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Carleson measures for the generalized Bergman spaces via a T(1)-type theorem

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Abstract. In this paper, we give a new characterization of Carleson measures for the generalized Bergman spaces. We show first that this problem is equivalent to a T(1)-type problem. Using an idea of Verdera (see [V]), we introduce a sort of curvature in the unit ball adapted to our kernel. We establish a good λ inequality which then yields the solution of this T(1)-type problem.

1. Introduction

Let n be a positive integer and let

$$\mathbb{C}^n=\mathbb{C}\times\ldots\times\mathbb{C}$$

denote the n-dimensional complex Euclidean space.

For $z=(z_1,\ldots,z_n)$ and $w=(w_1,\ldots,w_n)$ in $\mathbb{C}^n$, we write

$$z\cdot\overline{w}=\langle z,w\rangle=z_1\overline{w}_1+\ldots+z_n\overline{w}_n$$

and

$$|z|=\sqrt{|z_1|^2+\ldots+|z_n|^2}.$$

The open unit ball in $\mathbb{C}^n$ is the set

$$\mathbf{D}=\{z\in\mathbb{C}^n:|z|<1\}.$$

We use $\mathbf{H}(\mathbf{D})$ to denote the space of all holomorphic functions in $\mathbf{D}$. Let $\mathbf{S}=\partial\mathbf{D}$ be the boundary of $\mathbf{D}$. For $\alpha\in\mathbb{R}$, $\alpha>-n-1$, we define the *generalized*

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Bergman spaces $\mathbf{A}^2_\alpha$, see [ZZ], to consist of all holomorphic functions f in the unit ball $\mathbf{D}$ with the property that

$$\|f\|^2_\alpha = \sum_{m\in\mathbb{N}^n} |c(m)|^2 \frac{\Gamma(n+1+\alpha)m!}{\Gamma(n+1+|m|+\alpha)} < \infty,$$

where $f(z)=\sum_{m\in\mathbb{N}^n} c(m)z^m$ is the Taylor expansion of f.

For $\beta\in\mathbb{R}$, we define the *fractional radial derivative* of order β by

$$(I+\mathcal{R})^\beta f(z) := \sum_{m\in\mathbb{N}^n} (1+|m|)^\beta c(m)z^m.$$

One then easily observes, by means of Taylor expansion and Stirling's formula, that

$$\|f\|^2_\alpha \cong \int_{\mathbf{D}} |(I+\mathcal{R})^m f(z)|^2 (1-|z|^2)^{2m+\alpha}\, d\lambda(z), \tag{1}$$

where $2m+\alpha>-1$. One also observes that the finiteness of the right-hand side of (1) is independent of the choice of m. If we let $2\sigma=\alpha+n+1$ then we see by (1) that $\mathbf{A}^2_\alpha = B^\sigma_2$, where B^σ_2 is the analytic Besov–Sobolev spaces defined in [ARS]. Thus this scale of spaces includes the Drury–Arveson Hardy space $\mathbf{A}^2_{-n}$ [D], the usual Hardy space $\mathbf{H}^2(\mathbf{D})=\mathbf{A}^2_{-1}$ and the weighted Bergman spaces when $\alpha>-1$.

An interesting question about these spaces is to find their Carleson measures, that is characterize positive measures μ on $\mathbf{D}$ such that

$$\int_{\mathbf{D}} |f|^2\, d\mu \le C(\mu)\|f\|^2_\alpha, \quad f\in\mathbf{A}^2_\alpha. \tag{2}$$

(A measure μ which satisfies (2) is called a *Carleson measure* for $\mathbf{A}^2_\alpha$ or simply an $\mathbf{A}^2_\alpha$ Carleson measure, $\mu\in\mathrm{CM}(\mathbf{A}^2_\alpha)$.)

Viewing the space $\mathbf{A}^2_\alpha$ as defined by the relation (1), we see that the literature is now rich with solutions of this question for various values of α. The first case of interest was the case $\alpha=-1$ (the usual Hardy space). In [C], Carleson gave the characterization when $n=1$ and later in 1967, Hörmander [H] gave a solution for $n>1$. Stegenga [St] (when $n=1$), Cima and Wogen [CW] (when $n>1$) characterized Carleson measures for $\alpha>-1$.

For $p>0$ and $\beta\in\mathbb{R}$, the *Hardy–Sobolev space* $\mathbf{H}^p_\beta(\mathbf{D})$ is the set

$$\Big\{ f\in \mathrm{Hol}(\mathbf{D}) : \|f\|^p_\beta = \sup_{r<1} \int_{\mathbf{S}} |(I+\mathcal{R})^\beta f(r\xi)|^p\, d\sigma(\xi) < \infty \Big\}.$$

One can see, by means of Taylor expansion for example, that $\mathbf{A}^2_\alpha = \mathbf{H}^2_\beta(\mathbf{D})$ with $\beta=(-\alpha-1)/2$. For $\alpha\in(-n-1,-n)$, Cohn and Verbitsky [CV] gave a capacitary characterization of Carleson measures and recently Arcozzi, Rochberg and Sawyer [ARS] gave a different proof for a different (non-capacitary) characterization. It has been known, [AC] and [CO], that the capacitary condition is sufficient but not necessary for $\mu\in\mathrm{CM}(\mathbf{A}^2_\alpha)$ when $\alpha\geq -n$. For the special case of the Drury–Arveson Hardy space ($\alpha=-n$), [ARS] completely characterized their Carleson measures using certain tree conditions.

In this note, we present a sequence of conditions equivalent to $\mu\in\mathrm{CM}(\mathbf{A}^2_{-n})$, some implicit in [ARS], some new, and a completely different proof. Contrary to [ARS], the proof presented here works the same way in the whole range $\alpha\in(-n-1,-n]$.

To obtain our characterization, we show first that this problem is equivalent to a kind of T(1)-type problem associated with a Calderón–Zygmund type kernel and then we solve the T(1)-type problem which occurs. Recall that for a topological space X with a pseudodistance d, a kernel $K(x,y)$ is called an *n-Calderón–Zygmund kernel* (or simply a Calderón–Zygmund kernel) with respect to the pseudodistance d if

(a) there exists $C_1>0$ such that

$$|K(x,y)|\leq \frac{C_1}{d(x,y)^n},$$

and

(b) there exists $0<\delta\leq 1$ such that

$$|K(x,y)-K(x',y)|+|K(y,x)-K(y,x')|\leq C_2\frac{d(x,x')^\delta}{d(x,y)^{n+\delta}}$$

if $d(x,x')\leq C_3 d(x,y)$, $x,x',y\in X$.

