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Lipschitz continuity of the Green function in Denjoy domains

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Abstract. In this paper a Wiener-type characterization is presented of those boundary points of a Denjoy domain where the Green function is Lipschitz continuous. This property is linked with the splitting of a Euclidean boundary point into two minimal Martin boundary points.

1. Introduction

Let $x=(x',x_n)$ denote a typical point of Euclidean space $\mathbb{R}^n=\mathbb{R}^{n-1}\times\mathbb{R}$, $n\geq 2$, and let L denote the hyperplane $\mathbb{R}^{n-1}\times\{0\}$. By a *Denjoy domain* we mean a domain of the form $\Omega=\mathbb{R}^n\setminus E$, where E is a proper closed subset of L. We will assume that $0\in E$. If $n=2$, then we will further require that E be non-polar, to ensure that Ω possesses a Green function G_Ω.

Let $\mathcal{P}_E$ denote the cone of positive harmonic functions u on Ω that are bounded outside every neighbourhood of 0, and vanish continuously at every regular point of the boundary of Ω in $\mathbb{R}^n\cup\{\infty\}$ apart from 0. It is known (see [1] or [4]) that either all functions in $\mathcal{P}_E$ are proportional, or $\mathcal{P}_E$ is generated by two linearly independent minimal harmonic functions. (We recall that a positive harmonic function u on Ω is called *minimal* if any harmonic function v on Ω satisfying $0<v\leq u$ is proportional to u.) These two cases will be denoted by writing $\dim\mathcal{P}_E=1$ and $\dim\mathcal{P}_E=2$, respectively. They correspond to the situations in which Ω has one or two minimal Martin boundary points, respectively, associated with the Euclidean boundary point 0.

Let $0<\alpha<1$, and let $Q_{x'}$ denote the open cube with centre $(x',0)$ and side length $\alpha|x'|$, where all faces are parallel to the coordinate hyperplanes. Further, let $\beta_E(x')$ denote the harmonic measure of $\partial Q_{x'}$ for the open set $Q_{x'}\setminus E$, evaluated at $(x',0)$. If $(x',0)\in E$, then $\beta_E(x')$ is interpreted as 0. If E is 'large' near 0 then, intuitively, the potential theory of Ω near 0 should mirror that of two halfspaces bounded by a common hyperplane L. On the other hand, if E is 'small', then there is a single Martin kernel function associated with 0. The size of the set E is reflected in the size of the harmonic measure $\beta_E(x')$, which is small if E is large near $(x',0)$. Benedicks [4] obtained the following criterion involving $\beta_E(x')$ for distinguishing between these two cases. It was originally formulated for harmonic functions with pole at infinity rather than at the origin and for domains that are regular away from the pole, but the statement below follows using inversion and an approximation argument (cf. [5, p. 599]).

Theorem A. *For a Denjoy domain $\Omega=\mathbb{R}^n\setminus E$ with $0\in E$ the following statements are equivalent*:

(a) $\dim \mathcal{P}_E=2$;

(b)

$$\int_{\{|x'|\le 1\}} \frac{\beta_E(x')}{|x'|^{n-1}}\,dx' < \infty.$$

The purpose of this paper is to show that the harmonic measure condition (b) above can be reformulated as a Wiener-type criterion involving capacity, and that this condition also characterizes Lipschitz continuity of the Green function for Ω at the boundary point 0. To be more precise, let $x_0\in\Omega$. Then we will say that $G_\Omega(x_0,\cdot\,)$ is *Lipschitz continuous at* 0 if there exists a constant $C>0$ such that $G_\Omega(x_0,x)\le C|x|$ on some neighbourhood of 0, where $G_\Omega(x_0,\cdot\,)$ is defined to be 0 on E. Harnack's inequalities show that this definition is independent of the choice of x_0.

Let $\mathcal{C}(A)$ denote the Newtonian (or logarithmic, if $n=2$) capacity of a (Borel) set A. Also, let $\gamma\in\left(0,\frac{1}{3}\right)$ and $D(r)=\{(x',0):|x'|\le r\}$ and, for any $k=0,1,\ldots$, let $D_k=D(2^{-k})$ and

$$E_k=(E\cap D_k)\cup D(\gamma 2^{-k})\cup \overline{D_k\setminus D((1-\gamma)2^{-k})}.$$

Theorem 1. *For a Denjoy domain $\Omega=\mathbb{R}^n\setminus E$ with $0\in E$, and for any $x_0\in\Omega$, the following statements are equivalent*:

(a) $\dim \mathcal{P}_E=2$;

(b)

$$\begin{cases} \sum_{k=0}^{\infty} 2^{k(n-2)}[\mathcal{C}(D_k)-\mathcal{C}(E_k)]<\infty, & n\ge 3,\\ \sum_{k=0}^{\infty} 2^{k}[\mathcal{C}(D_k)-\mathcal{C}(E_k)]<\infty, & n=2; \end{cases}$$

(c) $G_\Omega(x_0,\cdot\,)$ *is Lipschitz continuous at* 0.

