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Gorenstein injective and projective modules and actions of finite-dimensional Hopf algebras

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Abstract. We study the stability of Gorenstein preenvelopes and precovers in the cases of H-extensions and smash products with H, where H is a Hopf algebra. We use these to define Gorenstein dimensions and give new examples of the so-called Gorenstein categories.

1. Introduction

In 1966 (in [15]) Auslander, motivated by Tate's observation on the existence of a complete projective resolution for the $\mathbb{Z}G$-module $\mathbb{Z}$, introduced a class of finitely generated modules having a certain complete resolution by projective modules. Then using these modules he defined the G-dimension (G ostensibly for Gorenstein) of finitely generated modules. It seems appropriate then to call the modules of G-dimension 0 the Gorenstein projective modules. In [8] Gorenstein projective modules (whether finitely generated or not) were defined. In the same paper the dual notion to that of a Gorenstein projective module was defined and so a relative theory of Gorenstein injective and projective modules was initiated (cf. [2] and [9] and their references). One of the main problems concerning Gorenstein injective and projective modules is to study the existence of covers and envelopes by these classes of modules, which allows one to study the so-called Gorenstein dimensions. Although there is a very nice relative theory over noetherian rings with nice homological properties (cf. [13]), Gorenstein dimensions may be defined over a more general class such as noetherian or coherent rings (cf. [10]) or those with a dualizing complex ([14]). More recently, in [5], the authors show the existence of a bound for the Gorenstein injective dimension in the category of quasi-coherent sheaves on certain projective schemes, which motivates them to introduce the concept of

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Gorenstein category without involving projective objects and using the global finitistic injective and projective dimensions (this one defined in terms of the vanishing of the Ext's). They show that a Grothendieck category with enough projectives is Gorenstein if and only if the global Gorenstein injective and projective dimensions are finite.

On the other hand, in [11], the authors began the study of the preserving covers problem, i.e., given a functor between two categories $\Phi\colon \mathcal{A}\to\mathcal{B}$ and an $\mathcal{F}$-cover $f\colon F\to A$ in $\mathcal{A}$, determine whether $\Phi(f)\colon \Phi(F)\to\Phi(A)$ is a $\Phi(\mathcal{F})$-cover in $\mathcal{B}$. Lately this problem was treated (among others) for the case of relative injective covers over Hopf extensions ([12]) and for Gorenstein modules over graded rings ([1]). In the aforementioned works, separability conditions of the involved functors (cf. [17]) are very useful.

The aim of this paper is to continue the study of the preserving property of covers for the case of H-extensions, where H is a Hopf algebra, and smash products of H with an algebra. Given a finite-dimensional Hopf algebra H over a field k and a k-algebra A, we show that the existence of Gorenstein injective preenvelopes and Gorenstein projective precovers are equivalent in A-Mod and $A\#H$-Mod under certain separability conditions of the extension $A\#H/A$, or in other words, Gorenstein injective and projective modules are, respectively, preenveloping and precovering classes equivalently in A-Mod and $A\#H$-Mod (Theorems 3.3 and 3.6). We also derive Gorenstein injective preenvelopes and Gorenstein projective precovers to the subalgebra of invariants A^H (Theorems 4.1 and 4.2). Finally, and as a natural consequence of the preserving of precovers and preenvelopes we define Gorenstein injective and projective dimensions in $A\#H$-Mod and A^H-Mod in terms of resolutions by these modules. Then we show that under some natural hypothesis, finiteness of global Gorenstein injective and projective dimensions in A-Mod, $A\#H$-Mod and A^H-Mod are equivalent, and thus obtaining new examples of Gorenstein categories from a given one (Theorem 5.4).

2. Some preliminaries

Let us start by giving a short introduction to the rich theory of Hopf algebras and their extensions. For more details and unexplained concepts we refer the reader to [16] and [4].

Throughout this paper k will denote a field. A *k-algebra* may be defined as a triple (A, M, u), where A is a vector space over k and $M\colon A\otimes A\to A$ and $u\colon k\to A$ are morphisms of k-vector spaces such that $M\circ(\mathrm{Id}_A\otimes M)=M\circ(M\otimes\mathrm{Id}_A)$ and if s_A and s'_A denote the isomorphisms $k\otimes A\cong A$ and $A\otimes k\cong A$, respectively, then $M\circ(u\otimes\mathrm{Id}_A)=s$ and $M\circ(\mathrm{Id}_A\otimes u)=s'$. A *$k$-coalgebra* is defined dually, i.e., a triple

