2$ it is of strong type if $2(\delta+1)/(2+\alpha)<p<\infty$ and restricted weak type when $p=2(\delta+1)/(2+\alpha)$.

For $\alpha<0$:

If $\delta\leq-\alpha/2-1$ there are no boundedness results except for $\delta=-\alpha/2-1$ when we have weak type $(1,1)$.

If $-\alpha/2-1<\delta\leq\alpha/2$ it is of strong type if $1<p<-2(\delta+1)/\alpha$, weak type $(1,1)$ and restricted weak type for $p=-2(\delta+1)/\alpha$.

If $\delta>\alpha/2$ it is of strong type for $2(\delta+1)/(2+\alpha)<p<-2(\delta+1)/\alpha$ and of restricted weak type at both end points $p=2(\delta+1)/(2+\alpha)$ and $p=-2(\delta+1)/\alpha$.

For $\alpha=0$:

If $\delta\leq-1$ there are no results.

If $-1<\delta\leq0$ it is of strong type $1<p<\infty$ and of weak type for $p=1$.

If $\delta>0$ it is of strong type for $\delta+1<p<\infty$ and of restricted weak type for $p=\delta+1$.

At this point we would like to remark that if we compare our results for the Riesz transforms with those obtained for the maximal operator of the heat semigroup ([6], [3]) we notice that besides the expected difference at $p=\infty$ we also obtain a weaker result at the end point $p=-2(\delta+1)/\alpha$. However we will show that it cannot be improved.

In fact the next theorem states that our results are sharp.

Theorem 2. *Let* $\alpha>-1$ *and* $\delta\in\mathbb{R}$. *Then*

(a) $\|\mathcal{R}^\alpha\|$ *is not strong type* $(1,1)$ *with respect to* $x^\delta\,dx$ *for any* δ.

(b) *If* $\alpha\neq0$ *and* $-\alpha p/2=\delta+1$ *for some* $p>1$, *then* $\|\mathcal{R}^\alpha\|$ *is not of weak type* (p,p) *with respect to* $x^\delta\,dx$.

(c) *If* $(\alpha+2)p/2=\delta+1$ *for some* $1<p<\infty$, *then* $\|\mathcal{R}^\alpha\|$ *is not of weak type* (p,p) *with respect to* $x^\delta\,dx$.

2. Preliminaries

The heat diffusion semigroup e^{-tL_α} is given by

$$W^\alpha(f,t,x)=\int_0^\infty U_\alpha(s,x,z)f(z)\,dz,$$

where $s=(1-e^{-t/2})/(1+e^{-t/2})$, and $U_\alpha(s,x,z)$ is

$$\frac{1}{2}\frac{1-s^2}{2s}e^{-1/4(s+1/s)(x^{1/2}-z^{1/2})^2}e^{-1/2(s+1/s)(xz)^{1/2}}I_\alpha\left(\frac{1-s^2}{2s}(xz)^{1/2}\right). \tag{2.1}$$

I_α is the modified Bessel function of order α (see [13] and [6]). We introduce the following notation which we will use frequently

$$M(s,x,z)=\frac{1}{2}\frac{1-s^2}{2s}e^{-1/4(s+1/s)(x^{1/2}-z^{1/2})^2}e^{-1/2(s+1/s)(xz)^{1/2}},$$

and, if we also set $w=((1-s^2)/2s)(xz)^{1/2}$, the kernel for the semigroup can be written as

$$U_\alpha(s,x,z)=M(s,x,z)I_\alpha(w).$$

To get the kernel of the Riesz–Laguerre transform R_α^+ we write

$$\begin{aligned} R_+^\alpha(x,z)&=\delta_x^\alpha(L^\alpha)^{-1/2}(x,z)=\delta_x^\alpha\frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_0^\infty e^{-tL^\alpha}(x,z)t^{1/2}\,\frac{dt}{t}\\ &=C\delta_x^\alpha\int_0^1 e^{-2\log((1+s)/(1-s))L^\alpha}(x,z)\left(\log\left(\frac{1+s}{1-s}\right)\right)^{-1/2}\frac{ds}{1-s^2}. \end{aligned}$$

Now, using the equality $I'_\alpha(r)=\alpha I_\alpha(r)/r+I_{\alpha+1}(r)$, the kernel of the Riesz transform can be written as

$$\begin{aligned} R_+^\alpha(x,z)&=C\left(\sqrt{x}\,\frac{d}{dx}+\frac{1}{2}\left(\sqrt{x}-\frac{\alpha}{\sqrt{x}}\right)\right)(L^\alpha)^{-1/2}(x,z)\\ &=C\int_0^1\left(\frac{1}{2}\frac{1-s^2}{2s}z^{1/2}U_{\alpha+1}(s,x,z)-\frac{1}{2}\frac{(1-s)^2}{2s}x^{1/2}U_\alpha(s,x,z)\right)\\ &\qquad\times\left(\log\left(\frac{1+s}{1-s}\right)\right)^{-1/2}\frac{ds}{1-s^2}\\ &=C\int_0^1 K_+^\alpha(s,x,z)\left(\log\frac{1+s}{1-s}\right)^{-1/2}\frac{ds}{1-s^2}. \end{aligned}$$

From Theorems 3 and 4, in the next section, and following the ideas of the proof of Proposition 3.2 in [14], it can be shown that for $f,g\in C_c^\infty((0,\infty))$ with disjoint compact supports, we have

$$\langle R_\pm^\alpha f,g\rangle=\int_0^\infty\int_0^\infty R_\pm^\alpha(x,z)f(x)g(z)\,dx\,dz. \tag{2.2}$$

We leave it to the reader to check the details yielding that the kernel for the other Riesz transform is given by

$$R_-^\beta(x,z)=C\int_0^1 K_-^\beta(s,x,z)\left(\log\left(\frac{1+s}{1-s}\right)\right)^{-1/2}\frac{ds}{1-s^2},$$

where

$$K_-^{\beta} = -\frac{1}{2}\frac{1-s^2}{2s}\left(z^{1/2}U_{\beta+1} - \frac{1+s}{1-s}x^{1/2}U_{\beta}\right) - \frac{\beta}{x^{1/2}}U_{\beta}. \tag{2.3}$$

3. Proof of the main result

We split the kernels of the Riesz transforms into their "local" and "global" parts. By the local part we mean the restriction of the kernel to the set

$$\{(x,z) : x/4 < z < 4x\}$$

and by the global part its restriction to the complementary set. In what follows we state some results for each of these parts and we show our main result, assuming they are true. The proofs of these theorems will be given in Sections 4 and 5, respectively.

Theorem 3. *Let $\alpha>-1$. We denote by $K(x,z)$ either the kernel $R_+^{\alpha}(x,z)$ or $R_-^{\alpha+1}(x,z)$. Then for x and z satisfying $x/4<z<4x$ we have*

(a) $|K(x,z)| \le C/|x-z|$;

(b)

$$\left|\frac{\partial}{\partial x}K(x,z)\right| + \left|\frac{\partial}{\partial z}K(x,z)\right| \le \frac{C}{|x-z|^2}.$$

Theorem 4. *Let $\alpha>-1$. We denote by $K(x,z)$ either the kernel $R_+^{\alpha}(x,z)$ or $R_-^{\alpha+1}(x,z)$. If we denote by T_{glob} the operator given by*

$$T_{\mathrm{glob}}f(x) = \int_{(0,\infty)} K(x,z)\chi_{\{z\le x/4\}\cup\{z\ge 4x\}} f(z)\,dz, \tag{3.1}$$

then, we have

(i) *T_{glob} is of strong type (p,p) with respect to $x^{\delta}\,dx$ as long as $-\alpha p/2<\delta+1<(\alpha+2)p/2$ and $1<p<\infty$;*

(ii) *T_{glob} is of weak type $(1,1)$ with respect to $x^{\delta}\,dx$ as long as $-\alpha/2\le\delta+1\le(\alpha+2)/2$ if $\alpha\ne 0$ and $0<\delta+1\le 1$ when $\alpha=0$;*

(iii) *if $\alpha\ne 0$ and $-\alpha p/2=\delta+1$ for some $1<p<\infty$, then T_{glob} is of restricted weak type (p,p) with respect to the measure $x^{\delta}\,dx$;*

(iv) *if $(\alpha+2)p/2=\delta+1$ for some $1<p<\infty$, then T_{glob} is of restricted weak type (p,p) with respect to $x^{\delta}\,dx$.*

Proof of Theorem 1. It is enough to prove the statements for each of the components of $\mathcal{R}^\alpha$. We consider the operator

$$R^\alpha_{+,\mathrm{loc}} f = R^\alpha_+ f - R^\alpha_{+,\mathrm{glob}} f, \tag{3.2}$$

where

$$R^\alpha_{+,\mathrm{glob}} f = \int_0^\infty R^\alpha_+(x,z)\chi_{\{z\le x/4\}\cup\{z\ge 4x\}} f(z)\,dz.$$

By Plancherel's theorem, R^α_+ is bounded on $L^2((0,\infty),dx)$. From Theorem 4 the operator $R^\alpha_{+,\mathrm{glob}}$ is also bounded on $L^2((0,\infty),dy)$. Therefore $R^\alpha_{+,\mathrm{loc}}$ is bounded on $L^2((0,\infty),dx)$, for any $\alpha>-1$. Moreover by (2.2) we have that for $f,g\in C^\infty_c((0,\infty))$ with disjoint compact supports

$$\langle R^\alpha_{+,\mathrm{loc}} f, g\rangle = \int_0^\infty \int_{x/4}^{4x} R^\alpha_+(x,z) f(x) g(z)\,dx\,dz.$$

Therefore by using Theorem 3 we have that $R^\alpha_{+,\mathrm{loc}}$ is a "local Calderón–Zygmund operator" as defined in [10]. Hence we can apply Theorem 4.3 (a) in [10] to conclude that $R^\alpha_{+,\mathrm{loc}}$ is bounded on $L^p((0,\infty),x^\delta\,dx)$ for any $\delta\in\mathbb{R}$ and $1<p<\infty$. Applying again Theorem 4 we get the boundedness of R^α_+ as stated in (a), (c) and (d).