Given a Calderón–Zygmund kernel K, we can define (at least formally) a *Calderón–Zygmund operator* associated with this kernel by

$$Tf(x)=\int_X K(x,y)f(y)\,d\mu(y).$$

One important question in the Calderón–Zygmund theory is to find a criterion for boundedness of a Calderón–Zygmund operator in $L^2(\mu)$.

Many authors studied this problem. When $X=\mathbb{R}^m$, $\mu=dx$ (the usual Lebesgue measure) and d is the Euclidean distance, a famous criterion was obtained by Journé and David [DJ] who also introduced the terminology T(1)-theorem. This criterion states that a Calderón–Zygmund operator is bounded in $L^2(d\mu)$ if and only if it is weakly bounded (in some sense), and the operator and its adjoint send the function 1 into BMO. This result was extended to spaces of homogeneous type in an unpublished work by R. Coifman. Later it was an interesting question to extend this T(1)-theorem to the case where the space is not of homogeneous type. (This essentially means that the measure μ does not satisfy the doubling condition). Several authors such as Tolsa, Nazarov, Treil, Volberg and Verdera [T1], [T2], [NTV] and [V] treated this situation in the setting of $\mathbb{R}^m$ with the Euclidean distance. One good example of a Calderón–Zygmund operator is the Cauchy integral operator. We say that the Cauchy integral operator is bounded in $L^2(d\mu)$ whenever, for some positive constant C, one has for every $\varepsilon>0$,

$$\int_{\mathbb{R}^m} |\mathcal{C}_\varepsilon(f\mu)|^2\,d\mu \le C\int_{\mathbb{R}^m} |f|^2\,d\mu, \quad f\in L^2(d\mu),$$

where

$$\mathcal{C}_\varepsilon(f\mu)(z)=\int_{|\zeta-z|>\varepsilon}\frac{f(\zeta)}{\zeta-z}\,d\mu(\zeta), \quad z\in\mathbb{C}.$$

Their result is that the Cauchy integral operator is bounded in $L^2(d\mu)$ if and only if

(i) $\mu(D)\le Cr(D)$ for each disc D with radius $r(D)$;

(ii) $\int_D |\mathcal{C}_\varepsilon(\chi_D\mu)|^2\,d\mu\le C\mu(D)$ for each disc D and $\varepsilon>0$.

We now return to the problem (2). We consider the kernel K_α defined by

$$K_\alpha(z,w)=\operatorname{Re}\biggl(\frac{1}{(1-z\cdot\overline{w})^{n+1+\alpha}}\biggr).$$

For a positive Borel measure μ in $\mathbf{D}$, we consider the operator T_α associated with this kernel defined by

$$T_\alpha f(z)=\int_{\mathbf{D}} f(w)K_\alpha(z,w)\,d\mu(w), \quad z\in\mathbf{D}.$$

We will prove that if on $\mathbf{D}$ we consider as in [B] the pseudodistance d defined by

$$d(z,w)=\bigl||z|-|w|\bigr|+\biggl|1-\frac{z}{|z|}\cdot\frac{\overline{w}}{|w|}\biggr|,$$

the kernel K_α is an $(n+1+\alpha)$-Calderón–Zygmund kernel in the unit ball $\mathbf{D}$ with respect to the pseudodistance d. Let $B=B(z,r)=\{w\in\mathbf{D}:d(z,w)<r\}$ be a "pseudoball" or simply a ball of center z and radius r. We are now ready to state our result.

Theorem 1.1. (Main theorem) *Suppose that* $\alpha\in(-n-1,-n]$ *and let* μ *be a positive Borel measure in* $\mathbf{D}$. *Then the following conditions are equivalent*:

(a) μ *is a Carleson measure for* $\mathbf{A}^2_\alpha$;

(b) T_α *is bounded in* $L^2(\mu)$;

(c) *There exists a constant* C *such that*

$$\mu(B(z,r))\le Cr^{n+1+\alpha}, \tag{i}$$

$$\int_B |T_\alpha(\chi_B)|^2\,d\mu\le C\mu(B) \tag{ii}$$

for each ball $B=B(z,r)$ *which touches the boundary of* $\mathbf{D}$.

This theorem is a T(1)-type theorem with respect to the Calderón–Zygmund operator T_α ((b)$\Leftrightarrow$(c)) and it shows the relation of this T(1)-type problem with Carleson measures for $\mathbf{A}^2_\alpha$ ((a)$\Leftrightarrow$(b)). Observe that the equivalency (a)$\Leftrightarrow$(b) is proved in [ARS, Lemma 24] in a general situation of Hilbert spaces of functions. Thus to prove Theorem 1.1, we will essentially prove the hard part (b)$\Leftrightarrow$(c). To prove the hard part, we will adapt to the unit ball the idea used by J. Verdera [V] to give an alternative proof of the T(1)-theorem for the Cauchy integral operator.

Let us mention that an analogue of condition (ii) here appears implicitly in [ARS]. Namely

$$\int_B T_\alpha(\chi_B)\,d\mu\le C\mu(B) \tag{3}$$

for each ball $B=B(z,r)$ which touches the boundary of $\mathbf{D}$. Since (3) implies (ii), our condition is much stronger. Moreover, by the Calderón–Zygmund theory, we know that the condition (b) is equivalent to the boundedness of T_α in $L^p(\mu)$ with $1<p<\infty$. Thus in condition (ii), we could replace 2 by p for any $p\in(1,\infty)$.

This paper is organized as follows. In Section 2 we gather some preliminaries including a key covering lemma, terminology and background. Section 3 is devoted to the study of the generalized Bergman spaces $\mathbf{A}^2_\alpha$ and the proof of (a)$\Leftrightarrow$(b). Section 4 contains the proof of the hard part of the main theorem. Section 5 deals with some extensions, comments and opens questions.