$(C, \triangle, \varepsilon)$, where C is a k-vector space and $\triangle\colon C \to C\otimes C$ and $\varepsilon\colon C\to k$ are morphisms of k-vector spaces such that $(\mathrm{Id}_C \otimes\triangle)\circ\triangle=(\triangle\otimes\mathrm{Id}_C)\circ\triangle$ and $(\varepsilon\otimes\mathrm{Id}_C)\circ\triangle=s_C$ and $(\mathrm{Id}_C \otimes\varepsilon)\circ\triangle=s'_C$. A *bialgebra* is then defined as a k-vector space H endowed with an algebra structure $H^a=(H, M, u)$ and a coalgebra structure $H^c=(H, \triangle, \varepsilon)$ and such that $\triangle$ and ε are algebra morphisms. Now given the vector space $\mathrm{Hom}_k(H^c, H^a)$ we may define an algebra structure on it given by the convolution product $f*g$ defined as $(f*g)(c)=\sum f(c_1)g(c_2)$, where $\triangle(c)=\sum c_{(1)}\otimes c_{(2)}$. Given a bialgebra H, a linear morphism $S\colon H\to H$ is called an *antipode* if S is the inverse of the identity map Id_H with respect to the convolution product. A bialgebra H is then a *Hopf algebra* if H has an antipode. So, from now on, H will denote a finite-dimensional Hopf algebra over k with comultiplication $\triangle\colon H\to H\otimes H$, counit $\varepsilon\colon H\to k$ and antipode $S\colon H\to H$.

A k-algebra A is called a *left H-module algebra* if A is a left H-module such that $h\cdot(ab)=\sum(h_{(1)}\cdot a)(h_{(2)}\cdot b)$ and $h\cdot 1_A=\varepsilon(h)1_A$ for all $a, b\in A$ and $h\in H$.

Given any left H-module M, the submodule of H-invariants is the set $M^H=\{m\in M : h\cdot m=\varepsilon(h)m \text{ for all } h\in H\}$ and analogously for right H-modules. If A is an H-module algebra, then A^H is a subalgebra of A.

The *smash product algebra* (or semidirect product) of A with H, denoted by $A\#H$, is the vector space $A\otimes H$, whose elements are denoted by $a\#h$ instead of $a\otimes h$, with multiplication given by $(a\#h)(b\#l)=\sum a(h_{(1)}\cdot b)\#h_{(2)}l$ for $a, b\in A$ and $h, l\in H$. The unit of $A\#H$ is $1\#1$ and we usually view ah as $a\#h$ and ha as $(1\#h)(a\#1)$.

The dual notion of H-module algebra is H-comodule algebra. A *right H-comodule* is a pair (M, ρ), where M is a k-vector space and $\rho\colon M\to M\otimes H$ is a linear morphism such that $(\mathrm{Id}_M \otimes\triangle)\circ\rho=(\rho\otimes\mathrm{Id}_M)\circ\rho$ and $(\mathrm{Id}_M \otimes\varepsilon)\circ\rho=s'_M$. An algebra A is a *right H-comodule algebra* if it is a right H-comodule, with structure map ρ such that $\rho(ab)=\sum a_{(0)}b_{(0)}\otimes a_{(1)}b_{(1)}\in A\otimes H$ for every $a, b\in A$. The category of H-modules is equivalent to the category of H^*-comodules.

A map $t\in H^*$ is called a *left integral* of H if $h^*t=h^*(1)t$ for any $h^*\in H^*$. Since H is finite-dimensional then it is shown that it has a non zero left integral t, and that the antipode S is bijective, with inverse S^{-1}. There is a unique element $\lambda\in H^*$ called the distinguished element of H^* such that $th=\lambda(h)t$ for all $h\in H$. Now if we denote by h^λ the element $\lambda\rightharpoonup h=\sum h_{(1)}\lambda(h_{(2)})$ and $B=A^H$ then we get two bimodule structures for a given H-module algebra A, namely, ${}_{A\#H}A_B$ given by $(a\#h)\rightharpoonup x=a(h\cdot x)$ and $x\leftharpoonup b=xb$ and ${}_BA_{A\#H}$ given by $b\rightharpoonup x=bx$ and $x\leftharpoonup(a\#h)=S^{-1}(h^\lambda)\cdot(xa)$.

We will say that $B\subset A$ is an *H-extension* and we will denote it by A/B if A is a right H-comodule algebra and B is the algebra of coinvariants, $B=A^{\mathrm{coH}}=\{a\in A: \rho(a)=a\otimes 1\}$. Since we are considering H finite-dimensional, then we get that A/B is an H-extension if and only if A is an H^*-module algebra with $B = A^{\mathrm{coH}} = A^{H^*}$. We will say then that whenever A is an H-module algebra, then A/A^H is an H^*-extension. An H-extension A/B is said to be *Galois* if the map $\beta\colon A\otimes_B A \to A\otimes H$ given by $\beta(a\otimes b) = \sum ab_{(0)}\otimes b_{(1)}$ is bijective.