The equivalence of (b) and (c) above extends a recent result of Carleson and Totik [5, Theorem 1.11] for the plane to all dimensions. Our approach is quite different from theirs.

When $n\geq 3$ it is natural to investigate the spine-like sets associated with the capacity condition Theorem 1(b). These are characterised below. Let $g\colon (0,1)\to (0,\infty)$ be increasing, and let

$$W_g=\left\{x\in(0,1)\times\mathbb{R}^{n-2}\times\{0\}:\sqrt{x_2^2+...+x_{n-1}^2}<g(x_1)\right\}.$$

Corollary 1. *Let $n\geq 3$ and $\Omega=\mathbb{R}^n\setminus E$, where $E=L\setminus W_g$, and let $x_0\in\Omega$. The following statements are equivalent*:

(a) $\dim\mathcal{P}_E=2$;

(b)

$$\int_0^1\frac{g(t)^{n-1}}{t^n}\,dt<\infty;$$

(c) $G_\Omega(x_0,\cdot\,)$ *is Lipschitz continuous at* 0.

One would not expect to be able to characterize Lipschitz continuity of the Green function in purely measure theoretic terms. Nevertheless, we give below a sharp sufficient condition of this nature. Let l_n denote n-dimensional measure.

Corollary 2. *Let $n\geq 2$, let $\Omega=\mathbb{R}^n\setminus E$ be a Denjoy domain with $0\in E$, and let $x_0\in\Omega$. If $\sum_{k=0}^{\infty}2^{nk}[l_{n-1}(D_k\setminus E_k)]^{n/(n-1)}<\infty$, then $G_\Omega(x_0,\cdot\,)$ is Lipschitz continuous at* 0.

Corollary 2 will be established by a capacitary estimate (Lemma 2 of Section 4) that may be of independent interest. When combined with Theorem 1 it yields a simpler proof of the main result of [8]. The sharpness of Corollary 2 is illustrated below. Let $B(x,r)$ denote the open ball in $\mathbb{R}^n$ of centre x and radius r.

Example. Let $\Omega=(\mathbb{R}^n\setminus L)\cup(\bigcup_{k=0}^{\infty}B(x^{(k)},r_k))$, where $x^{(k)}\in L\cap\partial B(0,2^{-k})$ and $r_k<2^{-k}$, and let $x_0\in\Omega$. Then $G_\Omega(x_0,\cdot\,)$ is Lipschitz continuous at 0 if and only if $\sum_{k=0}^{\infty}2^{nk}r_k^n<\infty$.

Corollary 3. *Let F be a closed subset of $[0,\infty)$ containing 0, let $\Omega=\mathbb{R}^n\setminus E$, where $E=\{x\in L:|x|\in F\}$ and $n\geq 3$, and let $x_0\in\Omega$. If $\sum_{k=0}^{\infty}2^{2k}(l_1([0,2^{-k}]\setminus F))^2<\infty$, then $G_\Omega(x_0,\cdot\,)$ is Lipschitz continuous at* 0.

The proof of Theorem 1 may be found in the next section. It relies on Theorem A and the notion of minimal thinness, an account of which may be found in [3, Chapter 9]. Chevallier [6] was the first to use minimal thinness in the study of

Denjoy domains, but we will exploit the connection in a different way. Corollaries 1 and 2 and the example will then be established in Sections 3 and 4. Corollary 3 can be proved in a manner analogous to the argument in Section 4, so we omit the details.

2. Proof of Theorem 1

2.1. The equivalence of conditions (a) and (b) in Theorem 1 will be established by showing that (b) is equivalent to the corresponding condition in Theorem A. In this section we consider the case where $n\geq 3$.