2. Preliminary results

In this section we collect a few results which will be useful to our purpose. These concern results on general homogeneous spaces and results for the special case of the unit ball $\mathbf{D}$.

2.1. Definition and properties of a space of homogeneous type

Definition 2.1. A *pseudodistance* on a set X is a map ρ from $X\times X$ to $\mathbb{R}^+$ such that

(1) $\rho(x,y)=0$ if and only if $x=y$;

(2) $\rho(x,y)=\rho(y,x)$;

(3) there exists a positive constant $K\ge 1$ such that, for all $x,y,z\in X$,

$$\rho(x,y)\le K(\rho(x,z)+\rho(z,y)) \quad \text{(the quasitriangular inequality).}$$

For $x\in X$ and $r>0$, the set $B(x,r)=\{y\in X:\rho(x,y)<r\}$ is called a *pseudoball* or simply a *ball* of center x and radius r.

Definition 2.2. A space of *homogeneous type* is a topological space X with a pseudodistance ρ and a positive Borel measure μ on X such that

(1) the balls $B(x,r)$ form a basis of open neighborhoods of x;

(2) (doubling property) there exists a constant $A>0$ such that, for all $x\in X$ and $r>0$, we have

$$0<\mu(B(x,2r))\le A\mu(B(x,r))<\infty.$$

(X,ρ,μ) is called a space of homogeneous type or simply a *homogeneous space.* We will often abusively call X a homogeneous space instead of (X,ρ,μ).

Homogeneous spaces have been treated by several authors such as Coifman and Weiss [CWe], and Stein [S]. We refer to them for further details.

We will use the following lemma to prove a key type of covering lemma, Lemma 2.4 below. It will be crucial in our argument later.

Lemma 2.3. *There exists a constant C_1 such that if $B(x_1,r_1)$ and $B(x_2,r_2)$ are two non-disjoint balls and if $r_1\le r_2$ then*

$$B(x_1,r_1)\subset B(x_2,C_1r_2).$$

Proof. Let $y\in B(x_1,r_1)\cap B(x_2,r_2)$. We have for $x\in B(x_1,r_1)$,

$$\begin{aligned}\rho(x,x_2)&\le K(\rho(x,y)+\rho(y,x_2))\\&\le K(K(\rho(x,x_1)+\rho(x_1,y))+\rho(y,x_2))<K(2Kr_1+r_2)<K(2K+1)r_2.\end{aligned}$$

We obtain the desired result if we set $C_1=K(2K+1)$. $\square$

Lemma 2.4. *Let (X,d,μ) be a homogeneous space. There exists positive constants K_1,K_2 and K_3 with $K_3>K_2>K(C_1+1)K_1$ such that for an open set*

O of X ($O \subsetneq X$), there exists a collection of balls $B_k := B(x_k, \rho_k)$ so that, if $B_k^* = B(x_k, K_1\rho_k)$, $B_k^{**} = B(x_k, K_2\rho_k)$ and $B_k^{***} = B(x_k, K_3\rho_k)$, then

(a) *the balls B_k are pairwise disjoint;*

(b) $O = \bigcup_k B_k^*$;

(c) $O = \bigcup_k B_k^{**}$;

(d) *for each k, $B_k^{***} \cap O^c \neq \varnothing$;*

(e) *a point $x \in O$ belongs to at most M balls B_k^{**} (bounded overlap property). Moreover, the constant M depends only on the constants K_1, K_2, A and K.*

Proof. Let O be an open set of X ($O \subsetneq X$). Let $\varepsilon = 1/16K^2C_1^2(1+C_1)$, where C_1 is the constant defined in Lemma 2.3. Consider the covering of O by the balls $B(x, \varepsilon\delta(x))$, where $\delta(x) = d(x, O^c)$, $x \in O$.

We have $d(x, O^c) > 0$ since O^c is a closed set. We now select a maximal disjoint subcollection of $\{B(x, \varepsilon\delta(x))\}_{x \in O}$; for this subcollection $B_1, B_2, ..., B_k, ...$ with $B_k := B(x_k, \varepsilon\delta(x_k)) = B(x_k, \rho_k)$, we shall prove assertions (a)–(e) above. We set

$$K_1 = \frac{1}{4K^2(C_1+1)\varepsilon}, \quad K_2 = \frac{1}{2\varepsilon K} \quad \text{and} \quad K_3 = \frac{2}{\varepsilon}.$$

Observe that our choice makes these constants satisfy our hypothesis. Note that (a) and (d) hold automatically by our choice of B_k. It is also clear that

$$B_k^* = B\left(x_k, \frac{\delta(x_k)}{4K^2(C_1+1)}\right) \subset B\left(x_k, \frac{\delta(x_k)}{2K}\right) = B_k^{**} \subset O.$$

What remains to be shown is that $O \subset \bigcup_k B_k^*$ (in this case (b) and (c) will be valid) and that (e) is true.

Let us prove that $O \subset \bigcup_k B_k^*$.

Let $x \in O$; by the maximality of the collection B_k,

$$B(x_k, \varepsilon\delta(x_k)) \cap B(x, \varepsilon\delta(x)) \neq \varnothing \quad \text{for some } k.$$

We claim that $\delta(x_k) \geq \delta(x)/4C_1$. If not, since $\varepsilon < 1/2C_1 < 1$, we have

$$B(x_k, 2\delta(x_k)) \cap B\left(x, \frac{\delta(x)}{2C_1}\right) \neq \varnothing.$$

Since $2\delta(x_k) < \delta(x)/2C_1$, by Lemma 2.3, $B(x_k, 2\delta(x_k)) \subset B(x, \delta(x)/2)$, which gives a contradiction since $B(x_k, 2\delta(x_k))$ meets O^c, while $B(x, \delta(x)/2) \subset O$. Using the fact that $4C_1\varepsilon\delta(x_k) \geq \varepsilon\delta(x)$, Lemma 2.3 gives that

$$x \in B(x, \varepsilon\delta(x)) \subset B(x_k, 4\varepsilon C_1^2\delta(x_k)) = B_k^*.$$

This proves (b) and (c).

We proceed to prove (e).