If A is an H-module algebra, then A may be considered as an H^*-comodule algebra. The category of Hopf modules ${}_A\mathcal{M}^{H^*}$ consists of all those left A-modules M with a right H^*-comodule structure satisfying the compatibility condition $\rho_M(am) = \sum a_{(0)}m_{(0)}\otimes a_{(1)}m_{(1)}$ for all $a\in A$ and $m\in M$. Since H is finite-dimensional we may identify ${}_A\mathcal{M}^{H^*}$ with the category $A\#H^{**}$-Mod$=A\#H$-Mod.

Now we recall the definition of some functors: the *induction functor* Ind: B-Mod$\to A\#H$-Mod given by $\mathrm{Ind}(M) = {}_{A\#H}A\otimes_B M$, the *coinduction functor* Coind: B-Mod$\to A\#H$-Mod given by $\mathrm{Coind}(M) = \mathrm{Hom}_B(A_{A\#H}, M)$ and finally, if we take invariants, we get the functor $(-)_0\colon {}_A\mathcal{M}^{H^*} \to A^H$-Mod, where $M_0 = \{m\in M : \rho_M(m) = m\otimes_1\} = \{m\in M : h\cdot m = \langle 1, h\rangle m \text{ for all } m\in H\}$. It can be checked that $M_0 \cong \mathrm{Hom}_A^{H^*}(A, M) \cong \mathrm{Hom}_{A\#H}(A, M)$, where $\mathrm{Hom}_A^{H^*}(A, M)$ denotes the group of left A-module and right H^*-comodule homomorphisms.

We finish this section by recalling the notions of precover, preenvelope and Gorenstein injective and projective modules. We refer to [9] for a general view of the theory. Given a class of A-modules $\mathcal{F}$, an *$\mathcal{F}$-precover* of an A-module M is a morphism $F\xrightarrow{\varphi} M$ with $F\in\mathcal{F}$ and such that if $F'\xrightarrow{f} M$ is a morphism with $F'\in\mathcal{F}$ then there is a morphism $F'\xrightarrow{g} F$ such that $\varphi g = f$. $\mathcal{F}$-preenvelopes are defined dually.

A left A-module M is called *Gorenstein injective* if there exists an exact sequence $\ldots\to E^{-1}\to E^0\to E^1\to\ldots$ of injective modules such that $M = \mathrm{Ker}(E^0\to E^1)$ and that it remains exact whenever $\mathrm{Hom}_A(E, -)$ is applied for every injective E. Gorenstein projective modules are defined dually.

3. Gorenstein injective and projective modules and smash products

In this section we study the property of preserving Gorenstein preenvelopes and precovers between a category of A-modules for a given H-module algebra and the category of $A\#H$-modules. We start with a natural result. In what follows, A will always denote an H-module algebra. If $M\in A\#H$-Mod then ${}_AM$ will denote the image of M by the restriction of the scalars functor ${}_A(-)\colon A\# H$-Mod$\to A$-Mod.

Lemma 3.1. (i) *If* $M\in A$-Mod *is Gorenstein injective* (*resp. Gorenstein projective*) *then* $A\#H\otimes_A M$ *is Gorenstein injective* (*resp. Gorenstein projective*).

(ii) *If* $M\in A\#H$-Mod *is Gorenstein injective* (*resp. Gorenstein projective*) *then* ${}_AM$ *is Gorenstein injective* (*resp. Gorenstein projective*).

Proof. (i) The functor $A\#H\otimes_A -$ is isomorphic to $\mathrm{Hom}_A(A\#H,-)$ since H is finite-dimensional and so we have a double adjunction $(A\#H\otimes_A -, {}_A(-))$ and $({}_A(-), A\#H\otimes_A -)$. Now let $\xi\equiv ...E^{-1}\to E^0\to E^1\to ...$ be exact in A-Mod with $M=\mathrm{Ker}(E^0\to E^1)$ and such that it remains exact whenever $\mathrm{Hom}_A(E,-)$ is applied for every $E\in A$-Mod injective. Since $A\#H$ is free as a right A-module (cf. [4, Proposition 6.1.7]) we get that $A\#H\otimes_A\xi$ is exact and $A\#H\otimes_A M=\mathrm{Ker}(A\#H\otimes_A E^0\to A\#H\otimes_A E^1)$. We also get that $A\#H\otimes_A E^i$ is injective for every i since ${}_A(-)$ is exact and $A\#H\otimes_A -$ is a right adjoint. Let us suppose finally that $E\in A\#H$-Mod is injective. Then

$$\mathrm{Hom}_{A\#H}(E, A\#H\otimes_A\xi)\cong\mathrm{Hom}_A({}_AE,\xi)$$

But ${}_AE$ is injective since it is a right adjoint of $A\#H\otimes_A -$ which is exact. Thus we get that $\mathrm{Hom}_A({}_AE,\xi)$ is exact since M is Gorenstein injective and so, $A\#H\otimes_A M$ is also Gorenstein injective.