Moreover as $R^\alpha_{+,\mathrm{loc}}$ is a "local Calderón–Zygmund operator" we can apply Theorem 4.3 (b) in [10] getting that $R^\alpha_{+,\mathrm{loc}}$ is bounded from $L^1((0,\infty),x^\delta\,dx)$ into $L^{1,\infty}((0,\infty),x^\delta\,dx)$ for any $\delta\in\mathbb{R}$. Therefore by using Theorem 4 (ii) we get (b). All the arguments are also valid for $R^{\alpha+1}_-$. □

4. Local region. Proof of Theorem 3

As we have seen in (2.1) the kernel $U_\alpha(s,x,z)$ can be expressed by

$$U_\alpha(s,x,z) = M(s,x,z) I_\alpha\left(\frac{1-s^2}{2s}(xz)^{1/2}\right), \tag{4.1}$$

where $M(s,x,z)=\frac{1}{2}((1-s^2)/2s)e^{-1/4(s+1/s)|x^{1/2}-z^{1/2}|^2}e^{-1/4(s+1/s)(xz)^{1/2}}$.

As in the local region $x\sim z$, the last exponential is equivalent to $e^{-c(s+1/s)x}$. Now we state some basic inequalities about the behaviour of I_α that we shall use often, (see for example [5]).

(a) If $0\le w\le 1$, then $I_\alpha(w)\simeq w^\alpha$.
(b) If $w\ge 1$ then $I_\alpha(w)\simeq e^w w^{-1/2}$.
(c) If $\alpha\ge -\frac{1}{2}$ then $I_\alpha(w)\le C_\alpha e^w w^{-1/2}$ for any $w\ge 0$.
(d) If $-1<\beta\le\alpha$, then $I_\alpha(w)\le C I_\beta(w)$.

From (b) it is easy to see that for $w=((1-s^2)/2s)(xz)^{1/2}\geq 1$ and $x\sim z$,

$$U_\alpha(s,x,z)\leq C\left(\frac{1-s^2}{2s}\right)^{1/2} e^{-1/4(s+1/s)|x^{1/2}-z^{1/2}|^2} e^{-csx}x^{-1/2}. \tag{4.2}$$

Before proceeding to prove Theorem 3 we state two lemmas which give some estimates useful to obtain the conditions on the size of the kernels and their derivatives.

Lemma 1. *Let $\gamma>\frac{1}{2}$ and $\varepsilon\geq 0$. Assume that*

$$L(s,x,z)=\frac{|z^{1/2}-x^{1/2}|^{2((\gamma-1)+\varepsilon)}}{x^{1/2+\varepsilon}s^\gamma}e^{-(1/4s)|x^{1/2}-z^{1/2}|^2}.$$

Then there exists a constant C such that for x and z in the local region

$$\int_0^{1/2} L(s,x,z)\left(\log\frac{1+s}{1-s}\right)^{-1/2} ds\leq\frac{C}{|x-z|}.$$

Proof. Using that $\log(1+s)/(1-s)\sim s$ for $0\leq s\leq 1/2$ and that $x\sim z$ we have

$$\begin{aligned}
&\int_0^{1/2} L(s,x,z)\left(\log\left(\frac{1+s}{1-s}\right)\right)^{-1/2} ds\\
&\qquad\leq\frac{|z^{1/2}-x^{1/2}|^{2[(\gamma-1)+\varepsilon]}}{x^{1/2+\varepsilon}}\int_0^{1/2}\frac{e^{-(1/4s)|x^{1/2}-z^{1/2}|^2}}{s^{\gamma-1/2}}\frac{ds}{s}\\
&\qquad\leq C\frac{|z^{1/2}-x^{1/2}|^{2[(\gamma-1)+\varepsilon]}}{|x^{1/2}-z^{1/2}|^{2\gamma-1}x^{1/2+\varepsilon}}\int_0^\infty e^{-u}u^{\gamma-1/2-1}\,du\\
&\qquad\leq C\frac{|z^{1/2}-x^{1/2}|^{2\varepsilon}}{|x^{1/2}-z^{1/2}|x^{1/2+\varepsilon}}\\
&\qquad= C\frac{|z^{1/2}-x^{1/2}|^{2\varepsilon}|z^{1/2}+x^{1/2}|}{|x-z|x^{1/2+\varepsilon}}\leq\frac{C}{|x-z|},
\end{aligned}$$

as we wanted. This completes the proof of the lemma. □

Lemma 2. *Let $\sigma\geq 0$ and γ such that $\gamma-\sigma>\frac{1}{2}$. Assume that*

$$L'(s,x,z)=\frac{|z^{1/2}-x^{1/2}|^{2(\gamma-3/2)}}{s^{\gamma-\sigma}x^{1+\sigma}}e^{-(1/4s)|x^{1/2}-z^{1/2}|^2}.$$

Then, in the local region $x\sim z$ we have

$$\int_0^{1/2} L'(s,x,z)\left(\log\left(\frac{1+s}{1-s}\right)\right)^{-1/2} ds\leq\frac{C}{|x-z|^2}.$$

Proof. By the estimate $\log((1+s)/(1-s))\sim s$ and the fact that in the local region $x\sim z$, we get

$$\begin{aligned}\int_0^{1/2} L'(s,x,z)\left(\log\left(\frac{1+s}{1-s}\right)\right)^{-1/2} ds \\ &\le C\frac{|z^{1/2}-x^{1/2}|^{2(\gamma-3/2)}}{x^{1+\sigma}}\int_0^{1/2}\frac{e^{-(1/4s)|x^{1/2}-z^{1/2}|^2}}{s^{\gamma-\sigma-1/2}}\frac{ds}{s}\\ &\le C\frac{|z^{1/2}-x^{1/2}|^{2(\gamma-3/2)}}{x^{1+\sigma}|z^{1/2}-x^{1/2}|^{2(\gamma-\sigma-1/2)}}\int_0^{\infty} e^{-u}u^{\gamma-\sigma-1/2}\frac{du}{u}\\ &\le C\frac{|z^{1/2}-x^{1/2}|^{2\sigma}}{x^{1+\sigma}|x^{1/2}-z^{1/2}|^{2}}\\ &\le C\frac{C}{|x-z|^2},\end{aligned}$$

which completes the proof of the lemma. □

Proof of Theorem 3. As we have seen in (2.3), given $\alpha>-1$, the kernels of both Riesz transforms before integration in s are:

$$K_+^{\alpha}=\frac{1}{2}\frac{1-s^2}{2s}\left(z^{1/2}U_{\alpha+1}-\frac{1-s}{1+s}x^{1/2}U_{\alpha}\right).$$

and

$$K_-^{\alpha+1}=-\frac{1}{2}\frac{1-s^2}{2s}\left(z^{1/2}U_{\alpha+2}-\frac{1+s}{1-s}x^{1/2}U_{\alpha+1}\right)-\frac{\alpha+1}{x^{1/2}}U_{\alpha+1}.$$

Therefore, if we call $b_s=\frac{1}{2}(1-s^2)/2s$ and a_s either $(1-s)/(1+s)$ or $(1+s)/(1-s)$, we get that we should analyse

$$H_1=b_s(z^{1/2}U_{\beta+1}-a_s x^{1/2}U_{\beta}) \quad\text{and}\quad H_2=-\frac{\alpha+1}{x^{1/2}}U_{\alpha+1},$$

since choosing in H_1, $a_s=(1-s)/(1+s)$ and $\beta=\alpha$, we get K_+^{α}, while $K_-^{\alpha+1}$ is obtained taking $a_s=(1+s)/(1-s)$, $\beta=\alpha+1$ in H_1 and then performing $-H_1+H_2$. Therefore it would be enough to get the desired estimates for those kernels. Setting $\psi(s)=(\log((1-s)/(1+s)))^{-1/2}/(1-s^2)$ we must show that in the local region $x\sim z$,

(a) $\int_0^1 |H_i(s,x,z)|\psi(s)\,ds\le C/|x-z|$, $i=1,2$,

(b) $\int_0^1 |(\partial/\partial x+\partial/\partial z)H_i(s,x,z)|\psi(s)\,ds\le C/|x-z|^2$, $i=1,2$.