Assume that $x\in\bigcap_{k=1}^{M} B_k^{**}=\bigcap_{k=1}^{M} B(x_k,K_2\rho_k)$. We have that

$$d(x_k,O^c)\le K(d(x,O^c)+d(x,x_k))\le K(d(x,O^c)+K_2\rho_k);$$

this implies that $d(x,O^c)\ge(d(x_k,O^c)-KK_2\rho_k)/K$. But

$$KK_2\rho_k=KK_2\varepsilon d(x_k,O^c)=\frac{d(x_k,O^c)}{2}$$

and thus $d(x,O^c)\ge d(x_k,O^c)/2K=\rho_k/2K\varepsilon$. Hence $\rho_k\le 2K\varepsilon d(x,O^c)$.

On the other hand,

$$d(x,O^c)\le K(d(x_k,O^c)+d(x,x_k))\le K\left(K_2\rho_k+\frac{\rho_k}{\varepsilon}\right)=K(K_2+\varepsilon^{-1})\rho_k.$$

So if $x\in B_k^{**}$, the radius ρ_k satisfies

$$\frac{d(x,O^c)}{K(K_2+\varepsilon^{-1})}\le\rho_k\le 2K\varepsilon d(x,O^c).$$

From this, we have that

$$B(x_k,\rho_k)\subset B(x,C_2d(x,O^c)),$$

where $C_2=2K^2(K_2+1)\varepsilon$. We also have, for each k,

$$B(x,C_2d(x,O^c))\subset B(x_k,C_3\rho_k),$$

where $C_3=K(C_2K(K_2+\varepsilon^{-1})+K_2)$. Thus

$$\bigcup_k B(x_k,\rho_k)\subset B(x,C_2d(x,O^c))$$

and

$$B(x,C_2d(x,O^c))\subset B(x_k,C_3\rho_k)\quad\text{for each } k.$$

Therefore, by the doubling property and the disjointness of B_k, we have that

$$\sum_{k=1}^{M}\mu(B(x_k,\rho_k))\le\mu(B(x,C_2d(x,O^c)))\le\mu(B(x_k,C_3\rho_k))\le C\mu(B(x_k,\rho_k)).$$

Thus $M\le C$ and we are done. □

One key result in the real variable theory is that, by means of the Besicovitch covering lemma, the usual central Hardy–Littlewood maximal function is bounded in $L^p(\mathbb{R}^m, d\mu)$, $1<p<\infty$, where the measure μ is not assumed to be doubling. Since the Besicovitch covering lemma is no longer true in general homogeneous spaces [KR], we will obtain the same L^p estimates for a certain "contractive" central Hardy–Littlewood maximal function to be defined later, via the following lemma,

Lemma 2.5. (ε-Besicovitch) *Let (X, d, μ) be a homogeneous space. Let E be a bounded set, fix a positive number M and denote by $\mathcal{F}$ the family of balls $B(a, r)$ with center $a \in E$ and radius $r \le M$. Then there exists a countable subfamily $\{B(a_k, r_k)\}_{k=1}^{\infty}$ of $\mathcal{F}$ with the following properties*:

(i) $E \subset \bigcup_{k=1}^{\infty} B(a_k, r_k)$;

(ii) *for all $0<\varepsilon<1$, the family $\mathcal{F}_\varepsilon = \{B(a_k, (1-\varepsilon)r_k)\}_{k=1}^{\infty}$ has bounded overlap, namely*

$$\sum_{k=1}^{\infty} \chi_{B(a_k,(1-\varepsilon)r_k)}(x) \le C \log \frac{1}{\varepsilon},$$

where C depends only on constants of X and χ_A denotes the characteristic function of the set A.

Proof. See [FGL, Lemma 3.1]. □

We now turn our attention to the special domain of interest, the unit ball

$$\mathbf{D} = \{z \in \mathbb{C}^n : |z| < 1\}.$$

In [B] a map d on $\mathbf{D} \times \mathbf{D}$ is defined by

$$d(z, w) = \begin{cases} \big||z|-|w|\big| + \left|1 - \dfrac{z}{|z|} \cdot \dfrac{\overline{w}}{|w|}\right|, & \text{if } z, w \in \mathbf{D}^*, \\ |z|+|w|, & \text{otherwise,} \end{cases}$$

where $\mathbf{D}^* = \mathbf{D} \setminus \{0\}$.

2.2. Properties of the pseudodistance d

Lemma 2.6. *The following assertions hold*:

(i) *d is a pseudodistance on $\mathbf{D}$*;

(ii) *d is invariant under rotations in $U(n)$.*

Proof. Assertion (i) follows essentially from the fact that the map $(\xi,\zeta)\mapsto|1-\xi\overline{\zeta}|^{1/2}$ is a distance in $\partial\mathbf{D}$ (see [R, Proposition 5.1.2]). Assertion (ii) follows from the fact that the inner product $z\cdot\overline{w}$ is invariant under rotations in $U(n)$. $\square$

Remark 2.7. The pseudodistance d at the boundary becomes the Korányi distance (see [R]). Hence, it is not equivalent to the usual "pseudohyberbolic" distance on $\mathbf{D}$.

The pseudoballs associated with this pseudodistance satisfy this important observation.

Lemma 2.8. *The pseudoball $B(z,r)=\{w\in\mathbf{D}:d(z,w)<r\}$ touches the boundary of $\mathbf{D}$ if and only if $r>1-|z|$.*

Proof. Fix a pseudoball $B(z,r)$. Let $\varepsilon=r-(1-|z|)$. Since we are interested in points which touch the boundary, we have to find conditions on points $w\in\overline{B(z,r)}$ such that $|w|>|z|$. For such w, we have $d(z,w)=|w|-|z|+|1-(z/|z|)\cdot(\overline{w}/|w|)|$. So

$$d(z,w)<r \quad\Longleftrightarrow\quad \left|1-\frac{z}{|z|}\cdot\frac{\overline{w}}{|w|}\right|<\varepsilon+1-|z|-|w|+|z|=\varepsilon+1-|w|. \tag{4}$$

From this we have our result. In fact, (4) shows that $B(z,r)$ touches the boundary of $\mathbf{D}$ if and only if $\varepsilon>0$. $\square$

These pseudoballs have close relations with the so called Korányi balls. Precisely, for $\xi\in\partial\mathbf{D}=\mathbf{S}$ and $\delta>0$, the Korányi ball of center ξ and radius δ is the set

$$Q_\delta(\xi):=Q(\xi,\delta)=\{z\in\mathbf{D}:|1-z\cdot\overline{\xi}|<\delta\}.$$

We have the following simple proposition.