(ii) Let $M\in A\#H$-Mod be Gorenstein injective and let ξ be a complete exact sequence as above. Then ${}_A\xi$ is exact and ${}_AM=\mathrm{Ker}({}_AE^0\to {}_AE^1)$ since ${}_A(-)$ is exact. We also get from the above that ${}_AE^i$ is injective for every i. Finally, let us assume that $E\in A$-Mod is injective. Then

$$\mathrm{Hom}_A(E,{}_A\xi)\cong\mathrm{Hom}_{A\#H}(A\#H\otimes_A E,\xi).$$

Now, since $A\#H\otimes_A E$ is injective, by the above we get that $\mathrm{Hom}_{A\#H}(A\#H\otimes_A E,\xi)$ is exact and therefore ${}_AM$ is Gorenstein injective.

The corresponding proofs for Gorenstein projective modules are totally analogous. □

The next result gives a relation between Gorenstein injective preenvelopes in A-Mod and $A\#H$-Mod. Using this we will show next that their existence are equivalent in both categories.

Proposition 3.2. (i) *If $f\colon M\to E$ is a Gorenstein injective preenvelope in $A\#H$-Mod, then ${}_Af\colon {}_AM\to {}_AE$ is a Gorenstein injective preenvelope in A-Mod.*

(ii) *If $f\colon M\to E$ is a Gorenstein injective preenvelope in A-Mod, then $A\#H\otimes_A f\colon A\#H\otimes_A M\to A\#H\otimes_A E$ is a Gorenstein injective preenvelope in $A\#H$-Mod.*

(iii) *Let $M\in A$-Mod and assume that $A\#H/A$ is separable. If $A\#H\otimes_A M\xrightarrow{f} E$ is a Gorenstein injective preenvelope, then $M\xrightarrow{\eta_M}{}_A(A\#H\otimes_A M)\xrightarrow{{}_Af}{}_AE$, where η_M denotes the unit of the adjunction $(A\#H\otimes_A -, {}_A(-))$, is a Gorenstein injective preenvelope in A-Mod.*

(iv) *If* $A\#H/A$ *is separable and* $M \in A$-Mod*, then* M *has a Gorenstein injective preenvelope if and only if* $A\#H \otimes_A M$ *has a Gorenstein injective preenvelope.*

(v) *Suppose that* $A\#H/A$ *is separable and* A/A^H *is* H^**-Galois and let* $M \in A\#H$-Mod*. If* ${}_A M \xrightarrow{f} E$ *is a Gorenstein injective preenvelope, then* $M \xrightarrow{\eta_M} A\#H \otimes_A {}_A M \xrightarrow{A\#H \otimes_A f} A\#H \otimes_A E$*, where* η_M *denotes the unit of the adjunction* $({}_A(-), A\#H \otimes_A -)$*, is a Gorenstein injective preenvelope.*

(vi) *Under the same conditions as* (v)*,* $M \in A\#H$-Mod *has a Gorenstein injective preenvelope if and only if* ${}_A M$ *has a Gorenstein injective preenvelope.*

Proof. (i) and (ii) follow from Lemma 3.1 and [1, Proposition 2.5].

(iii) Since $A\#H/A$ is separable, the functor $A\#H \otimes_A -$ is separable and then, by [1, Proposition 2.6] using the adjunction $(A\#H \otimes_A, {}_A(-))$, we get the desired result.

(iv) This is a consequence of (ii) and (iii).

(v) In this case ${}_A(-)$ is separable by [18, Corollary 4.7] and so it follows as (iii).

(vi) This is a consequence of (i) and (v). □

Now as a consequence we get that the class of Gorenstein injective modules is preenveloping equivalently in the categories A-Mod and $A\#H$-Mod. The proof is analogous to that of [1, Theorem 2.9].

Theorem 3.3. *Suppose that* $A\#H/A$ *is separable. Then every* $M \in A\#H$-Mod *has a Gorenstein injective preenvelope if and only if every* $M \in A$-Mod *has a Gorenstein injective preenvelope.*

Corollary 3.4. *Let* A *be a* k*-algebra and* G *a finite group acting on* A *with* $1 \in \mathrm{tr}(Z(A))$*. Then every* A*-module has a Gorenstein injective preenvelope if and only if every* $A*G$*-module has a Gorenstein injective preenvelope.*

Proof. A is a kG-module algebra, $A*G = A\#H$ and $1 \in \mathrm{tr}(Z(A))$ is equivalent to the fact that $A*G/A$ is separable. □

Dual arguments now give the following results about Gorenstein projective precovers.