We split the first integral in $0\le s\le\frac{1}{2}$ and $\frac{1}{2}\le s\le 1$. In the latter case we observe that all the terms in H_1 and H_2 are bounded in absolute value by $Cx^{1/2}U_\alpha$ with α that may be assumed to be negative. In fact for H_1 we just use that $U_\gamma\le CU_{\gamma'}$

when $\gamma' \le \gamma$ and that $x \sim z$, while for H_2 we use that $I_{\alpha+1}(w) \le C w I_\alpha(w)$, and so

$$\begin{aligned}\frac{1}{x^{1/2}} U_{\alpha+1}(s,x,z) &= \frac{1}{x^{1/2}} M(s,x,z) I_{\alpha+1}\left(\frac{1-s^2}{2s}(xz)^{1/2}\right)\\ &\le \frac{C}{x^{1/2}} U_\alpha(s,x,z) \frac{1-s^2}{2s}(xz)^{1/2} \le C x^{1/2} U_\alpha(s,x,z).\end{aligned}$$

Now, for $w=((1-s^2)/2s)(xz)^{1/2} \le 1$ and $\frac{1}{2} \le s \le 1$ we have

$$\begin{aligned}x^{1/2} U_\alpha &\le C(1-s^2) x^{1/2} \left(\frac{1-s^2}{2s}(xz)^{1/2}\right)^\alpha e^{-c|x^{1/2}-z^{1/2}|^2} e^{-cx}\\ &\le C(1-s^2) x^{1/2+\alpha} e^{-c|x^{1/2}-z^{1/2}|^2} e^{-cx} \le C \frac{(1-s^2)^{1+\alpha}}{x^{1/2}} e^{-c|x^{1/2}-z^{1/2}|^2},\end{aligned} \tag{4.3}$$

where we used that $x \sim z$ and $e^{-cx} \le C x^{-(1+\alpha)}$.

Similarly for $w \ge 1$, using (4.2), we get for some $C>0$,

$$x^{1/2} U_\alpha \le C(1-s^2)^{1/2} e^{-c|x^{1/2}-z^{1/2}|^2} e^{-cx} \le \frac{(1-s^2)^{1/2}}{x^{1/2}} e^{-c|x^{1/2}-z^{1/2}|^2}. \tag{4.4}$$

Therefore, letting $\gamma = \min\{\alpha+1, \frac{1}{2}\}$, we have

$$\begin{aligned}\int_{1/2}^1 x^{1/2} U_\alpha(s,x,z)\psi(s)\,ds &\le \frac{C}{x^{1/2}} e^{-c|x^{1/2}-z^{1/2}|^2} \int_{1/2}^1 (1-s^2)^\gamma \left(\log\frac{1+s}{1-s}\right)^{-1/2} \frac{ds}{1-s^2}\\ &\le \frac{C}{x^{1/2}|x^{1/2}-z^{1/2}|} \le \frac{C|x^{1/2}+z^{1/2}|}{x^{1/2}|x-z|} \le \frac{C}{|x-z|}.\end{aligned}$$

Now we consider $0 \le s \le \frac{1}{2}$. This time we observe that all terms in H_1 and H_2 are bounded by $(C/s)x^{1/2}U_\alpha$. For this kernel, in the region $w=((1-s^2)/2s)(xz)^{1/2} \le 1$, using that $I_\alpha(w) \sim w^\alpha$, we have

$$\frac{C}{s} x^{1/2} U_\alpha(s,x,z) \le \frac{C}{s x^{1/2}} \left(\frac{x}{s}\right)^{1+\alpha} e^{-cx/s} e^{-(1/4s)|x^{1/2}-z^{1/2}|^2}.$$

Therefore, applying Lemma 1 with $\varepsilon=0$ and $\gamma=1$ we are done.

The most delicate part concerns H_1 in the region $w \ge 1$. Here we will use the cancellation by means of the asymptotic formula ([5], p. 123)

$$I_\alpha(w) = \frac{e^w}{(2\pi w)^{1/2}} \left[1 + O_\alpha\left(\frac{1}{w}\right)\right].$$

First we write H_1 as

$$H_1(s,x,z) = b_s(z^{1/2}-x^{1/2}) U_{\beta+1} + b_s x^{1/2}(U_{\beta+1} - a_s U_\beta) = H_1^1 + H_1^2.$$

The first term follows easily since using the behavior of I_β at infinity, H_1^1 is bounded by

$$C\frac{|z^{1/2}-x^{1/2}|}{s^2}\frac{s^{1/2}}{(xz)^{1/4}}e^{-(1/4s)|x^{1/2}-z^{1/2}|^2}\leq C\frac{|z^{1/2}-x^{1/2}|}{x^{1/2}s^{3/2}}e^{-(1/4s)|x^{1/2}-z^{1/2}|^2},$$

and we may apply Lemma 1 with $\gamma=\frac{3}{2}$ and $\varepsilon=0$. For the second we use the asymptotic formula to get

$$|I_{\beta+1}(w)-a_sI_\beta(w)|\leq\frac{e^w}{(2\pi w)^{1/2}}\left((1-a_s)+\frac{C}{w}\right).$$

But, for both possible values of a_s it holds that $1-a_s\leq Cs$ for $0\leq s\leq\frac{1}{2}$. Therefore since in our situation $w\simeq x/s$, we get

$$|H_1^2|\leq\frac{Cx^{1/2}}{s}|U_{\beta+1}-a_sU_\beta|\leq\frac{C}{s}e^{-(1/4s)|x^{1/2}-z^{1/2}|^2}e^{-csx}\frac{x^{1/2}}{s}\left(\frac{s}{x}\right)\left(s+\frac{s}{x}\right)$$

$$\leq Ce^{-(1/4s)|x^{1/2}-z^{1/2}|^2}\left(\frac{e^{-csx}}{s^{1/2}}+\frac{1}{s^{1/2}x}\right)\leq C\frac{e^{-(1/4s)|x^{1/2}-z^{1/2}|^2}}{sx^{1/2}},\tag{4.5}$$

where we used that $e^{-csx}\leq C(sx)^{-1/2}$ and that in the region $w\geq1$, since $w\sim x/s$, we have $1/x^{1/2}\leq C/s^{1/2}$. An application of Lemma 1 with $\varepsilon=0$ and $\gamma=1$ leads to the desired bound for H_1^2.

Finally we take care of H_2 in $0\leq s\leq\frac{1}{2}$ and $w\geq1$. But, using that for $w\geq1$, $I_{\alpha+1}(w)\leq Ce^w$, we get just as above

$$H_2(s,x,z)\leq\frac{C}{sx^{1/2}}e^{-(1/4s)|x^{1/2}-z^{1/2}|^2},$$

and we are done.

Now we turn to the boundedness of the derivatives of H_1 and H_2.

First we analyze the derivatives with respect to z. Straightforward calculations show that we have to deal with the following type of terms:

Case 1.

$$\frac{\beta+3}{4}\frac{1-s^2}{2s}\frac{1}{z^{1/2}}U_{\beta+1}.$$

Case 2.

$$\frac{1-s^2}{4s}\left(-\frac{1-s^2}{4s}x^{1/2}U_\beta-\frac{1}{4}\left(s+\frac{1}{s}\right)z^{1/2}U_{\beta+1}\right).$$

Case 3.

$$\frac{1-s^2}{4s}\left(\frac{1}{4}a_s\left(s+\frac{1}{s}\right)x^{1/2}U_\beta-\frac{1-s^2}{4s}a_sxz^{-1/2}U_{\beta+1}\right).$$

Case 4.

$$-\frac{1-s^2}{4s}\frac{\beta}{2}a_s\frac{x^{1/2}}{z}U_\beta.$$

Case 5.

$$-\frac{(\alpha+1)}{4}\frac{s^2+1}{s}\frac{1}{x^{1/2}}U_{\alpha+1}+(\alpha+1)\frac{1-s^2}{2s}\frac{1}{z^{1/2}}U_{\alpha+2}+(\alpha+1)^2\frac{1}{zx^{1/2}}U_{\alpha+1}.$$

With such notation, $(\partial/\partial z)K_+^\alpha$ is the sum of the first four terms with $a_s=(1-s)/(1+s)$ and $\beta=\alpha$, while $(\partial/\partial z)K_-^{\alpha+1}$ is the sum of Case 5 with all the others with $a_s=(1+s)/(1-s)$ and $\beta=\alpha+1$. As before, for $\frac12\le s\le1$, all the terms above are bounded by one of the type

$$\frac{C}{x^{1/2}}U_\alpha(s,x,z)e^{\varepsilon x},$$

for $\varepsilon>0$ small enough and we may assume that $\alpha<0$. This is clear for terms having $x^{-1/2}$, $z^{-1/2}$ or $x^{1/2}z^{-1}$. For those with $x^{1/2}$, $z^{1/2}$ or $xz^{-1/2}$ we use that $x\sim z$ and that $x^{1/2}\le Ce^{\varepsilon x}/x^{1/2}$. Finally, the last term of Case 5 follows from the inequality $I_{\alpha+1}(w)\le CwI_\alpha(w)$ valid for any $w\ge0$.

Now, for the above kernel we have that for $w=((1-s^2)/2s)(xz)^{1/2}\le1$, $x\sim z$ and $\frac12\le s\le1$,

$$\begin{aligned}\frac{1}{x^{1/2}}U_\alpha e^{\varepsilon x}&\le C\frac{1-s^2}{x^{1/2}}e^{\varepsilon x}e^{-cx}e^{-c|x^{1/2}-z^{1/2}|^2}(1-s^2)^\alpha x^\alpha\\&\le C\frac{(1-s^2)^{1+\alpha}x^{1/2}e^{-c'x}}{x^{1-\alpha}}\frac{1}{|x^{1/2}-z^{1/2}|^{2(1+\alpha)}}\\&\le C\frac{(1-s^2)^{1+\alpha}}{x^{1-\alpha}}\frac{|x^{1/2}-z^{1/2}|^{-2\alpha}}{|x-z|^2}|x^{1/2}+z^{1/2}|^2\\&\le C\frac{(1-s^2)^{1+\alpha}}{|x-z|^2},\end{aligned}$$

as long as $0<\varepsilon<c$ and $-1<\alpha\le0$.