Proposition 2.9. *There exist positive constants a_1 and a_2 such that, for every pseudoball $B(z,r)$ which touches the boundary of $\mathbf{D}$,*

$$Q\left(\frac{z}{|z|},r\right)\subset B(z,a_1r) \quad\text{and}\quad B(z,r)\subset Q\left(\frac{z}{|z|},a_2r\right).$$

For $\alpha>-1$, let $d\lambda_\alpha(z)=(1-|z|^2)^\alpha\,d\lambda(z)$, where $d\lambda(z)$ is the usual Lebesgue measure of $\mathbb{C}^n\sim\mathbb{R}^{2n}$. We then have the following result.

Lemma 2.10. *For each fixed $\alpha>-1$, the triplet $(\mathbf{D},d,d\lambda_\alpha)$ is a homogeneous space.*

Proof. Since d is already a pseudodistance on $\mathbf{D}$, we need only prove that $d\lambda_\alpha$ is a doubling measure. One can prove that for $0<R<3$, $\zeta=(r,0,...,0)$, $0<r<1$,

$$\lambda_\alpha(B(\zeta,R))\cong R^{n+1}\max(R,1-r)^\alpha. \tag{5}$$

This ends the proof of the lemma. □

Remark 2.11. This lemma shows that we can apply Lemmas 2.4 and 2.5 in $(\mathbf{D},d,d\lambda_\alpha)$.

We will make use of the following properties of d, whose proof is immediate.

Lemma 2.12. *For every* $z\in\mathbf{D}$ *and* r_0, $0<r_0<1$, *if we let* $z_0=(r_0,0,...,0)$ *we have that*

(1) $|1-z_1\cdot r_0|\geq\frac{1}{3}d(z,z_0)$;
(2) $|z_1-r_0|\leq d(z,z_0)$;
(3) $\sum_{k=2}^n|z_k|^2\leq 2d(z,z_0)$;
(4) $|1-z\cdot z_0|\leq 1-r_0^2+d(z,z_0)$.

For $\alpha>-n-1$ fixed, set $k=n+1+\alpha$. We consider the kernel K_α given by

$$K_\alpha(z,w)=\operatorname{Re}\left(\frac{1}{(1-z\cdot\overline{w})^k}\right).$$

The next result shows some important properties of this kernel.

Proposition 2.13. (1) *There exists a constant* C_3 *such that for all* $z,w\in\mathbf{D}$,

$$|K_\alpha(z,w)|\leq\frac{C_3}{d(z,w)^k}.$$

(2) *There exist two constants* C_1,C_2 *such that for all* $z,w,\zeta\in\mathbf{D}$ *satisfying*

$$d(z,\zeta)>C_1d(w,\zeta),$$

we have

$$|K_\alpha(z,w)-K_\alpha(z,\zeta)|\leq C_2\frac{d(w,\zeta)^{1/2}}{d(z,\zeta)^{k+1/2}}.$$

Before proving this, let us observe that this proposition shows that the kernel K_α is a k-Calderón–Zygmund kernel with respect to the pseudodistance d. We will give the proof of this proposition for the kernel $H_\alpha(z,w)=1/(1-z\cdot\overline{w})^k$ instead. The case K_α is then a consequence.

Proof. Assertion (1) follows from (1) of Lemma 2.12 and the invariance under rotation. Indeed for $z,w\in\mathbf{D}$, if Δ is the rotation such that $\Delta(w)=(|w|,0,...,0)$ then we have

$$d(z,w)=d(\Delta(z),\Delta(w))\le 3|1-\Delta(z)\overline{\Delta(w)}|\le 3|1-z\cdot\overline{w}|. \tag{6}$$

So $d(z,w)^k\le 3^k|1-z\cdot\overline{w}|^k$. Take $C_3=3^k$.

Let us prove (2). By the invariance under rotation, we can suppose that $\zeta=(r_0,0,...,0)$. We use the identity

$$H_\alpha(z,w)-H_\alpha(z,\zeta)=\int_0^1\frac{kz\cdot(\overline{w}-\overline{\zeta})}{(1-z\cdot\overline{w}-tz(\overline{\zeta}-\overline{w}))^{k+1}}\,dt$$

to obtain that

$$|H_\alpha(z,w)-H_\alpha(z,\zeta)|\le\int_0^1\frac{k|z\cdot(\overline{w}-\overline{\zeta})|}{|1-z\cdot\overline{w}-tz(\overline{\zeta}-\overline{w})|^{k+1}}\,dt. \tag{7}$$

We have that

$$\begin{aligned}|z\cdot(\overline{w}-\overline{\zeta})|&\le|z_1\cdot(\overline{w}_1-r_0)|+\left(\sum_{k=2}^n|z_k|^2\right)^{1/2}\left(\sum_{k=2}^n|w_k|^2\right)^{1/2}\\&\le|w_1-r_0|+\left(\sum_{k=2}^n|z_k|^2\right)^{1/2}\left(\sum_{k=2}^n|w_k|^2\right)^{1/2}.\end{aligned}$$

So by Lemma 2.12 and (6), we have that

$$\begin{aligned}|z\cdot(\overline{w}-\overline{\zeta})|&\le 2d(w,\zeta)^{1/2}(d(w,\zeta)^{1/2}+d(z,\zeta)^{1/2})\\&\le\frac{4}{\sqrt{C_1}}d(w,\zeta)^{1/2}d(z,\zeta)^{1/2}\le\frac{C}{\sqrt{C_1}}d(w,\zeta)^{1/2}|1-z\cdot\overline{\zeta}|^{1/2}.\end{aligned} \tag{8}$$

This shows that for C_1 large enough, we have $|z\cdot(\overline{w}-\overline{\zeta})|\le\frac{1}{2}|1-z\cdot\overline{\zeta}|$. On the other hand, observe that

$$|1-z\cdot\overline{w}-tz(\overline{\zeta}-\overline{w})|=|1-z\cdot\overline{\eta}|,$$

where $\eta=(1-t)w+t\zeta$. Since

$$|(1-z\cdot\overline{\zeta})-(1-z\cdot\overline{\eta})|=|z\cdot(\overline{\eta}-\overline{\zeta})|$$

and

$$|(1-z\cdot\overline{\zeta})-(1-z\cdot\overline{\eta})|=(1-t)|z\cdot(\overline{w}-\overline{\zeta})|\le|z\cdot(\overline{w}-\overline{\zeta})|,$$

we conclude that for large C_1, $|1-z\cdot\overline{\eta}|>\frac{1}{2}|1-z\cdot\overline{\zeta}|$.