Proposition 3.5. (i) *If* $P \xrightarrow{g} N$ *is a Gorenstein projective precover in* $A\#H$-Mod*, then* ${}_A P \xrightarrow{Ag} {}_A N$ *is a Gorenstein projective precover.*

(ii) *If* $P \xrightarrow{g} N$ *is a Gorenstein projective precover in* A-Mod*, then*

$$A\#H \otimes_A P \xrightarrow{A\#H \otimes_A g} A\#H \otimes_A N$$

is a Gorenstein projective precover in $A\#H$-Mod*.*

(iii) *Let* $N\in A$-Mod *and assume that* $A\#H/A$ *is separable. If* $P\xrightarrow{g}A\#H\otimes_A N$ *is a Gorenstein projective precover, then* ${}_AP\xrightarrow{Ag}{}_A(A\#H\otimes_A N)\xrightarrow{\varepsilon_N}N$*, where* ε_N *denotes the counit of the adjunction* $({}_A(-), A\#H\otimes_A -)$*, is a Gorenstein projective precover.*

(iv) *Let* $N\in A$-Mod *and assume that* $A\#H/A$ *is separable. Then* N *has a Gorenstein projective precover if and only if* $A\#H\otimes_A N$ *has a Gorenstein projective precover.*

(v) *Suppose that* $A\#H/A$ *is separable and* A/A^H *is* H^**-Galois and let* $N\in A\#H$-Mod*. If* $P\xrightarrow{g}{}_AN$ *is a Gorenstein projective precover, then*

$$A\#H\otimes_A P\xrightarrow{A\#H\otimes_A g}A\#H\otimes_A {}_AN\xrightarrow{\varepsilon_N}N,$$

where ε_N *denotes the counit of the adjunction* $(A\#H\otimes_A -, {}_A(-))$*, is a Gorenstein projective precover.*

(vi) *Under the same conditions as (v),* $N\in A\#H$-Mod *has a Gorenstein projective precover if and only if* ${}_AN$ *has a Gorenstein projective precover.*

Theorem 3.6. *Let us suppose that* $A\#H/A$ *is separable. Then every* $N\in A\#H$-Mod *has a Gorenstein projective precover if and only if every* $N\in A$-Mod *has a Gorenstein projective precover.*

4. Gorenstein injective and projective modules and Hopf invariants

The aim of this section is to study how Gorenstein injective and projective modules and preenvelopes and precovers by these classes of modules behave under taking Hopf invariants.

Theorem 4.1. *Suppose that* $A\#H/A$ *is separable,* ${}_{A^H}A$ *is projective,* A_{A^H} *is flat and that any injective* $E\in A^H$-Mod *has finite projective dimension. If every* A*-module has a Gorenstein injective preenvelope, then any* A^H*-module has a Gorenstein injective preenvelope.*

Proof. By Theorem 3.3 every $A\#H$-module has a Gorenstein injective preenvelope.

On the other hand, by [18, Corollary 2.9], A^H-Mod is equivalent to a quotient category of $A\#H$-Mod with respect to a localizing subcategory Λ. If $\mathcal{A}=\{\mathrm{Hom}_{A^H}(A,M):M\in A^{H\text{-Mod}}\}$, then $\mathrm{Coind}(-)=\mathrm{Hom}_{A^H}(A,-)\colon A^{H\text{-Mod}}\to\mathcal{A}$ is an equivalence with inverse $(-)_0\colon \mathcal{A}\to A^H$-Mod. Now, since an equivalence of categories preserves Gorenstein injective preenvelopes, we only have to find a $\mathrm{Coind}(GI)$-preenvelope for any object in $\mathcal{A}$, where GI denotes the class of Gorenstein injective modules.

So let $\mathrm{Coind}(M)\in\mathcal{A}$ and $f\colon \mathrm{Coind}(M)\to E$ be a Gorenstein injective preenvelope in $A\#H$-Mod and let $E'\in A^H$-Mod be Gorenstein injective. Then there is an exact sequence $\xi\equiv\ldots\to E^{-1}\to E^0\to E^1\to\ldots$ of injective A^H-modules with $E'=\mathrm{Ker}(E^0\to E^1)$ and such that it remains exact when $\mathrm{Hom}_{A^H}(I,-)$ is applied for every injective I. Then $\mathrm{Coind}(\xi)$ is exact and $\mathrm{Coind}(E')=\mathrm{Ker}(\mathrm{Coind}(E^0)\to\mathrm{Coind}(E^1))$. Furthermore $\mathrm{Coind}(E^i)$ is injective for every i since $\mathrm{Coind}(-)$ is a right adjoint of $(-)_0$ which is exact by [18, Proposition 2.2] and [18, Corollary 2.9]. Finally, if $I\in A\#H$-Mod is injective we get that $\mathrm{Hom}_{A\#H}(I,\mathrm{Coind}(\xi))\cong\mathrm{Hom}_{A^H}(I_0,\xi)$ is exact since I_0 is injective and thus $\mathrm{Coind}(E')$ is Gorenstein injective. Now, if we consider $g\colon \mathrm{Coind}(M)\to\mathrm{Coind}(E')$ we get $h\colon E\to\mathrm{Coind}(E')$ such that $hf=g$. But by [3, Proposition 1.6] there is a unique morphism $h'\colon \mathrm{Coind}(E_0)\to\mathrm{Coind}(E')$ such that $h'\delta_E=h$, where $\delta_E\colon E\to\mathrm{Coind}(E_0)$ is the localization of E with respect to the localizing subcategory Λ. Thus $\delta_E f\colon \mathrm{Coind}(M)\to\mathrm{Coind}(E_0)$ is a $\mathrm{Coind}(GI)$-preenvelope whenever E_0 is Gorenstein injective.