Let

$$A_k=\{x':2^{-k}\leq 3|x'|\leq 2^{1-k}\}\quad\text{and}\quad A_k^*=\{x':\gamma 2^{-k}\leq |x'|\leq (1-\gamma)2^{-k}\}.$$

Given $\gamma\in\left(0,\frac{1}{3}\right)$, we choose

$$\alpha=\min\left\{\frac{1-3\gamma}{\sqrt{n-1}},\frac{2\gamma}{(1-\gamma)\sqrt{n-1}}\right\}$$

in the definition of the cubes $Q_{x'}$. This choice ensures that $Q_{x'}\cap L\subset A_k^*\times\{0\}$ whenever $x'\in A_k$, and $Q_{x'}\cap L\subset D_k$ whenever $x'\in A_k^*$, whence

$$Q_{x'}\cap E=Q_{x'}\cap E_k\quad\text{whenever } x'\in A_k, \tag{2.1}$$

and

$$Q_{x'}\cap E\subseteq Q_{x'}\cap E_k\quad\text{whenever } x'\in A_k^*. \tag{2.2}$$

Also, if K is any compact set in $\mathbb{R}^n$, we denote by v_K the capacitary (Newtonian) potential of K, and by μ_K the associated Riesz measure.

The function $1-v_{E_k}$ is subharmonic and bounded above by 1 on $\mathbb{R}^n$, and vanishes quasi-everywhere (that is, apart from a polar set) on E_k. If $x'\in A_k^*$, we thus have $1-v_{E_k}=0$ quasi-everywhere on $Q_{x'}\cap E$, by (2.2), and clearly $1-v_{E_k}\leq 1$ on $\partial Q_{x'}$, in particular. It follows from the maximum principle, applied on $Q_{x'}\setminus E$, that

$$1-v_{E_k}(x',0)\leq\beta_E(x'),\quad x'\in A_k^*. \tag{2.3}$$

Now $d\mu_{D_0}(x',x_n)=f(|x'|)\,dx'\,d\delta_0$, where δ_0 is the Dirac measure at 0 in $\mathbb{R}$ and $f\colon[0,1)\to(0,\infty)$ is continuous. (This can be shown using Green's theorem and the fact that the function $x'\mapsto\lim_{t\to 0+}(1-v_{D_0}(x',t))/t$ is positive and continuous on $\{x' : |x'| < 1\}$, by [3, Lemma 8.5.1].) Letting $c_1 = \max_{[0,1-\gamma]} f$, we can thus use dilation to see that

$$d\mu_{D_k} \le 2^k c_1 \, dx' \, d\delta_0 \quad \text{on } D((1-\gamma)2^{-k}). \tag{2.4}$$

This, together with (2.3), yields

$$\begin{aligned}\int_{A_k^*} \frac{\beta_E(x')}{|x'|^{n-1}} \, dx' &\ge \frac{1}{2^k c_1} \int_{A_k^* \times \{0\}} \frac{\beta_E(x')}{|x'|^{n-1}} \, d\mu_{D_k}(x) \\ &\ge \frac{1}{2^k c_1} \int_{A_k^* \times \{0\}} \frac{1 - v_{E_k}(x)}{|x'|^{n-1}} \, d\mu_{D_k}(x) \\ &\ge c_2 2^{k(n-2)} \int_{A_k^* \times \{0\}} (1 - v_{E_k}(x)) \, d\mu_{D_k}(x),\end{aligned}$$

where $c_2 = c_1^{-1}(1-\gamma)^{1-n}$. Since $E_k \supseteq D_k \setminus (A_k^* \times \{0\})$, it follows that $1 - v_{E_k} = 0$ on $D_k \setminus (A_k^* \times \{0\})$, and so

$$\begin{aligned}\int_{A_k^*} \frac{\beta_E(x')}{|x'|^{n-1}} \, dx' &\ge c_2 2^{k(n-2)} \int_{D_k} \left(1 - \int_{E_k} \frac{d\mu_{E_k}(y)}{|x-y|^{n-2}}\right) d\mu_{D_k}(x) \\ &= c_2 2^{k(n-2)} \left(\mathcal{C}(D_k) - \int_{E_k} \int_{D_k} \frac{d\mu_{D_k}(x)}{|x-y|^{n-2}} \, d\mu_{E_k}(y)\right) \\ &= c_2 2^{k(n-2)} \left(\mathcal{C}(D_k) - \int_{E_k} v_{D_k}(y) \, d\mu_{E_k}(y)\right).\end{aligned}$$

Now, $v_{D_k} = 1$ on D_k and $E_k \subseteq D_k$, so

$$\int_{E_k} v_{D_k}(y) \, d\mu_{E_k}(y) = \mathcal{C}(E_k)$$

and hence

$$\int_{A_k^*} \frac{\beta_E(x')}{|x'|^{n-1}} \, dx' \ge c_2 2^{k(n-2)} [\mathcal{C}(D_k) - \mathcal{C}(E_k)].$$

Although the sets A_k^* are not disjoint, any point $x' \in \mathbb{R}^{n-1}$ can belong to at most a fixed finite number (depending on γ) of these sets. Thus it is now clear that Theorem A(b) implies Theorem 1(b).