Therefore, from (7), (8) and (6), we have that

$$|H_\alpha(z,w)-H_\alpha(z,\zeta)|\le 2^{k+1}k\frac{C}{\sqrt{C_1}}\frac{|1-z\cdot\overline{\zeta}|^{1/2}d(w,\zeta)^{1/2}}{|1-z\cdot\overline{\zeta}|^{k+1}}\le C_2\frac{d(w,\zeta)^{1/2}}{d(z,\zeta)^{k+1/2}}. \quad \square$$

3. The generalized Bergman spaces $\mathbf{A}^2_\alpha$

In this section, we define the space $\mathbf{A}^2_\alpha$. We give some properties of this space. Finally we show that the Carleson measures problem for these space is equivalent to the T(1)-type problem associated with the Calderón–Zygmund kernel K_α.

Definition 3.1. Let $\alpha\in\mathbb{R}$, $\alpha>-n-1$. We denote by $\mathbf{A}^2_\alpha$ the space of all holomorphic functions f in the unit ball $\mathbf{D}$ with the property that

$$\|f\|^2_\alpha=\sum_{m\in\mathbb{N}^n}|c(m)|^2\frac{\Gamma(n+1+\alpha)m!}{\Gamma(n+1+|m|+\alpha)}<\infty,$$

where $f(z)=\sum_{m\in\mathbb{N}^n}c(m)z^m$ is the Taylor expansion of f.

Theorem 3.2. *The space* $\mathbf{A}^2_\alpha$ *is equipped with an inner product such that the associated reproducing kernel is given by*

$$B_\alpha(z,w)=\frac{1}{(1-z\cdot\overline{w})^{n+1+\alpha}}.$$

Proof. For $f(z)=\sum_{m\in\mathbb{N}^n}c(m)z^m$ and $g(z)=\sum_{m\in\mathbb{N}^n}d(m)z^m$, define the product by

$$\langle f,g\rangle_\alpha=\sum_{m\in\mathbb{N}^n}c(m)\overline{d(m)}\frac{\Gamma(n+1+\alpha)m!}{\Gamma(n+1+|m|+\alpha)}.$$

This clearly defines an inner product in $\mathbf{A}^2_\alpha$. Let $f\in\mathbf{A}^2_\alpha$ with $f(z)=\sum_{m\in\mathbb{N}^n}c(m)z^m$. Since

$$B_\alpha(z,w)=\sum_{m\in\mathbb{N}^n}\frac{\Gamma(n+1+|m|+\alpha)}{\Gamma(n+1+\alpha)m!}z^m\overline{w}^m,$$

we have for $w\in\mathbf{D}$,

$$\begin{aligned}\langle f,B_\alpha(\,\cdot\,,w)\rangle_\alpha&=\sum_{m\in\mathbb{N}^n}c(m)\frac{\Gamma(n+1+|m|+\alpha)}{\Gamma(n+1+\alpha)m!}w^m\frac{\Gamma(n+1+\alpha)m!}{\Gamma(n+1+|m|+\alpha)}\\&=\sum_{m\in\mathbb{N}^n}c(m)w^m=f(w).\end{aligned}$$

We are done. $\square$

Remark 3.3. The space $\mathbf{A}^2_\alpha$ is a Hilbert space with the Hilbert norm $\|\cdot\|_\alpha$.

The following proposition gives a norm of a certain element in $\mathbf{A}^2_\alpha$. The proof is a straightforward computation.

Proposition 3.4. *Let $w\in\mathbf{D}$ and set $f(z)=1/(1-z\cdot\overline{w})^s$. If $2s>n+1+\alpha$ then $f\in\mathbf{A}^2_\alpha$. Moreover,*

$$\|f\|^2_\alpha \cong \frac{1}{(1-|w|^2)^{2s-n-1-\alpha}}.$$

We recall that we want to characterize positive Borel measures μ on $\mathbf{D}$ such that

$$\int_{\mathbf{D}} |f|^2\,d\mu \le C(\mu)\|f\|^2_\alpha, \quad f\in\mathbf{A}^2_\alpha. \tag{9}$$

As we have mentioned in the introduction the solution of this question is well known for $\alpha\ge-1$. The result from these cases is the following theorem.

Theorem 3.5. (Carleson, Hörmander, Stegenga, Cima and Wogen) *Let $\alpha\ge-1$ and let μ be a positive Borel measure on $\mathbf{D}$. The following conditions are equivalent*:

(a) *There exists a positive constant C such that*

$$\mu(Q_\delta(\xi))\le C\delta^{n+1+\alpha} \tag{10}$$

for all $\xi\in\mathbf{S}$ and all $\delta>0$;

(b) *The measure μ is an $\mathbf{A}^2_\alpha$ Carleson measure.*

The actual range $\alpha\in(-n-1,-1)$ is more difficult. A characterization of Carleson measures in terms of capacity when $\alpha\in(-n-1,-n)$ is an old result of Cohn and Verbitsky [CV]. Recently Arcozzi, Rochberg and Sawyer [ARS] obtained another characterization for all α in $(-n-1,-n]$. Our approach gives a new characterization of Carleson measures for this case. It seems likely that our characterization could be extended to the remaining range $\alpha\in(-n,-1)$. However, we have not yet succeeded to do this.

We must observe that for $\alpha>-n-1$, condition (10) remains a necessary condition for Carleson measures for $\mathbf{A}^2_\alpha$. This can be seen by using Proposition 3.4, (9) and the following result [ZZ, Theorem 45].