On the other hand if $w\ge1$ we get

$$\begin{aligned}\frac{1}{x^{1/2}}U_\alpha e^{\varepsilon x}&\le C\frac{(1-s^2)^{1/2}}{x}e^{-c|x^{1/2}-z^{1/2}|^2}e^{\varepsilon x}e^{-cx}\\&\le C\frac{(1-s^2)^{1/2}}{x}\frac{1}{|x^{1/2}-z^{1/2}|^2}\le C\frac{(1-s^2)^{1/2}}{|x-z|^2}.\end{aligned}$$

In both cases we arrive at the desired estimate since the integrals against $\psi(s)$ are finite. Now we proceed with the case $0\le s\le\frac12$. Here again we notice that all the terms in the Cases 1, 4 and 5 may be bounded by one of the following kernels

$$A_1=x^{1/2}U_\alpha, \quad \text{and} \quad A_2=\frac{1}{sx^{1/2}}U_\alpha,$$

and we may assume that $-1<\alpha<0$. To bound A_1 for $w=((1-s^2)/2s)(xz)^{1/2}\le 1$ we observe that in our situation $x\le Cs\le C$, and hence

$$\begin{aligned} x^{1/2}U_\alpha &\le \frac{C}{s}x^{1/2}\frac{(xz)^{\alpha/2}}{s^\alpha}e^{-(1/4s)|x^{1/2}-z^{1/2}|^2} \\ &\le \frac{C}{xs^{3/2}}x^{\alpha+3/2}s^{1/2-\alpha}e^{-(1/4s)|x^{1/2}-z^{1/2}|^2} \le \frac{C}{xs^{3/2}}e^{-(1/4s)|x^{1/2}-z^{1/2}|^2}. \end{aligned}$$

On the other hand, for $w\ge 1$,

$$x^{1/2}U_\alpha \le \frac{C}{s}x^{1/2}\frac{s^{1/2}}{(xz)^{1/4}}e^{-(1/4s)|x^{1/2}-z^{1/2}|^2}e^{-csx} \le \frac{C}{xs^{3/2}}e^{-(1/4s)|x^{1/2}-z^{1/2}|^2},$$

since $e^{-csx}\le Cs^{-1}x^{-1}$ and $x\sim z$.

Applying now Lemma 2 we get the desired boundedness for A_1. As for A_2, in $w\le 1$,

$$\frac{1}{sx^{1/2}}U_\alpha \le \frac{C}{s^2x^{1/2}}\left(\frac{x}{s}\right)^\alpha e^{-(c/s)x}e^{-(1/4s)|x^{1/2}-z^{1/2}|^2} \le \frac{C}{s^{\alpha+3/2}x^{1-\alpha}}e^{-(1/4s)|x^{1/2}-z^{1/2}|^2},$$

where we have used that $e^{-cx/s}\le C(x/s)^{-1/2}$. Now for $w\ge 1$ we get

$$\frac{1}{sx^{1/2}}U_\alpha \le \frac{C}{s^2x^{1/2}}\frac{s^{1/2}}{x^{1/2}}e^{-(1/4s)|x^{1/2}-z^{1/2}|^2} = \frac{C}{s^{3/2}x}e^{-(1/4s)|x^{1/2}-z^{1/2}|^2}.$$

With these estimates the result follows for A_2 by applying Lemma 2 with $\sigma=-\alpha\ge 0$ and $\gamma=\frac{3}{2}$ in the first case, and with $\sigma=0$ and $\gamma=\frac{3}{2}$ in the second one.

It remains to take care of Cases 2 and 3 for $0\le s\le\frac{1}{2}$. We rewrite their sum as

$$\begin{aligned} \text{Case 2}+\text{Case 3} &= \frac{1-s^2}{16}((a_sxz^{-1/2}-z^{1/2})U_{\beta+1}+(a_s-1)x^{1/2}U_\beta) \\ &\quad+\frac{1-s^2}{16s^2}\left((a_s+1)x^{1/2}U_\beta-\left(1+a_s\frac{x}{z}\right)z^{1/2}U_{\beta+1}\right)=N+L. \end{aligned}$$

Regarding N, each term can be bounded again by A_1 as before. As for L the worst term is the first one with $\beta=\alpha$. Furthermore, if we are in the region $w=((1-s^2)/2s)(xz)^{1/2}\le 1$, we have $1/s\le C/x$ and therefore

$$\frac{C}{s^2}x^{1/2}U_\alpha \le \frac{C}{sx^{1/2}}U_\alpha,$$

which turns out to be A_2.

Therefore to finish the boundedness of the derivatives with respect to z, it only remains to get the right bounds for L in $0\le s\le \frac{1}{2}$ and $w\ge 1$. Here we have to take care of the cancellation, so we will use again the asymptotic formula as we did before. First let us write L as

$$L=\frac{1-s^2}{16s^2}\Big((a_s+1)x^{1/2}U_\beta-z^{1/2}(a_s+1)U_{\beta+1}+z^{1/2}a_s\Big(1-\frac{x}{z}\Big)U_{\beta+1}\Big).$$

Now we use that $U_\gamma=Me^w w^{-1/2}(1+C_\gamma/w)$, where $w=((1-s^2)/2s)(xz)^{1/2}\ge 1$ and plug this formula into the above expression leading to

$$\begin{aligned}L&=\frac{1-s^2}{16s^2}M\frac{e^w}{w^{1/2}}\bigg[(a_s+1)(x^{1/2}-z^{1/2})+a_s z^{1/2}\Big(1-\frac{x}{z}\Big)+C\frac{(1+a_s)}{w}(x^{1/2}+a_s z^{1/2})\bigg]\\&=L_1+L_2+L_3.\end{aligned}$$

First we observe that

$$|L_3|\le \frac{Cx^{1/2}}{s^2}\frac{e^{-(1/4s)|x^{1/2}-z^{1/2}|^2}}{sw^{3/2}}\le \frac{C}{s^{3/2}x}e^{-(1/4s)|x^{1/2}-z^{1/2}|^2},$$

and hence an application of Lemma 2 with $\gamma=\frac{3}{2}$ and $\sigma=0$ gives the result. Now we write the remaining sum as

$$\begin{aligned}L_1+L_2&=\frac{1-s^2}{16s^2}M\frac{e^w}{w^{1/2}}\bigg((x^{1/2}-z^{1/2})+a_s\frac{x^{1/2}}{z^{1/2}}(z^{1/2}-x^{1/2})\bigg)\\&=\frac{1-s^2}{16s^2}M\frac{e^w}{w^{1/2}}(x^{1/2}-z^{1/2})\bigg(1-a_s\frac{x^{1/2}}{z^{1/2}}\bigg)\\&=\frac{1-s^2}{16s^2}M\frac{e^w}{w^{1/2}}(x^{1/2}-z^{1/2})\Big((1-a_s)+\frac{a_s}{z^{1/2}}(z^{1/2}-x^{1/2})\Big)\\&=G_1+G_2.\end{aligned}$$

Using again that in any case $1-a_s\simeq s$, we get for G_1,

$$\begin{aligned}|G_1|&\le \frac{C}{s^2}e^{-(1/4s)|x^{1/2}-z^{1/2}|^2}e^{-csx}\Big(\frac{s}{x}\Big)^{1/2}|x^{1/2}-z^{1/2}|\\&\le \frac{C}{s^2x}|x^{1/2}-z^{1/2}|e^{-(1/4s)|x^{1/2}-z^{1/2}|^2},\end{aligned}$$

where for the last inequality we used that $e^{-csx}\le C(xs)^{-1/2}$. Therefore we may apply Lemma 2 this time with $\gamma=2$ and $\sigma=0$.

Finally for G_2,

$$\begin{aligned}|G_2|&\le \frac{C}{s^3}e^{-(1/4s)|x^{1/2}-z^{1/2}|^2}\Big(\frac{s}{x}\Big)^{1/2}\frac{|x^{1/2}-z^{1/2}|^2}{x^{1/2}}\\&=\frac{C}{s^{5/2}x}|x^{1/2}-z^{1/2}|^2e^{-(1/4s)|x^{1/2}-z^{1/2}|^2},\end{aligned}$$

and again the desired estimate follows by Lemma 2 with $\gamma=\frac{5}{2}$ and $\sigma=0$.

To finish the proof of the theorem we should deal with the derivatives with respect to x of both kernels. They are quite similar, so we only outline the calculations.