For the converse we recall the following elementary fact [4, Lemma 7].

Lemma A. *Let $\beta_E^*(x')$ denote the harmonic measure of the set*

$$T_{x'} = \partial Q_{x'} \cap \left\{ y : |y_n| = \frac{\alpha |x'|}{2} \right\}$$

for $Q_{x'}\setminus E$ *evaluated at* $(x',0)$. *Then*

$$\beta_E^*(x')\le\beta_E(x')\le n\beta_E^*(x'),\quad x'\in\mathbb{R}^{n-1}.$$

We can use Lemma A, and then (2.1) and the maximum principle on $Q_{x'}\setminus E$, to see that, for $x'\in A_k$,

$$\beta_E(x')\le n\beta_E^*(x')\le n\frac{1-v_{E_k}(x',0)}{\min_{T_{x'}}(1-v_{E_k})}\le n\frac{1-v_{E_k}(x',0)}{\min_{T_{x'}}(1-v_{D_k})}\le c_3(1-v_{E_k}(x',0)),$$

where, by a dilation argument, c_3 depends only on n and α. Letting $c_4=\min_{[0,2/3]}f$, we can argue as previously to see that

$$\begin{aligned}\int_{A_k}\frac{\beta_E(x')}{|x'|^{n-1}}\,dx'&\le 3^{n-1}\Big(\frac{c_3}{c_4}\Big)2^{k(n-2)}\int_{D_k}(1-v_{E_k}(x))\,d\mu_{D_k}(x)\\&=3^{n-1}\Big(\frac{c_3}{c_4}\Big)2^{k(n-2)}[\mathcal{C}(D_k)-\mathcal{C}(E_k)].\end{aligned}$$

Since $\bigcup_{k=0}^{\infty}(A_k\times\{0\})=D\big(\frac{2}{3}\big)\setminus\{0\}$, Theorem 1(b) implies Theorem A(b).

2.2. In order to show the equivalence of conditions (a) and (b) in Theorem 1 when the dimension is 2, we recall that the Robin constant $r(K)$, of a non-polar compact set K, is related to logarithmic capacity by the equation $\mathcal{C}(K)=e^{-r(K)}$, whence

$$\mathcal{C}(D_k)-\mathcal{C}(E_k)=\mathcal{C}(D_k)(1-e^{r(D_k)-r(E_k)}).$$

Since $\mathcal{C}(D_k)=2^{-k-1}$, it is easy to see that the two-dimensional case of the capacity condition Theorem 1(b) can be reformulated as

$$\sum_{k=0}^{\infty}[r(E_k)-r(D_k)]<\infty.$$

Let g_K denote the Green function for $\mathbb{R}^2\setminus K$ with pole at infinity, and let ν_K denote the equilibrium measure for K. Then

$$g_K(x)=r(K)+\int_K\log|x-y|\,d\nu_K(y),\quad x\in\mathbb{R}^2\setminus K, \tag{2.5}$$

and

$$\int_K\log|x-y|\,d\nu_K(y)=-r(K)\quad\text{quasi-everywhere on } K. \tag{2.6}$$

Defining

$$c_5=\sup_{|x|\le 2}g_{\overline{D_0\setminus A_0^*}}(x),$$

we can use dilation, (2.2) and the maximum principle on $Q_{x'}\setminus E$ to see that

$$c_5^{-1} g_{E_k}(x',0) \le \beta_E(x'), \quad x' \in A_k^*.$$

Also, there is a positive constant c_6, depending only on γ, such that

$$d\nu_{D_k} \le 2^k c_6 \, dx' \, d\delta_0 \quad \text{on } D((1-\gamma)2^{-k}).$$

Hence

$$\int_{A_k^*} \frac{\beta_E(x')}{|x'|} \, dx' \ge \frac{1}{(1-\gamma)c_5c_6} \int_{D_k} g_{E_k}(x) \, d\nu_{D_k}(x) = \frac{1}{(1-\gamma)c_5c_6} [r(E_k) - r(D_k)],$$

using (2.5), (2.6) and the fact that ν_K is a probability measure. (We were able to pass from an integral over A_k^* to one over D_k because $g_{E_k}=0$ on $D_k\setminus(A_k^*\times\{0\})$.) It is now clear that Theorem A(b) implies Theorem 1(b). The converse is proved by similar reasoning.