So let $E\in A\#H$-Mod be Gorenstein injective and let $\xi\equiv\ldots\to E^{-1}\to E^0\to E^1\to\ldots$ be exact with $E=\mathrm{Ker}(E^0\to E^1)$, E^i injective for every i and such that $\mathrm{Hom}_{A\#H}(I,-)$ leaves it exact for every injective $I\in A\#H$-Mod. Then ξ_0 is an exact sequence of injective modules and $E_0=\mathrm{Ker}(E_0^0\to E_0^1)$. Finally $\mathrm{Hom}_{A^H}(I,\xi_0)$ is exact since I has finite projective dimension. □

Remark 1. Gorenstein categories (cf. [5]) are an example where injective objects have finite projective dimension. These include categories of modules over Iwanaga–Gorenstein rings, i.e., left and right noetherian rings such that both left and right injective dimensions of the ring are finite, [13]. In [7, Theorem 1.2] it is shown that when R is Iwanaga–Gorenstein then the fixed ring R^G by the action of a finite group G is also Iwanaga–Gorenstein or more generally, in [12, Theorem 4.3], if A is an H-module algebra which is Iwanaga–Gorenstein, then A^H is Iwanaga–Gorenstein.

Theorem 4.2. *Let t be an integral in H and suppose that $AtA=A\#H$, $A\#H/A$ is separable and ${}_{A^H}A$ is projective. If every finitely generated A-module has a finitely generated Gorenstein projective precover, then every finitely generated A^H-module has a Gorenstein projective precover.*

Proof. Let $X\in A^H$-Mod be finitely generated. So there is an exact sequence ${A^H}^{(n)}\to X\to 0$ for some integer n. Then $A^{(n)}\cong A\otimes_{A^H}A^{(n)}\to A\otimes_{A^H}X\to 0$ is exact and since A is finitely generated in $A\#H$-Mod, also $A\otimes_{A^H}X$ is finitely generated. Now, by [4, Proposition 6.1.7] $A\#H$ is free in Mod-A and so $A\#H\otimes_A -$ is exact, which gives that ${}_A(-)$ preserves projectives, and so, if $(A\#H)^{(n)}\to A\otimes_{A^H}X\to 0$ is an exact sequence for some integer n, then ${}_A(A\#H)^{(n)}\to {}_A(A\otimes_{A^H}X)\to 0$ is also exact since ${}_A(-)$ is an exact functor and so ${}_A(A\otimes_{A^H}X)$ is finitely generated. So let $f\colon P\to {}_A(A\otimes_{A^H}X)$ be a finitely generated Gorenstein projective precover. Then, by Lemma 3.1, $A\#H\otimes_A P$ is finitely generated Gorenstein projective. Therefore, by [11, Proposition 6]

$$A\#H\otimes_A P \xrightarrow{1\otimes_A f} A\#H\otimes_A A\otimes_{A^H} X \xrightarrow{\varepsilon_A\otimes_{A^H}1} A\otimes_{A^H} X$$

is a Gorenstein projective precover in $A\#H$-Mod. Now [11, Proposition 4] gives that

$$(A\#H\otimes_A P)_0 \xrightarrow{\varepsilon_A\otimes_{A^H}1\circ 1\otimes_A f} (A\otimes_{A^H}X)_0\cong X$$

is a Gorenstein projective precover in A^H-Mod whenever $(A\#H\otimes_A P)_0$ is Gorenstein projective.