This time the kind of terms involved are:

Case $1'$.

$$\frac{1}{2}\frac{1-s^2}{2s}\frac{\beta+1}{2}\frac{a_s}{x^{1/2}}U_\beta.$$

Case $2'$.

$$\frac{1}{2}\frac{1-s^2}{2s}\left(-\frac{1+s^2}{4s}z^{1/2}U_{\beta+1}+\frac{1-s^2}{4s}\frac{z}{x^{1/2}}U_\beta\right).$$

Case $3'$.

$$\frac{1}{2}\frac{1-s^2}{2s}\left(a_s\frac{1+s^2}{4s}x^{1/2}U_\beta-a_s\frac{1-s^2}{4s}z^{1/2}U_{\beta+1}\right).$$

Case $4'$.

$$\frac{\alpha+1}{2x^{3/2}}U_{\alpha+1}.$$

Case $5'$.

$$\frac{\alpha+1}{x^{1/2}}\frac{\partial}{\partial x}U_{\alpha+1}.$$

This time $(\partial/\partial x)R^\alpha_+$ is the sum of the first three terms with $a_s=(1-s)/(1+s)$ and $\beta=\alpha$ and $(\partial/\partial x)R^\alpha_-$ is the sum of the five cases with $a_s=(1+s)/(1-s)$ and $\beta=\alpha+1$. Case $1'$ is equivalent to Case 4 since $x\simeq z$. Case $5'$ is similar to Case 5 since this came out from $((\alpha+1)/x^{1/2})(\partial/\partial z)U_{\alpha+1}$. As $U_{\alpha+1}$ is a symmetric function in x and z, and we did not use any cancellation in this case, all the terms will be the same after using that $x\sim z$.

Regarding Cases $2'$ and $3'$ we observe that each term individually is equivalent to a similar term in Cases 2 and 3. So the only difference might be in the argument involving cancellation. However, putting together the terms with $1/s^2$ from Cases $2'$ and $3'$, we would arrive at the following expression instead of L

$$\begin{aligned}L'&=\frac{1-s^2}{16s^2}\left[\left(a_s+\frac{z}{x}\right)x^{1/2}U_\beta-(a_s+1)z^{1/2}U_{\beta+1}\right]\\&=\frac{1-s^2}{16s^2}\left[(a_s+1)x^{1/2}U_\beta-(a_s+1)z^{1/2}U_{\beta+1}-\left(1-\frac{z}{x}\right)x^{1/2}U_\beta\right].\end{aligned}$$

The asymptotic formula for U_γ gives rise again to three terms. The last one has the same behaviour as L_3. The other two, say L_1' and L_2', turn into

$$\begin{aligned} L_1'+L_2' &= \frac{1-s^2}{16s^2} M \frac{e^w}{w^{1/2}} \Big[(a_s+1)(x^{1/2}-z^{1/2}) - \Big(x^{1/2}-\frac{z}{x^{1/2}}\Big)\Big] \\ &= \frac{1-s^2}{16s^2} M \frac{e^w}{w^{1/2}} (x^{1/2}-z^{1/2}) \Big(a_s - \frac{z^{1/2}}{x^{1/2}}\Big) \\ &= \frac{1-s^2}{16s^2} M \frac{e^w}{w^{1/2}} (x^{1/2}-z^{1/2}) \Big[(a_s-1) + \frac{x^{1/2}-z^{1/2}}{x^{1/2}}\Big]. \end{aligned}$$

Each of these terms are now quite similar to G_1 and G_2 above.

So it remains to deal with Case 4′ which is actually of a new kind. Anyhow, for $w\ge1$ we use that $I_{\alpha+1}(w)\le CwI_\alpha(w)\le Cw^{1/2}e^w$ to get

$$\begin{aligned} \frac{1}{x^{3/2}} U_{\alpha+1} &\le \frac{C}{x^{3/2}} \frac{1-s^2}{s} e^{-(1/4s)|x^{1/2}-z^{1/2}|^2} \Big(\frac{1-s^2}{2s}\Big)^{1/2} (xz)^{1/4} \\ &\le C \frac{(1-s^2)^{3/2}}{s^{3/2}x} e^{-(1/4s)|x^{1/2}-z^{1/2}|^2}. \end{aligned}$$

Then for the integral between 0 and $\frac12$ we may use Lemma 2 with $\gamma=\frac32$ and $\sigma=0$.

As for the other range of s we get

$$\begin{aligned} \int_{1/2}^1 \frac{1}{x^{3/2}} U_{\alpha+1}(s,x,z)\psi(s)\,ds &\le \frac{C}{x} e^{-c|x^{1/2}-z^{1/2}|^2} \int_{1/2}^1 (1-s^2)^{3/2}\psi(s)\,ds \\ &\le \frac{C}{x|x^{1/2}-z^{1/2}|^2} \le \frac{C}{|x-z|^2}. \end{aligned}$$

On the other hand the argument for $w\le1$ relies strongly on the fact that $\alpha+1>0$. More precisely, assuming the worst case $-1<\alpha\le0$, we have

$$\begin{aligned} \frac{1}{x^{3/2}} U_{\alpha+1} &\le \frac{C}{x^{3/2}} \frac{1-s^2}{2s} e^{-(1/4s)|x^{1/2}-z^{1/2}|^2} \Big(\frac{1-s^2}{2s}(xz)^{1/2}\Big)^{\alpha+1} \\ &\le \frac{C}{x^{1/2-\alpha}} \frac{(1-s^2)^{2+\alpha}}{s^{2+\alpha}} e^{-(1/4s)|x^{1/2}-z^{1/2}|^2} \\ &\le \frac{C(1-s^2)^{3/2+\alpha}}{x^{1-\alpha}s^{3/2+\alpha}} e^{-(1/4s)|x^{1/2}-z^{1/2}|^2}, \end{aligned}$$

where we have used that for $w\le1$, $(1-s^2)/s\le C/x$. Therefore, for $0\le s\le\frac12$ we may apply Lemma 2 with $\gamma=\frac32$ and $\sigma=-\alpha\ge0$, since $\gamma-\sigma=\frac32+\alpha>\frac12$.

Finally, for the integral on the interval $\left[\frac{1}{2},1\right]$, since $1+\alpha>0$ we have

$$\begin{aligned}\int_{1/2}^{1}\frac{1}{x^{3/2}}U_{\alpha+1}(s,x,z)\psi(s)\,ds &\le \frac{C}{x^{1-\alpha}|x^{1/2}-z^{1/2}|^{2(1+\alpha)}}\int_{1/2}^{1}(1-s^2)^{3/2+\alpha}\psi(s)\,ds\\ &\le \frac{C|x^{1/2}-z^{1/2}|^{-2\alpha}}{x^{1-\alpha}|x^{1/2}-z^{1/2}|^2}\le C\frac{x^{-\alpha}|x^{1/2}+z^{1/2}|^2}{x^{1-\alpha}|x-z|^2}\le\frac{C}{|x-z|^2},\end{aligned}$$

since $-\alpha\ge 0$. This completes the proof of the theorem. □

5. Global region. Proof of Theorem 4

We consider the modified Hardy operators

$$\begin{aligned}H_0^\beta f(x)&=x^{-\beta-1}\int_0^x f(y)y^\beta\,dy, \quad x>0,\\ H_\infty^\beta f(x)&=x^\beta\int_x^\infty f(y)y^{-\beta-1}\,dy, \quad x>0.\end{aligned}$$

We shall use the following known results, whose proofs can be found in [2] and [8].

Lemma 3. *Let* $\beta>-1$.

(a) *If* $1<p<\infty$ *and* $\delta<(\beta+1)p-1$, H_0^β *is of strong type* (p,p) *with respect to* $x^\delta\,dx$.

(b) H_0^β *is of weak type* $(1,1)$ *with respect to* $x^\delta\,dx$ *as long as* $\delta\le\beta$.

(c) *If* $1<p<\infty$ *and* $\delta=(\beta+1)p-1$, H_0^β *is of restricted weak type* (p,p) *with respect to* $x^\delta\,dx$.

Lemma 4. *Let* $\beta>-1$.

(a) *If* $1<p<\infty$ *and* $\delta>-\beta p-1$ *then* H_∞^β *is of strong type* (p,p) *with respect to* $x^\delta\,dx$.

(b) H_∞^β *is of weak type* $(1,1)$ *with respect to* $x^\delta\,dx$ *as long as* $\delta\ge-\beta-1$ *when* $\beta\ne0$ *and* $\delta>-1$ *when* $\beta=0$.

(c) H_∞^β *is of restricted weak type* (p,p) *with respect to* $x^\delta\,dx$ *when* $\delta=-\beta p-1$ *and* $p>1$ *when* $\beta\ne0$.

Proof of Theorem 4. Our aim is to prove that on the global region each of the terms involved in R_+^α or $R_-^{\alpha+1}$ are bounded, up to a constant, by the following sum of operators

$$H_0^{1+\alpha/2}+H_0^{\alpha/2}+H_\infty^{\alpha/2}+\widetilde{U}_{\alpha,\mathrm{glob}}^*, \tag{5.1}$$

where $\widetilde{U}_\alpha^*$ is a minor modification of U_α^*, the maximal operator of the semigroup associated to our system of Laguerre functions. More precisely

$$\widetilde{U}_\alpha^* f(x) = \sup_{0<s\le 1} \int \widetilde{U}_\alpha(s,x,z) f(z)\,dy,$$

where $\widetilde{U}_\alpha = \widetilde{M} I_\alpha$, with $\widetilde{M} = \frac{1}{2}((1-s^2)/2s) e^{-1/8(s+1/s)|x^{1/2}-z^{1/2}|^2} e^{-1/2(s+1/s)(xz)^{1/2}}$.