2.3. The equivalence of conditions (a) and (c) in Theorem 1 involves arguments based on minimal thinness. The following lemma will prove useful. Let $b(x',r)$ denote the open ball in $\mathbb{R}^{n-1}$ of centre x' and radius r.

Lemma B. *There is a positive constant C, depending only on $\varkappa\in(0,1)$ and n, with the following property: if u is nonnegative and subharmonic on $b(y',r)\times(-r,r)$ and harmonic off L, then*

$$u(x) \le C\Big[u\Big(y', \frac{r}{2}\Big) + u\Big(y', -\frac{r}{2}\Big)\Big] \quad \textit{when } x \in b(y', \varkappa r) \times \Big(-\frac{r}{2}, \frac{r}{2}\Big). \tag{2.7}$$

Lemma B holds because (following the argument of [4, p. 54]) Harnack's inequalities yield the existence of a constant $C'>0$, depending on $\varkappa$ and n, such that

$$u(x) \le \frac{C' r^{n-1}}{x_n^{n-1}} \Big[u\Big(y', \frac{r}{2}\Big) + u\Big(y', -\frac{r}{2}\Big)\Big] \quad \text{when } x \in b\Big(y', \frac{1+\varkappa}{2} r\Big) \times \Big(-\frac{r}{2}, \frac{r}{2}\Big),$$

whence (2.7) holds in view of Domar [7, Theorem 2] and the subharmonicity of u.

Let $h(x)=x_n|x|^{-n}$ when $x\in\mathbb{R}^n\setminus\{0\}$, and let I_f denote the Poisson integral in the halfspace $H=\{(x',x_n):x_n>0\}$ of a nonnegative measurable function f on L. In what follows, any function in $\mathcal{P}_E$ is deemed to have the value 0 on $E\setminus\{0\}$.

Let $x_0=(0',1)$ and suppose, firstly, that Theorem 1(a) holds. It is easy to observe from [4, Theorem 3] that this is equivalent to saying that there is a function u in $\mathcal{P}_E$ with a representation of the form

$$u(x) = \begin{cases} ah(x) + I_u(x), & x_n>0, \\ -bh(x) + I_u(x', -x_n), & x_n<0, \end{cases} \tag{2.8}$$

where $a, b \in [0, \infty)$ are not both zero. Hence the function

$$u_+ = \frac{u+bh}{a+b},$$

which has the representation

$$u_+(x) = h^+(x) + I_{u_+}(x', |x_n|), \quad x_n \neq 0,$$

belongs to $\mathcal{P}_E$, where $h^+ = \max\{h, 0\}$. Let $u_-(x) = u_+(x', -x_n)$. Since $\mathcal{P}_E$ is finite-dimensional, there must be a minimal harmonic minorant of u_+ (respectively, u_-) of the form (2.8) with $a > 0 = b$ (respectively, $b > 0 = a$). Since $\dim \mathcal{P}_E = 2$, it follows that u_+ and u_- are themselves minimal.

For any positive superharmonic function w on Ω and any set $A \subset \Omega$, we define

$$R_w^A = \inf\{s : s \text{ is positive and superharmonic on } \Omega \text{ and } s \geq w \text{ on } A\}.$$

Let $J = \Omega \cap \{(x', x_n) : x_n \leq 0\}$. Then $R_{u_+}^J(x) = I_{u_+}(x', |x_n|)$ when $x_n \neq 0$, and so we have $u_+ - R_{u_+}^J = h > 0$ on H. It follows that J is minimally thin with respect to the minimal harmonic function u_+ on Ω. We claim that any sequence of points in H that converges to the Martin boundary point of Ω associated with u_+ must converge to 0 in the Euclidean topology. For, otherwise, there would be a sequence $\{x^{(k)}\}_{k=0}^{\infty}$ in H such that

$$\liminf_{k\to\infty} |x^{(k)}| > 0 \text{ and } \frac{G_\Omega(x, x^{(k)})}{G_\Omega(x_0, x^{(k)})} \to \frac{u_+(x)}{u_+(x_0)} \quad \text{as } k \to \infty,$$

whence u_+ would be bounded near 0, by Lemma B, and we would arrive at the contradictory conclusion that $u_+ \equiv 0$.