So let P be Gorenstein projective in $A\#H$-Mod. Then we get an exact sequence $\xi\equiv\ldots\to P^{-1}\to P^0\to P^1\to\ldots$ of projective modules with $P=\mathrm{Ker}(P^0\to P^1)$ and such that it remains exact whenever $\mathrm{Hom}_{A\#H}(-,F)$ is applied for every projective F. Now, since $(-)_0$ is exact by [18, Proposition 2.2] and [18, Corollary 2.9] and $\mathrm{Hom}_{A^H}(A,-)$ is exact by hypothesis, we get that $(-)_0$ preserves projectives and so ξ_0 is an exact sequence of projective modules with $P_0=\mathrm{Ker}(P_0^0\to P_0^1)$.

On the other hand, by [16, Proposition 4.4.4] ${}_{A^H}A$ is finitely generated and projective by hypothesis and so $A^{H^{(n)}}=A\oplus F$, which gives that $\mathrm{Hom}_{A^H}(A,A^H)$ is a direct summand of $\mathrm{Hom}_{A^H}(A^{H^{(n)}},A^H)\cong A^{H^{(n)}}$ that is projective in $A\#H$-Mod. Therefore

$$\mathrm{Hom}_{A^H}(\xi_0,A^H)\cong \mathrm{Hom}_{A\#H}(\xi,\mathrm{Hom}_{A^H}(A,A^H))$$

is exact and so ξ_0 remains exact whenever $\mathrm{Hom}_{A^h}(-,F)$ is applied for every projective F. Thus P_0 is Gorenstein projective. □

We will finish this section by showing other situations where Gorenstein injective preenvelopes and Gorenstein projective precovers are preserved.

Proposition 4.3. *Let us assume that $A\#H/A$ is separable and that A/A^H is H^*-Galois. Then every A-module has a Gorenstein injective preenvelope* (*resp. Gorenstein projective precover*) *if and only if every A^H-module has a Gorenstein injective preenvelope* (*resp. Gorenstein projective precover*).

Proof. By hypothesis, A^H-Mod and $A\#H$-Mod are equivalent categories. So we only have to apply Theorems 3.3 and 3.6 and the fact that Gorenstein injective preenvelopes and Gorenstein projective precovers are preserved by equivalences of categories. □

Corollary 4.4. *Let $A=\sum_{g\in G} A_g$ be a strongly graded k-algebra over a finite group G. Suppose that A/A_e is separable. Then every A_e-module has a Gorenstein injective preenvelope (resp. Gorenstein projective precover) if and only if every A-module has a Gorenstein injective preenvelope (resp. Gorenstein projective precover).*

Proof. A is a $(kG)^*$-module algebra. Since A is strongly graded, $A/A^{(kG)^*}$ is kG-Galois. Thus $A\#(kG)^*/A$ is separable if and only if A/A_e is separable by [18, Corollary 4.7] and so Proposition 4.3 applies. □

5. Gorenstein dimensions

The existence of Gorenstein injective preenvelopes and Gorenstein projective precovers for every module allows us to define Gorenstein injective and Gorenstein projective dimensions naturally. Although these dimensions have been defined in [2] over rings where Gorenstein projective precovers are not known to exist, their existence allows us to characterize these dimensions in terms of derived functors. So we recall from [9] the following definition.

Definition 5.1. Let A be any ring and $M\in A$-Mod. We will say that M has *Gorenstein injective* (resp. *Gorenstein projective*) *dimension* less than or equal to n if there exists an exact sequence

$$0 \longrightarrow M \longrightarrow E^0 \longrightarrow E^1 \longrightarrow \dots \longrightarrow E^n \longrightarrow 0$$
$$(\text{resp. } 0 \longrightarrow P_n \longrightarrow P_1 \longrightarrow \dots \longrightarrow P_0 \longrightarrow M \longrightarrow 0)$$

with every E^i being Gorenstein injective (resp. every P_i being Gorenstein projective). We will say that $\mathrm{Gid}(M)=n$ (resp. $\mathrm{Gpd}(M)=n$) if there is no such shorter sequence. In the case when there is no such finite sequence, we will say that $\mathrm{Gid}(M)=\infty$ (resp. $\mathrm{Gpd}(M)=\infty$).

As was pointed out above, a consequence of the existence of Gorenstein injective and Gorenstein projective resolutions is that we can define right derived functors of $\mathrm{Hom}_A(M,-)$ and $\mathrm{Hom}_A(-,N)$, respectively, (see comments in pp. 169–170 of [9]). Since they do not have to coincide on $\mathrm{Hom}_A(M,N)$, we will denote them by $\mathrm{GIExt}^i_A(M,-)$ and $\mathrm{GPExt}^i_A(-,N)$.

Now we have the following relations between Gorenstein dimensions in A-Mod and $A\#H$-Mod for a given H-module algebra.