Therefore the only difference with U_α is the $\frac{1}{8}$ in the first exponential of $\widetilde{M}$ instead of $\frac{1}{4}$. We refer to [3] in order to check that such a change does not affect the estimates obtained there. In particular all the results of Theorem 1 remain true for $\widetilde{U}_\alpha^*$. Moreover we point out that, as in the case of U_α^*, the operator

$$\widetilde{U}_{\alpha,\mathrm{glob}}^* f(x) = \sup_{0<s\le 1} \int \widetilde{U}_\alpha(s,x,z) \chi_G(x,z) f(z)\,dz,$$

with $G = \{(x,z) : (x,z)/x < z/4 \text{ or } x > 4z\}$ satisfies the inequality

$$\widetilde{U}_{\alpha,\mathrm{glob}}^* f \le C(H_0^{\alpha/2} f + H_\infty^{\alpha/2} f)$$

for any $\alpha > -1$ and $f \ge 0$.

Next we check that the sum of operators given in (5.1) satisfies all the statements of Theorem 4.

First, to get (i), according to Theorem 1 of [3] and Lemmas 3 and 4 , if $1<p<\infty$ and $-2p/\alpha-1<\delta<(\alpha+2)p/2-1$, the sum $H_0^{\alpha/2}+H_\infty^{\alpha/2}+\widetilde{U}_\alpha^*$ is bounded on $L^p(x^\delta)$ as stated. The remaining operator $H_0^{1+\alpha/2}$ gives no further restriction on δ since it is bounded for $\delta<(2+\alpha/2)p-1$.

Now, for $p=1$, using the same results, the last three terms are of weak type for $-\alpha/2-1 \le \delta \le \alpha/2$ when $\alpha \ne 0$ and with strict inequality on the left-hand side when $\alpha=0$. Regarding $H_0^{1+\alpha/2}$, it is of weak type $(1,1)$ with respect to $x^\delta\,dx$ for the wider range $\delta \le 1+\alpha/2$. So (ii) holds.

Finally, for $1<p<\infty$ and $\delta=(\alpha+2)p/2-1$, the operators $\widetilde{U}_\alpha^*$ and $H_0^{\alpha/2}$ are of restricted weak type and the two others are of strong type, while, when $1<p<\infty$ and $\delta=-\alpha p/2-1$, $\alpha\ne 0$, $H_\infty^{\alpha/2}$ is of restricted weak type, $\widetilde{U}_\alpha^*$ is of weak type and the remaining ones are of strong type (p,p) with respect to $x^\delta\,dx$.

Therefore, in order to complete the proof we have to bound the terms of R_+^α and $R_-^{\alpha+1}$ by the stated operators.

Let us remind that the kernels of the Riesz transforms are given by

$$\int_0^1 K_+^\alpha(s,x,z)\psi(s)\,ds \quad \text{and} \quad \int_0^1 K_-^{\alpha+1}(s,x,z)\psi(s)\,ds,$$

with $\psi(s)=(\log(1+s)/(1-s))^{-1/2}/(1-s^2)$ and

$$K_+^\alpha=\frac{1}{2}\frac{1-s^2}{2s}\left(z^{1/2}U_{\alpha+1}-\frac{1-s}{1+s}x^{1/2}U_\alpha\right),$$
$$K_-^{\alpha+1}=-\frac{1}{2}\frac{1-s^2}{2s}\left(z^{1/2}U_{\alpha+2}-\frac{1+s}{1-s}x^{1/2}U_{\alpha+1}\right)-\frac{\alpha+1}{x^{1/2}}U_{\alpha+1}.$$

First we notice that in the global region $G=\{(z,z):z<x/4 \text{ or } z>4x\}$ we have

$$|x^{1/2}-z^{1/2}|^2\geq C\max\{x,z\}\simeq x+z.$$

Now, for the first term in K_+^α we have for $(x,z)\in G$,

$$\begin{aligned}\frac{1}{2}\frac{1-s^2}{2s}z^{1/2}U_{\alpha+1}&=\frac{1}{2}\frac{1-s^2}{2s}z^{1/2}MI_{\alpha+1}\\&\leq C\frac{1-s^2}{s}z^{1/2}e^{(1/8s)|x^{1/2}-z^{1/2}|^2}\widetilde{M}I_{\alpha+1}\\&\leq C\frac{1-s^2}{s}e^{-cx/s}z^{1/2}\widetilde{U}_{\alpha+1},\end{aligned}$$

and then for $f\geq 0$, interchanging the order of integration we have

$$\int_0^\infty \chi_G(x,z)f(z)z^{1/2}\int_0^1\frac{1-s^2}{2s}U_{\alpha+1}(s,x,z)\psi(s)\,ds\,dz\leq\frac{C}{x^{1/2}}\widetilde{U}^*_{\alpha+1,\mathrm{glob}}(z^{1/2}f)(x),$$

since $\int_0^1 e^{-cx/s}\psi(s)\,ds/s\leq Cx^{-1/2}$.

As we remarked above, $\widetilde{U}^*_{\alpha+1,\mathrm{glob}}\leq C(H_0^{(\alpha+1)/2}+H_\infty^{(\alpha+1)/2})$. By looking at the definition of these operators we get that

$$x^{-1/2}\widetilde{U}^*_{\alpha+1,\mathrm{glob}}(z^{1/2}f)(x)\leq C(H_0^{1+\alpha/2}f+H_\infty^{\alpha/2}f)(x),$$

and the claim is true for this piece of R_+^α. For the second term in K_+^α, proceeding as above, its absolute value is bounded in the global region by

$$C\frac{(1-s)^2}{s}x^{1/2}U_\alpha\leq C\frac{(1-s)^2}{s}x^{1/2}e^{-cx/s}\widetilde{U}_\alpha,$$

and hence for $f\geq 0$, interchanging the order of integration, we obtain

$$\int_0^\infty \chi_G(x,z)f(z)\int_0^1\frac{1-s^2}{s}x^{1/2}U_\alpha(s,x,z)\psi(s)\,ds\leq C\widetilde{U}^*_\alpha f(x),$$

and we are done with this term.

Regarding $R_-^{\alpha+1}$, the first term in $K_-^{\alpha+1}$ is bounded in absolute value by the first term in K_+^{α}, since $U_{\alpha+2}\le U_{\alpha+1}$. As for the second term in $K_-^{\alpha+1}$ we have the bound

$$C\frac{x^{1/2}}{s}U_{\alpha+1}\le c\frac{xz^{1/2}}{s}\frac{1-s^2}{s}U_\alpha,$$

using that $I_{\alpha+1}(w)\le wI_\alpha(w)$. In the global region we use that $U_\alpha\le Ce^{-c(x+z)/s}\widetilde{U}_\alpha$ to conclude that the above kernel, in the global region, is bounded by

$$C(1-s^2)\frac{x}{s^{3/2}}\left(\frac{z}{s}\right)^{1/2}e^{-c(x+z)/s}\widetilde{U}_\alpha\le C(1-s^2)\frac{x}{s^{3/2}}e^{-cx/s}\widetilde{U}_\alpha.$$

Then, integration against $\psi(s)$ and $f(z)$ leads to the bound $\widetilde{U}_\alpha^* f(x)$, since $\int_0^1(1-s^2)s^{-3/2}e^{-cx/s}\psi(s)\,ds\le C/x$.

Finally, for the third term in $K_-^{\alpha+1}$, our aim is to prove that for $(x,z)\in G$,

$$\int_0^1 x^{-1/2}U_{\alpha+1}(s,x,z)\psi(s)\,ds\le C\frac{x^{\alpha/2}z^{(\alpha+1)/2}}{(x+z)^{\alpha+3/2}}. \tag{5.2}$$

In fact in the global region and for $w=((1-s^2)/2s)(xz)^{1/2}\le 1$, we have the bound

$$x^{-1/2}U_{\alpha+1}(s,x,z)\le C\left(\frac{1-s^2}{s}\right)^{2+\alpha}x^{\alpha/2}z^{(\alpha+1)/2}e^{-c(x+z)/s},$$

while for $w\ge 1$, using the corresponding bound for $I_{\alpha+1}$ and that $(((1-s^2)/2s)(xz)^{1/2})^{\alpha+3/2}\ge 1$, we arrive at the same estimate.

Then for $(x,z)\in G$,

$$\int_0^1 x^{-1/2}U_{\alpha+1}(s,x,z)\psi(s)\,ds\le Cx^{\alpha/2}z^{(\alpha+1)/2}\int_0^1\left(\frac{1-s^2}{s}\right)^{2+\alpha}e^{-c(x+z)/s}\psi(s)\,ds.$$

Splitting the integral from 0 to $\frac{1}{2}$ and from $\frac{1}{2}$ to 1, as usual, it is easily checked that it behaves like $(x+z)^{-(\alpha+3/2)}$, proving the claim.

Now, we notice that the right-hand side of (5.2) is bounded either by $Cx^{-\alpha/2-1}z^{\alpha/2}$ or by $Cx^{\alpha/2}z^{-\alpha/2-1}$, since $\alpha+1>0$.

Therefore the kernel corresponding to the third term of $K_-^{\alpha+1}$ is bounded in the global region by

$$C(\chi_{\{z\le x\}}x^{-\alpha/2-1}z^{\alpha/2}+\chi_{\{z\ge x\}}x^{\alpha/2}z^{-\alpha/2-1}),$$

giving rise to the operators $H_0^{\alpha/2}$ and $H_\infty^{\alpha/2}$, respectively. This completes the proof of the theorem. □

Remark 1. We point out that the operator $H_\infty^{\alpha/2}$ appeared in our argument just twice. First when we bound the first term using $\widetilde{U}^*_{\alpha+1,\mathrm{glob}}$ and later when estimating the third term of $R_-^{\alpha+1}$. In fact $\widetilde{U}^*_{\alpha+1,\mathrm{glob}}$ may be bounded as in [3] by a smaller operator $T_{\alpha+1}$ which is of weak type (p,p) for $p=-2/\alpha(\delta+1)$, $1<p<\infty$. So the only term in our argument which might not be of such a weak type is the last term of $R_-^{\alpha+1}$.