Hence, by [3, Theorem 9.5.2],

$$\limsup_{\substack{x\to 0\\ x\in H}} \frac{G_H(x_0, x)}{G_\Omega(x_0, x)} > 0, \tag{2.9}$$

and so, by [3, Theorem 9.3.3(ii)], $G_\Omega(x_0, x)/G_H(x_0, x)$ has a finite minimal fine limit at 0 with respect to H. In fact, since $G_H(x_0, x)/x_n$ has a finite limit as $x \to 0$ in H, we see that $G_\Omega(x_0, x)/x_n$ has a finite minimal fine limit at 0 with respect to H, and hence an equal nontangential limit at 0 (see [3, Theorem 9.7.4]). This argument can also be applied using u_- in place of u_+, so we can now use Lemma B to deduce that Theorem 1(c) holds.

Conversely, suppose that condition (c) holds. By [1, Théorème 2], there is a minimal harmonic function u_0 on Ω such that

$$\frac{G_\Omega(\,\cdot\,, (0', t))}{G_\Omega(x_0, (0', t))} \to u_0, \text{ as } t \downarrow 0, \quad \text{on } \Omega. \tag{2.10}$$

Let $0<\varepsilon<1$ and $V_\varepsilon=\Omega\setminus\overline{B}(0,\varepsilon)$, and let λ_x^U denote harmonic measure for an open set U and a point $x\in U$. By Lemma B and Harnack's inequalities there is a positive constant $C(\Omega,\varepsilon)$ such that

$$G_\Omega(x,(0',t))\le C(\Omega,\varepsilon)G_\Omega(x_0,(0',t)),\quad x\in\Omega\cap\partial B(0,\varepsilon),\ 0<t<\frac{\varepsilon}{2}. \tag{2.11}$$

Since

$$G_\Omega(x,(0',t))=\int_{\Omega\cap\partial B(0,\varepsilon)} G_\Omega(\,\cdot\,,(0',t))\,d\lambda_x^{V_\varepsilon},\quad x\in V_\varepsilon,\ 0<t<\varepsilon,$$

we can use (2.10) and dominated convergence (see (2.11)) to deduce that

$$u_0(x)=\int_{\Omega\cap\partial B(0,\varepsilon)} u_0\,d\lambda_x^{V_\varepsilon}\le C(\Omega,\varepsilon)\lambda_x^{V_\varepsilon}(\Omega\cap\partial B(0,\varepsilon)),\quad x\in V_\varepsilon.$$

It follows that $u_0\in\mathcal{P}_E$. Since

$$\limsup_{t\downarrow 0}\frac{G_H(x_0,(0',t))}{G_\Omega(x_0,(0',t))}>0,$$

by hypothesis, we can apply [3, Theorem 9.5.2] again to see that J is minimally thin with respect to u_0 in Ω. Thus $I_{u_0}=R_{u_0}^J\ne u_0$ on H, and so we can appeal to the observation concerning (2.8) to conclude that $\dim\mathcal{P}_E=2$, as required.

3. Proof of Corollary 1 and an estimate for harmonic measure

We define the union of cones

$$K_A=\bigcup_{(z',0)\in A}\{(x',x_n):x_n>|x'-z'|\},\quad A\subset L.$$

If $x=(x',x_n)\in H\setminus K_A$, then the $(n-1)$-dimensional ball $B((x',0),x_n)\cap L$ does not meet A. It follows that there is a constant $C(n)\in(0,1)$ such that, for any Borel subset A of L,

$$\lambda_x^H(A)\le C(n),\quad x\in H\setminus K_A. \tag{3.1}$$

More generally, if $\delta\in(0,1)$, there is a constant $C(n,\delta)\in(0,1)$ such that

$$\lambda_x^H(A)\le C(n,\delta)\quad\text{if } |B((x',0),x_n)\cap A|<\delta|B((x',0),x_n)|. \tag{3.2}$$

Let $n\ge 3$ and

$$W(\beta,\rho)=\left\{x\in B(0,\beta)\cap L:\sqrt{x_2^2+\ldots+x_{n-1}^2}<\rho\right\},\quad 0<\rho\le\beta<1.$$

This is a core of radius ρ cut through an $(n-1)$-dimensional ball of radius β. We form a region ω by joining two half-balls of radius 1 along the core $W(1-\gamma,\rho)$: the result, if ρ is small, is the unit ball with most of the hyperplane L removed except for a narrow opening $W(1-\gamma,\rho)$. Formally,

$$\omega=(B(0,1)\setminus L)\cup W(1-\gamma,\rho).$$

We now estimate the harmonic measure of the unit sphere $\partial B(0,1)$ for ω on the core $W(1-\gamma,\rho)$.