Corollary 5.2. *Let* $M\in A\#H$-Mod *and* $N\in A$-Mod. *Then*
(i) $\mathrm{Gid}({}_AM)\le\mathrm{Gid}(M)$ (*resp.* $\mathrm{Gpd}({}_AM)\le\mathrm{Gpd}(M)$)*;*
(ii) $\mathrm{Gid}(A\#H\otimes_A N)\le\mathrm{Gid}(N)$ (*resp.* $\mathrm{Gpd}(A\#H\otimes_A N)\le\mathrm{Gpd}(N)$).

Proof. (i) Let $0\to M\to E^0\to E^1\to\ldots\to E^n\to 0$ be a Gorenstein injective resolution of $M\in A\#H$-Mod. Then by Lemma 3.1 and [1, Proposition 2.5] $0\to{}_AM\to{}_AE^0\to{}_AE^1\to\ldots\to{}_AE^n\to 0$ is a Gorenstein injective resolution of ${}_AM$.

(ii) This is analogous using that $A\#H\otimes_A-$ preserves Gorenstein injectives.

The proofs for Gorenstein projective dimensions are also analogous. □

Now global Gorenstein injective and global Gorenstein projective dimensions of a ring A are defined as usual. We will denote them by $\mathrm{glGid}(A)$ and $\mathrm{glGpd}(A)$, respectively.

Proposition 5.3. *Let the extension* $A\#H/A$ *be separable. If* $\mathrm{glGid}(A\#H)<\infty$ (*resp.* $\mathrm{glGpd}(A\#H)<\infty$) *then* $\mathrm{glGid}(A)<\infty$ (*resp.* $\mathrm{glGid}(A)<\infty$).

Proof. Let us suppose that $\mathrm{glGid}(A\#H)=n$ and let $M\in A$-Mod. Then $\mathrm{Gid}(A\#H\otimes_A M)\le n$. Then by Corollary 5.2, $\mathrm{Gid}({}_A(A\#H\otimes_A M))\le n$ and so $\mathrm{GIExt}_A^{n+1}(L,{}_A(A\#H\otimes_A M))=0$ for every $L\in A$-Mod. But now, $({}_A(-),A\#H\otimes_A -)$ is an adjoint pair with $A\#H\otimes_A-$ separable and so, by [11, Proposition 5], the epimorphism ${}_A(A\#H\otimes_A M)\xrightarrow{\varepsilon_M}M$ splits. Then $\mathrm{GIExt}_A^{n+1}(L,M)=0$ for every $L\in A$-Mod since $\mathrm{GIExt}_A^i(L,-)$ preserves finite direct sums and thus $\mathrm{Gid}(M)\le n$.

The proof for the global Gorenstein projective dimension is analogous. □

Theorem 5.4. *Let* $A\#H/A$ *be separable and suppose that* A/A^H *is* H^**-Galois. Then* $\mathrm{glGid}(A)<\infty$ *if and only if* $\mathrm{glGid}(A\#H)<\infty$ *which is true if and only if* $\mathrm{glGid}(A^H)<\infty$ (*resp.* $\mathrm{glGpd}(A)<\infty$ *if and only if* $\mathrm{glGpd}(A\#H)<\infty$ *which is true if and only if* $\mathrm{glGpd}(A^H)<\infty$).

Proof. That $\mathrm{glGid}(A\#H)<\infty$ if and only if $\mathrm{glGid}(A^H)<\infty$ follows from the fact that in this case A^H-Mod and $A\#H$-Mod are equivalent categories.

If $\mathrm{glGid}(A\#H)<\infty$ then $\mathrm{glGid}(A)<\infty$ follows from Proposition 5.3.

Conversely, since $A\#H/A$ is separable and A/A^H is H^*-Galois, we get by [18, Corollary 4.7] that the functor ${}_A(-)$ is separable. If we consider the adjoint pair $({}_A(-),A\#H\otimes_A-)$, then we get by [11, Proposition 5] that the natural map $\eta_M: M\to A\#H\otimes_{A}{}_AM$ is a split monomorphism for every $M\in A\#H$-Mod. Now, if $M\in A\#H$-Mod, then $\mathrm{Gid}({}_AM)\le k$ for an integer k, and so, by Corollary 5.2 $\mathrm{Gid}(A\#H\otimes_{A}{}_AM)\le k$. Thus, analogously to the proof of Proposition 5.3 we get that $\mathrm{Gid}(M)\le k$. □

Remark 2. In [5, Theorem 2.8] it is shown that a Grothendieck category with enough projectives $\mathcal{A}$ is Gorenstein if and only if glGpd($\mathcal{A}$) and glGid($\mathcal{A}$) are both finite. Therefore the last theorem gives new examples of Gorenstein categories from a given one. An immediate example of a Gorenstein category is R-Mod with R an Iwanaga–Gorenstein ring (cf. [9, Section 9.1]). In this way, the last result extends [7, Theorem 1.2] and [12, Theorem 4.3]. Examples of non-noetherian Gorenstein rings can be found in [6].

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