6. Sharpness of the results. Proof of Theorem 2

(a) It is enough to show the negative result for R_+^α. Suppose it were bounded on $L^1(x^\delta)$. For ε small, choosing $I_\varepsilon=(1+2\varepsilon,2)$ $J_\varepsilon=(1,1+\varepsilon)$ and $f_\varepsilon=\chi_{J_\varepsilon}$ we would have

$$C_1\int_{I_\varepsilon}\left|\int_1^{1+\varepsilon} R_+^\alpha(x,z)\,dz\right|x^\delta\,dx \le C\|f_\varepsilon\|_{L^1(x^\delta)}\le C_2\varepsilon.$$

Now if we split the kernel as

$$R_+^\alpha(x,z)=A(x,z)+B(x,z),$$

having in mind that $x\sim 1$, we would arrive at

$$C_1\int_{I_\varepsilon}\left|\int_{J_\varepsilon} A(x,z)\,dz\right|dx\le C_2\left(\varepsilon+\int_{I_\varepsilon}\int_{J_\varepsilon}|B(x,z)|\,dz\,dx\right).$$

In order to get a contradiction we must make a clever partition of the kernel. First observe that for $(x,z)\in I_\varepsilon\times J_\varepsilon$,

$$1<z<1+\varepsilon<1+2\varepsilon<x<2.$$

In particular (x,z) belong to the local region. Moreover, as

$$w=\frac{1-s^2}{2s}(xz)^{1/2}\ge 1$$

if and only if $0\le s\le(\sqrt{1+xz}-1)/\sqrt{xz}=a_{xz}$ and in our situation $1\le\sqrt{xz}\le 2$, we have that $a_{xz}\ge\gamma=(\sqrt{2}-1)/2$.

Then, for $(x,z)\in I_\varepsilon\times J_\varepsilon$ and $0<s<\gamma$, we must have $w\ge 1$ and in that case we may use

$$I_\alpha(w)\simeq C\frac{e^w}{w^{1/2}},\quad \alpha>-1.$$

With the notation used in the proof of Theorem 3 we set

$$A(x,z)=\int_0^\gamma H_1^1(s,x,z)\psi(s)\,ds,$$

and hence

$$B(x,z)=\int_\gamma^1 H_1^1(s,x,z)\psi(s)\,ds+\int_0^1 H_1^2(s,x,z)\psi(s)\,ds.$$

First we bound $A(x,z)$ from below. Recalling that

$$H_1^1(s,x,z)=\frac{1-s}{1+s}(z^{1/2}-x^{1/2})U_{\alpha+1}(s,x,z),$$

we see that H_1^1 does not change sign on the integration domain and by the above remark, for $0<\gamma<s$ we have

$$U_{\alpha+1}(s,x,z)\geq \frac{C}{s^{3/2}}e^{-(1/4s)|x^{1/2}-z^{1/2}|^2},$$

as the other exponentials either cancel or are bounded from below when $1\leq x$, $z\leq 2$.

Therefore

$$-A(x,z)\geq (x^{1/2}-z^{1/2})\int_0^\gamma \frac{1}{s}e^{-(1/4s)|x^{1/2}-z^{1/2}|^2}\,\frac{ds}{s}.$$

Changing variables we get that the integral behaves like $|x^{1/2}-z^{1/2}|^{-2}$ and hence for $(x,z)\in I_\varepsilon\times J_\varepsilon$ we have

$$-A(x,z)\geq \frac{C}{x-z},$$

and then since $x-z\leq x-1$

$$\int_{I_\varepsilon}\left|\int_{J_\varepsilon} A(x,z)\,dz\right|dx\geq C\int_{1+2\varepsilon}^2 \frac{\varepsilon}{x-1}\,dx=C\varepsilon\log\frac{1}{2\varepsilon}.$$

Now we bound B from above. Coming back to the proof of Theorem 3 all the terms in $H=H_1+H_2$ are bounded for $\gamma\leq s\leq 1$ by $(1-s^2)^\sigma e^{-C|x^{1/2}-z^{1/2}|^2}$ for some $\sigma>0$ (see (4.3) and (4.5)).

Since in our case $x\geq 1$ we obtain for $(x,z)\in I_\varepsilon\times J_\varepsilon$,

$$\left|\int_\gamma^1 H_1(s,x,z)\psi(s)\,ds\right|\leq C.$$

To take care of H_1^2, for $0<s<\gamma$ we refer to (4.5) where we got for $w\geq 1$,

$$|H_1^2(s,x,z)|\leq Ce^{-(1/4s)|x^{1/2}-z^{1/2}|^2}\left(\frac{e^{-csx}}{s^{1/2}}+\frac{1}{s^{1/2}x}\right).$$

Using that in our situation $x\geq 1$ we arrive at the bound

$$\frac{C}{s^{1/2}}e^{-(1/4s)|x^{1/2}-z^{1/2}|^2}.$$

As we observed, for $0<s<\gamma$, $w=((1-s^2)/2s)(xz)^{1/2}$ must be greater than one so that

$$\begin{aligned}\int_0^\gamma |H_1^2(s,x,z)|\psi(s)\,ds &\leq \int_0^\gamma e^{-(1/4s)|x^{1/2}-z^{1/2}|^2}\,\frac{ds}{s}\leq \int_{(1/\gamma)|x^{1/2}-z^{1/2}|^2}^\infty e^{-u}\,\frac{du}{u}\\ &\leq C\left(1+\log\frac{C}{|x-z|}\right),\end{aligned}$$

since $|x^{1/2}-z^{1/2}|\sim|x-z|$ in our situation. In this way we get

$$\begin{aligned}\int_{I_\varepsilon}\int_{J_\varepsilon}|B(x,z)|\,dz\,dx &\leq C\varepsilon+\varepsilon\int_{1+2\varepsilon}^2 \log\left(\frac{C}{x-\varepsilon-1}\right)dx\\ &\leq C\varepsilon\left(1+\int_0^1\log\frac{c}{u}\,du\right)\leq C\varepsilon.\end{aligned}$$

Putting together all the estimates, if R_+^α were of strong type with respect to $x^\delta\,dx$ we would arrive at $\log(c/\varepsilon)\leq C$ for all ε small enough. This proves statement (a).

(b) As we remarked after proving Theorem 4, see Remark 1, the only piece that was not bounded for an operator of weak type (p,p) for $p=-2(\delta+1)/\alpha$, $\alpha\neq 0$ and $p>1$, was the one coming from the third term in $R_-^{\alpha+1}$, namely $x^{-1/2}U_{\alpha+1}$ and over the region $z\geq 4x$. Since the local parts of both Riesz operators is bounded on $L^p(x^\delta)$ for any $p>1$ and any δ, the operator $\|\mathcal{R}^\alpha\|$ will not be of weak type for such p if and only if the operator with kernel

$$H(x,z)=\chi_{\{z\geq 4x\}}\int_0^1 x^{-1/2}U_{\alpha+1}(s,x,z)\psi(s)\,ds,$$

it is not of weak type. Since it is non-negative we are going to bound it from below for $(x,z)\in D=\{(x,z):z\geq 4x,\ 0\leq x\leq\frac14,\ 0\leq z\leq 1\}$.

Let us start observing that $w=((1-s^2)/2s)(xz)^{1/2}\leq 1$ if and only if $s\geq a_{xz}=\sqrt{xz}/(1+\sqrt{x,z})$. Since in D, $a_{xz}\leq\sqrt{xz}$ for $\sqrt{xz}\leq s$, we may use the estimate $I_{\alpha+1}(w)\simeq w^{\alpha+1}$. Therefore

$$H(x,z)\geq C\chi_{\{z\geq 4x\}}x^{-1/2}\int_{\sqrt{xz}}^{3/4}M(s,x,z)\left(\frac{1-s^2}{2s}(xz)^{1/2}\right)^{\alpha+1}\psi(s)\,ds.$$

For the exponentials in M we observe that $(s+1/s)|x^{1/2}-z^{1/2}|^2 \le C((x+z)/s+1)$ and also $(s+1/s)(xz)^{1/2} \le C((x+z)/s+1)$. Moreover, since s is away from 1, we get

$$H(x,z) \ge C x^{\alpha/2} z^{(\alpha+1)/2} \chi_{\{z\ge 4x\}} \int_{\sqrt{xz}}^{3/4} \frac{e^{-c(x+z)/s}}{s^{\alpha+3/2}} \frac{ds}{s}.$$

Performing the change of variables $(x+z)/s=u$ the integral turns into

$$\frac{C}{(x+z)^{\alpha+3/2}} \int_{4/3(x+z)}^{(x+z)/\sqrt{xz}} e^{-u} u^{\alpha+3/2} \frac{du}{u} \ge \frac{C}{(x+z)^{\alpha+3/2}}.$$

Since $(x+z)/\sqrt{xz}=(x/z)^{1/2}+(z/x)^{1/2}\ge 2$ and $\frac{4}{3}(x+z)\le\frac{5}{3}\le 2$.