Lemma 1. *There is a positive constant $C(n,\gamma)$, such that*

$$\lambda_x^\omega(\partial B(0,1))\le C(n,\gamma)\rho,\quad x\in W(1-\gamma,\rho),\ 0<\rho\le 1-\gamma.$$

Proof. There is no loss of generality in supposing that $\rho<2^{-5/2}\gamma$, so that we have $1-\gamma+2^{3/2}\rho<1-\gamma/2$. We denote the half-ball $B(0,1)\cap H$ by V, and let $h=\lambda_x^\omega(\partial B(0,1))$. Then

$$h(x)=\lambda_x^V(H\cap\partial B(0,1))+\int_{W(1-\gamma,\rho)} h\,d\lambda_x^V,\quad x\in V. \tag{3.3}$$

We let $m=\sup_{W(1-\gamma,\rho)} h$ and define the region

$$U=\{(x',x_n):(x',|x_n|)\in K_{W(1-\gamma,\rho)} \text{ and } |x_n|<2\rho\}\cup W(1-\gamma,\rho),$$

which contains a union of double-sided truncated cones with vertices on $W(1-\gamma,\rho)$. We will estimate h on the boundary of U in terms of m. By the symmetry of h and of U, it suffices to consider $x\in H\cap\partial U$, in which case we can use (3.3) and estimate separately each of the two terms on the right-hand side.

In view of (3.1) and (3.2) there is a constant $C_1\in(0,1)$, depending only on n, such that

$$\lambda_x^H(W(1-\gamma,\rho))\le C_1,\quad x\in H\cap\partial U.$$

Hence

$$\int_{W(1-\gamma,\rho)} h\,d\lambda_x^V\le\int_{W(1-\gamma,\rho)} h\,d\lambda_x^H\le C_1 m,\quad x\in H\cap\partial U.$$

Since there is a positive constant C_2, depending only on n and γ, such that

$$\lambda_x^V(H\cap\partial B(0,1))\le C_2x_n,\quad x\in H\cap B\Big(0,1-\frac{\gamma}{2}\Big),$$

and since $U \subset B(0,1-\gamma/2)$ by our assumption concerning ρ, we deduce from (3.3) that

$$h(x) \le 2C_2\rho + C_1 m, \quad x \in H \cap \partial U.$$

As mentioned earlier, this last estimate holds on the whole boundary of U so that, by the maximum principle applied to h on U,

$$m \le 2C_2\rho + C_1 m.$$

This completes the proof of the lemma, since we now have the desired estimate $m \le 2C_2(1-C_1)^{-1}\rho$. □

Now let Ω be as in the statement of Corollary 1, and suppose that condition (b) of that result holds. Further, let h_k denote the harmonic measure of $\partial B(0,2^{-k})$ for the domain

$$\omega_k = (B(0,2^{-k}) \setminus L) \cup W((1-\gamma)2^{-k}, \min\{(1-\gamma)2^{-k}, g(2^{-k})\}).$$

By Lemma 1 and dilation we see that

$$h_k \le C(n,\gamma) 2^k g(2^{-k}) \quad \text{on } \omega_k \cap L. \tag{3.4}$$

Since $1 - v_{E_k} \le h_k$ on ω_k by the maximum principle, we see that

$$\begin{aligned} \mathcal{C}(D_k) - \mathcal{C}(E_k) &= \int_{D_k} 1 \, d\mu_{D_k} - \int_{E_k} v_{D_k} \, d\mu_{E_k} \\ &= \int_{D_k} (1 - v_{E_k}) \, d\mu_{D_k} \le \int_{\omega_k \cap L} h_k \, d\mu_{D_k} \le C_3 2^k g(2^{-k})^{n-1}, \end{aligned}$$

using (3.4) and (2.4), where C_3 depends only on n and γ. Since, by hypothesis, $\sum_{k=0}^{\infty} [2^k g(2^{-k})]^{n-1} < \infty$, we now see that Theorem 1(b) holds, and so $G_\Omega(x_0, \cdot)$ is Lipschitz continuous at 0.