Theorem 3.6. *Let $\alpha\in\mathbb{R}$ be such that $n+1+\alpha>0$ and let μ be a positive Borel measure on $\mathbf{D}$. Then the following conditions are equivalent*:

(a) *There exists a positive constant* C *such that*

$$\mu(Q_\delta(\xi)) \leq C\delta^{n+1+\alpha}$$

for all $\xi\in\mathbf{S}$ *and all* $\delta>0$;

(b) *For each* $s>0$ *there exists a positive constant* C *such that*

$$\sup_{z\in\mathbf{D}} \int_{\mathbf{D}} \frac{(1-|z|^2)^s \, d\mu(w)}{|1-z\cdot\overline{w}|^{n+1+\alpha+s}} \leq C < \infty; \tag{11}$$

(c) *For some* $s>0$ *there exists a positive constant* C *such that the inequality in* (11) *holds.*

The equivalence (a)⇔(b) of Theorem 1.1, as we have mentioned in the introduction, is a consequence of [ARS, Lemma 24]. We state it below as a proposition. Recall that the Calderón–Zygmund operator T_α is given by

$$T_\alpha f(z) = \int_{\mathbf{D}} f(w) K_\alpha(z,w)\, d\mu(w), \quad z\in\mathbf{D},$$

where the kernel K_α is defined by

$$K_\alpha(z,w) = \mathrm{Re}\left(\frac{1}{(1-z\cdot\overline{w})^{n+1+\alpha}}\right).$$

Proposition 3.7. (see [ARS]) *Suppose that* $n+1+\alpha>0$ *and let* μ *be a positive Borel measure on* $\mathbf{D}$. *Then the following conditions are equivalent*:

(a) *The measure* μ *is an* $\mathbf{A}^2_\alpha$ *Carleson measure*;

(b) *The operator* T_α *is bounded in* $L^2(\mu)$.

4. Proof of the equivalence (b)⇔(c) in Theorem 1.1

This section is devoted to the proof of the T(1)-type theorem, that is the characterization of positive Borel measures μ on $\mathbf{D}$ such that the operator T_α is bounded in $L^2(\mu)$. To get the equivalence (b)⇔(c) in Theorem 1.1, it suffices to prove the following theorem.

Theorem 4.1. *Let* $k=n+1+\alpha$ *and let* μ *be a positive Borel measure on* $\mathbf{D}$. *Then the following conditions are equivalent*:

(1) *The operator* T_α *is bounded in* $L^2(\mu)$;

(2) *The operator* T_α *is bounded in* $L^p(\mu)$ *for some* $p>2$;

(3) (i) *There exists a constant* $C>0$ *such that*

$$\mu(B(z,r)) \le Cr^k \tag{12}$$

for all pseudoballs $B(z,r)$ *which touch the boundary, and*

(ii) *there exists a constant* $C>0$ *such that*

$$\int_B \left(\int_B \operatorname{Re}\left(\frac{1}{(1-z\cdot\overline{w})^k} \right) d\mu(w) \right)^2 d\mu(z) \le C\mu(B)$$

for all pseudoballs B *which touch the boundary.*

4.1. Proof of the implication (1)⇒(3) in Theorem 4.1

Assertion (i) follows from the discussion after Theorem 3.5 and the fact that the sets $B(z,r)$ and $Q_r(z/|z|)$ are comparable when $B(z,r)$ touches the boundary (in the sense of Proposition 2.9), we also make use of Proposition 3.7. Assertion (ii) is obtained by testing the boundedness on the characteristic function $f=\chi_B$.

4.2. Related maximal functions

Definition 4.2. We say that a measure μ satisfies *the growth condition* when μ satisfies inequality (12).

We proceed now to prove that (i) and (ii) are sufficient for the boundedness of T_α for some $p>2$, that is the proof of the implication (3)⇒(2). We focus our attention on the special case $\alpha=-n$. We then set $T=T_{-n}$ and $K=K_{-n}$.

As we have mentioned in the introduction, we follow the same idea as in [V]. Indeed, we first introduce a sort of curvature which plays the role of the Menger curvature. This curvature is adapted to our domain and has a close relation with our operator. Next, we proceed to construct for every ball which touches the boundary, a subset in this ball satisfying some properties (see Lemma 4.9 below). This is the first crucial step of our proof. Finally, the next crucial step is to prove an appropriate good λ inequality without resorting to a doubling property on μ. Lemma 2.4 is used in those steps.

We suppose that a measure μ satisfies the growth condition. In our estimates we use two variants of the central Hardy–Littlewood maximal operator acting on a complex Radon measure ν, namely,

$$M\nu(z) = \sup_{r>1-|z|} \frac{|\nu|(B(z,r))}{r},$$

and for a positive constant $\rho\geq 1$,

$$M_\mu^\rho \nu(z)=\sup_{r>1-|z|}\frac{|\nu|(B(z,r))}{\mu(B(z,\rho r))},\quad z\in\operatorname{supp}\mu,$$

where $B(z,r)$ is the pseudoball centered at z of radius r which touches the boundary and $\operatorname{supp}\mu$ is the closed support of μ.

Proposition 4.3. *Let μ be a positive Borel measure which satisfies the growth condition. For every $\rho>1$, there exists a positive constant $C(\rho)$ such that for any $f\in L^p(\mu)$,*

$$\int_{\mathbf{D}} M_\mu^\rho(f\mu)^p\,d\mu\leq C\int_{\mathbf{D}}|f|^p\,d\mu,\quad 1<p<\infty. \tag{13}$$

Proof. Fix $\rho>1$. Let $E_\lambda^\rho=\{z\in\mathbf{D}:M_\mu^\rho(f\mu)(z)>\lambda\}$. Observe first that, for each $z\in E_\lambda^\rho$, there exists a pseudoball $B(z,r_z)$ such that

$$\mu(B(z,\rho r_z))\leq\frac{1}{\lambda}\int_{B(z,r_z)}|f|\,d\mu. \tag{14}$$

Consider the family $\mathcal{F}=\{B(z,\rho r_z)\}_{z\in E_\lambda^\rho}$. Applying Lemma 2.5 to this family with $\varepsilon=1-1/\rho$, we obtain a subfamily $\{B(z_k,\rho r_k)\}_{k=1}^\infty$ of $\mathcal{F}$ such that $E_\lambda^\rho\subset\bigcup_{k=1}^\infty B(z_k,\rho r_k)$, and the family $\{B(z_k,r_k)\}_{k=1}^\infty$ has bounded overlaps. Therefore, from (14) and this bounded overlap property, we have

$$\mu(E_\lambda^\rho)\leq\sum_{k=1}^\infty\mu(B(z_k,\rho r_k))\leq\frac{1}{\lambda}\sum_{k=1}^\infty\int_{B(z_k,r_k)}|f|\,d\mu\leq\frac{C(\rho)}{\lambda}\int_{\mathbf{D}}|f|\,d\mu.$$