Altogether, using $x+z\le\frac{5}{4}z$,

$$\chi_D(x,z)H(x,z) \ge C x^{\alpha/2} z^{-\alpha/2-1} \chi_D(x,z).$$

Then our claim would be true if we can show that for some $f\in L^p(x^\delta)$ with $p=-2(\delta+1)/\alpha>1$, supported on $(0,1)$, the function

$$h(x) = \chi_{(0,1/4)}(x) x^{\alpha/2} \int_x^1 f(z) z^{-\alpha/2-1}\, dz$$

does not belong to $L^{p,\infty}(x^\delta)$.

Take now $f(z)=\chi_{(0,1/4)}(z)z^{\alpha/2}/\log(1/z)$. Clearly we have $f(z)\in L^p(x^\delta)$ for $p=-2(\delta+1)/\alpha$. Also, in this case we have

$$h(x) = \chi_{(0,1/4)} x^{\alpha/2} \log\log\frac{1}{x}.$$

We would like to show that this function does not belong to $L^{p,\infty}(x^\delta)$. Since $p=-2(\delta+1)/\alpha$ is positive we distinguish two cases: $\alpha>0$ and $\delta+1<0$, or $\alpha<0$ and $\delta+1>0$. In either case the function is monotone near the origin, increasing in the first case and decreasing in the second one. Now if we call $\mu_h(\lambda)$ the distribution function of h with respect to $x^\delta\, dx$, that is

$$\mu_h(\lambda) = \int_{\{h(x)>\lambda\}} x^\delta\, dx,$$

straightforward calculations give that $\mu_h(\lambda)\simeq h^{-1}(\lambda)^{\delta+1}$ for small λ in the first case and for large λ in the second case. Therefore if h were in $L^{p,\infty}(x^\delta)$ we would have

$$h^{-1}(\lambda) \ge C\lambda^{2/\alpha} \text{ for } \lambda\to 0, \quad \text{or} \quad h^{-1}(\lambda)\le C\lambda^{2/\alpha} \text{ for } \lambda\to\infty.$$

Since $\delta+1<0$ in the first case and $\delta+1>0$ in the second one. This means that

$$\lambda \ge h(c\lambda^{2/\alpha}),\ \lambda\to 0, \quad \text{or, alternatively} \quad \lambda\le h(c\lambda^{2/\alpha}),\ \lambda\to\infty,$$

h^{-1} being increasing or decreasing. Therefore either $\log\log(1/c\lambda^{2/\alpha})\to 0$ when $\lambda\to 0$, or $\log\log(1/c\lambda^{2/\alpha})\to\infty$ when $\lambda\to\infty$. Since both conclusions are false the claim is proved.

(c) It is enough to show that R_+^α is not of weak type (p,p) for p such that $1<p<\infty$ and $(\alpha+2)/2p=\delta+1$. Let us observe that in this case, since $\alpha+2$ is always positive, $\delta+1$ must be positive.

First, since the local part is strong type (p,p) for any $1<p<\infty$ and any δ, we may restrict ourselves to consider the global part. Following the estimates given in the proof of Theorem 4 we see that the first term of $R^\alpha_{+,\mathrm{glob}}$ was bounded by the operators $H_0^{1+\alpha/2}$ and $H_\infty^{\alpha/2}$. But, according to Lemma 3, $H_0^{1+\alpha/2}$ is of strong type (p,p) with respect to x^δ as long as $\delta+1<(2+\alpha/2)p$.

Since in our case $p=2(\delta+1)/(\alpha+2)$, the last inequality is clearly satisfied. As for $H_\infty^{\alpha/2}$, it is strong type, according to Lemma 4, as long as $\delta+1>-\alpha p/2$, which in our case means $1>-\alpha/(\alpha+2)$ which is true since $\alpha>-1$.

Therefore it would be enough to show that the remaining term of $R^\alpha_{+,\mathrm{glob}}$ is not of weak type. To do that let us take $f(z)=\chi_{(0,1/4)}(z)(z^{-1-\alpha/2})/\log(1/z)$. Clearly $f\in L^p(x^\delta)$ for $p=2(\delta+1)/(\alpha+2)$.

Also,

$$\int_0^\infty z^{\alpha/2} f(z)\,dz=\infty.$$

Then, if we call this piece of the operator $R^+_{\alpha,2}$, we have that

$$\begin{aligned} R^+_{\alpha,2}f(x) &= \int_0^{1/4}\left(\int_0^1 \frac{1-s}{1+s}x^{1/2}U_\alpha(s,x,z)\psi(s)\,ds\right)f(z)\,dz \\ &\geq C\int_0^{1/4}\int_{1/4}^{1/2} x^{1/2}U_\alpha(s,x,z)\,ds f(z)\,dz. \end{aligned}$$

Observe that if we restrict our positive function $R^+_{\alpha,2}f(x)$ to $1<x<2$, together with the inequalities $0<z<\frac14$ and $\frac14<s<\frac12$ it gives that $w=((1-s^2)/2s)(xz)^{1/2}\leq 1$ and then having in mind that all the exponentials on U_α are bounded we get for these values of x,

$$R^+_{\alpha,2}f(x)\geq C\int_0^{1/4}\int_{1/4}^{1/2}\left(\frac{1-s^2}{2s}(xz)^{1/2}\right)^\alpha ds f(z)\,dz \geq C\int_0^{1/4} z^{\alpha/2}f(z)\,dz=\infty.$$

Hence $R^+_{\alpha,2}f$ is infinite on some interval, proving the assertion.

7. Some relations with other systems of Laguerre functions

As we said in the introduction, Riesz transforms associated to a different system of Laguerre functions have been considered by Nowak and Stempak in a recent paper (see [11]). In this section we briefly describe the relationship between their results and ours. Let $k=(k_1,...,k_d)\in\mathbb{N}^d$, $\mathbb{N}=\{0,1,...\}$, and $\alpha=(\alpha_1,...,\alpha_d)\in(-1,\infty)^d$ be multi-indices. The Laguerre functions φ_k^α on $\mathbb{R}_+^d$ are defined as

$$\varphi(x)=\varphi_{k_1}^{\alpha_1}(x_1)...\varphi_{k_d}^{\alpha_d}(x_d),\quad x=(x_1,...,x_d)\in\mathbb{R}_+^d,$$

where $\varphi_{k_i}^{\alpha_i}$ are the one-dimensional Laguerre functions

$$\varphi_{k_i}^{\alpha_i}(x_i)=\sqrt{2x_i}\mathcal{L}_{k_i}^{\alpha_i}(x_i^2). \tag{7.1}$$

Given $\alpha\in\left[-\frac{1}{2},\infty\right)^d$, the Riesz transforms R_j^α are defined for functions $f\in L^2$ as

$$R_j^\alpha f=-2\sum_{k\in\mathbb{N}^d}\left(\frac{k_j}{4|k|+2|\alpha|+2d}\right)^{1/2}\langle f,\varphi_k^\alpha\rangle\varphi_{k-e_j}^{\alpha+e_j},\quad j=1,...,d,$$

with the convention $\varphi_{k-e_j}^{\alpha+e_j}=0$ if $k_j-1<0$.

In dimension one, if we change in the above expression φ_k^α to $\mathcal{L}_k^\alpha$ and $\varphi_{k-1}^{\alpha+1}$ to $\mathcal{L}_{k-1}^{\alpha+1}$, we recover the definition given in the present paper of the operators R_+^α. It is therefore clear, using relation (7.1), that for $1<p<\infty$, and real numbers γ and δ, related by

$$\gamma=2\delta-\frac{p}{2}+1,$$

the inequalities

$$\int_0^\infty |R^\alpha f(x)|^p x^\gamma\,dx\le C\int_0^\infty |f(x)|^p x^\gamma\,dx$$

and

$$\int_0^\infty |R_+^\alpha f(y)|^p y^\delta dy\le C\int_0^\infty |f(y)|^p y^\delta dy$$

are equivalent. In particular The δ-interval that appears in our Theorem 1 (a) leads to the following γ-interval for the φ-setting

$$-\left(\alpha+\tfrac{1}{2}\right)p-1<\gamma<\left(\alpha+\tfrac{3}{2}\right)p-1. \tag{7.2}$$

In other words we get strong power weighted results for operators associated to a given Laguerre orthonormal system from a corresponding weighted result for a different orthonormal system. This fact has also appeared in [3] in relation with the heat semigroup and has been systematically analyzed in [1] for a larger class of operators and for more Laguerre function systems. Roughly speaking, one could say that an exhaustive knowledge of the weighted L^p boundedness of the operators associated to a particular Laguerre system, implies a complete knowledge of the boundedness of the corresponding operators on the other Laguerre orthonormal systems. In the particular case we are considering now, the range of γ described in (7.2) for $\alpha \geq -\frac{1}{2}$, is wider than the A_p-range $-1<\gamma<p-1$, admitted in Theorem 3.4 of [11]. Furthermore, boundedness results are also given for $-1<\alpha<-\frac{1}{2}$. On the other side, it is worth pointing out that in [11] results for non-necessarily power weights are obtained. Also multi-dimensional results are treated for $\alpha \in \left[-\frac{1}{2}, \infty\right)^d$, $\alpha_i \notin \left(-\frac{1}{2}, \frac{1}{2}\right)$.

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