We already know that (c) implies (a), by Theorem 1, so it remains to check that (a) implies (b). As in Section 2.3, it follows from condition (a) that the function $x \mapsto x_n |x|^{-n}$ has a positive harmonic majorant u in Ω. By Harnack's inequalities, we see that $u \ge C_4 2^{nk} g(2^{-k-1})$ on

$$\left\{ x \in D_k \setminus D_{k+1} : \sqrt{x_2^2 + \ldots + x_{n-1}^2} < \tfrac{1}{2} g(2^{-k-1}) \right\},$$

where C_4 depends only on n. Since the function

$$(x', 0) \longmapsto \liminf_{\substack{y \to (x',0) \\ y \in \Omega}} u(y)$$

is locally integrable on L, we see that $\sum_{k=0}^{\infty}[2^k g(2^{-k})]^{n-1}<\infty$, whence Corollary 1(b) holds.

4. Proof of Corollary 2 and an estimate for capacity

4.1. Corollary 2 will follow immediately from Theorem 1 using a dilation argument once we have established the lemma below. An alternative argument that leads to the two-dimensional case of this lemma is outlined on p. 566 of [5].

Lemma 2. *Let $n\geq 2$, let W be a relatively open subset of $D(1-\gamma)$ and let $E=D(1)\setminus W$. There is a positive constant $C(n,\gamma)$ such that*

$$\mathcal{C}(D(1))-\mathcal{C}(E)\leq C(n,\gamma)l_{n-1}(W)^{n/(n-1)}.$$

Proof. To see this, we choose ρ such that $l_{n-1}(D(\rho))=2l_{n-1}(W)$. Thus $\rho=C(n)l_{n-1}(W)^{1/(n-1)}$. We may assume, without loss of generality, that $\rho<2^{-3/2}\gamma$. Let

$$U=\{(x',x_n):(x',|x_n|)\in K_W \text{ and } |x_n|<\rho\}\cup W,$$

and let h denote the harmonic measure of the set $\partial B(0,1)$ for the domain $\omega=(B(0,1)\setminus L)\cup W$. In view of (3.1) and (3.2) there is a constant $C_5\in(0,1)$, depending only on n and γ, such that $\lambda_x^H(W)\leq C_5$ when $x\in H\cap\partial U$. Arguing as in the proof of Lemma 1, we deduce that

$$h(x)\leq C_6\rho+C_5\sup_W h,\quad x\in H\cap U,$$

where C_6 depends only on n and γ, and hence $\sup_W h\leq C_6(1-C_5)^{-1}\rho$.

If $n\geq 3$ then, since $1-v_E\leq h$ on $B(0,1)\setminus E$, we see as before that

$$\begin{aligned}\mathcal{C}(D(1))-\mathcal{C}(E)&\leq\int_W h\,d\mu_{D(1)}\\&\leq C(n,\gamma)l_{n-1}(W)^{1/(n-1)}\mu_{D(1)}(W)\leq C(n,\gamma)l_{n-1}(W)^{1+1/(n-1)},\end{aligned}$$

by (2.4).

If $n=2$, we use an analogous argument based on equilibrium measures along the lines of that in Section 2.2. Using the notation of that section, we note that the Robin constants of two compact sets K_1 and K_2, with $K_1\subset K_2$, satisfy

$$r(K_1)-r(K_2)=\int_{K_2\setminus K_1} g_{K_1}\,d\nu_{K_2}.$$

Thus, since $g_E \leq C_7 h$ in $D(1)$, we see that

$$r(E) - r(D(1)) = \int_W g_E \, d\nu_{D(1)} \leq C_8 l_1(W) \nu_{D(1)}(W) \leq C_9 (l_1(W))^2,$$

where C_7, C_8 and C_9 depend at most on γ. □

4.2. The "if" assertion in the example follows easily from Lemma 2. The "only if" assertion follows from Theorem 1 and Harnack's inequality, arguing as in the final paragraph of Section 3.

Note. (Added in August 2007.) V. V. Andrievskii, in the preprint [2], has found a different capacitary/metric covering condition on $L \setminus E$ that characterizes the case where $\dim \mathcal{P}_E = 2$ when $n = 2$, and has also related this condition to the boundary behaviour of the Green function. His methods are based on conformal mappings and estimation of the modules of families of curves.

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