Hence, M_μ^ρ is of weak type $(1,1)$. We obtain the desired result from the obvious L^∞ estimate and the Marcinkiewicz interpolation theorem. □

Remark 4.4. Observe that for some constant $C(\rho)>0$, we have

$$M\nu(z)\leq C(\rho)M_\mu^\rho\nu(z),\quad z\in\operatorname{supp}\mu, \tag{15}$$

so (13) remains true if we replace M_μ^ρ by M.

The weak estimate is valid if one replaces $f\mu$ by any finite measure ν.

Lemma 4.5. *Let μ be a positive Borel measure which satisfies the growth condition. Given $\beta>0$, there exists a constant C such that for z^0, $R>1-|z^0|$ and a positive measurable function f,*

$$R^\beta\int_{d(z^0,w)>R}\frac{f(w)\,d\mu(w)}{d(z^0,w)^{1+\beta}}\leq CM(f\mu)(z)$$

for all $z \in B(z^0, R)$*. In particular, we have*

$$\mu^\beta(B(z^0,R)) \int_{d(z^0,w)>R} \frac{d\mu(w)}{|1-z^0\overline{w}|^{1+\beta}} \le C.$$

Proof. Fix $\beta>0$, z^0, $R>1-|z^0|$ and a positive measurable function f. Let $z\in B(z^0,R)$. We have

$$\begin{aligned}\int_{d(z^0,w)>R} \frac{f(w)\,d\mu(w)}{d(z^0,w)^{1+\beta}} &\le \sum_{k=0}^{\infty} \int_{2^kR<d(z^0,w)\le 2^{k+1}R} \frac{f(w)\,d\mu(w)}{d(z^0,w)^{1+\beta}}\\ &\le \sum_{k=0}^{\infty} \frac{1}{(2^kR)^{1+\beta}} \int_{d(z^0,w)<2^{k+1}R} f(w)\,d\mu(w)\\ &\le \sum_{k=0}^{\infty} \frac{1}{(2^kR)^{1+\beta}} \int_{d(z,w)<2K2^{k+1}R} f(w)\,d\mu(w)\\ &\le CM(f\mu)(z)\sum_{k=0}^{\infty} \frac{2^kR}{(2^kR)^{1+\beta}}\\ &\le CR^{-\beta}M(f\mu)(z).\end{aligned}$$

By using (6), the second claim of the lemma follows from the fact that

$$\mu^\beta(B(z^0,R)) \le CR^\beta, \quad \frac{1}{|1-z^0\overline{w}|^{1+\beta}} \le \frac{C}{d(z^0,w)^{1+\beta}}, \quad \text{and}$$
$$M(f\mu)(z^0) \le 1 \quad \text{for } f\equiv 1. \quad \square$$

For a Radon measure ν set, for $z\in\mathbf{D}$,

$$T^*\nu(z) = \int_{\mathbf{D}} \frac{d|\nu|(w)}{|1-z\cdot\overline{w}|}.$$

Lemma 4.6. *Let* Ω *be an open pseudoball which touches the boundary and let* μ *be a positive Borel measure on* $\mathbf{D}$ *satisfying the growth condition.*

If we set $\nu=\chi_{\Omega^c}\mu$*, then*

$$\int_\Omega T^*(f\nu)^2\,d\mu \le C\int_{\mathbf{D}} |f|^2\,d\nu, \quad f\in L^2(\nu).$$

Proof. It is enough to prove that for some $\eta>0$, there exists $\rho>1$, $\gamma>0$ and $C>0$ such that

$$\mu(\{z\in\Omega : T^*(f\nu)(z) > (1+\eta)t\}) \le C\mu(\{z\in\Omega : M^\rho_\mu(f\nu)(z) > \gamma t\}). \tag{16}$$

Indeed if (16) is true, then, by (13),

$$\begin{aligned}\int_\Omega T^*(f\nu)^2\,d\mu &= \int_0^\infty \mu(\{z\in\Omega: T^*(f\nu)(z)>t\})2t\,dt\\ &= (1+\eta)^2\int_0^\infty \mu(\{z\in\Omega: T^*(f\nu)(z)>(1+\eta)t\})2t\,dt\\ &\le C\int_0^\infty \mu(\{z\in\Omega: M_\mu^\rho(f\nu)(z)>\gamma t\})2t\,dt\\ &\le C\int_\Omega M_\mu^\rho(f\nu)^2(z)\,d\mu(z)\\ &\le C\int_{\mathbf{D}} |f|^2\,d\nu.\end{aligned}$$

To prove (16), we applied Lemma 2.4 to the open set

$$E_t=\{z\in\Omega: T^*(f\nu)(z)>t\}.$$

We will obtain (16) once we prove that for each j,

$$\mu(\{z\in B_j^*: T^*(f\nu)(z)>(1+\eta)t \ \text{ and } \ M_\mu^\rho(f\nu)(z)\le\gamma t\})=0, \tag{17}$$

where B_j^* is a term in the first decomposition of the open set E_t with respect to Lemma 2.4.

In fact we will have

$$\begin{aligned}&\mu(\{z\in\Omega: T^*(f\nu)(z)>(1+\eta)t\})\\ &\qquad\le\sum_j \mu(\{z\in B_j^*: T^*(f\nu)(z)>(1+\eta)t\})\\ &\qquad\le\sum_j \mu(\{z\in B_j^*: T^*(f\nu)(z)>(1+\eta)t \text{ and } M_\mu^\rho(f\nu)(z)\le\gamma t\})\\ &\qquad\quad+\sum_j \mu(\{z\in B_j^*: M_\mu^\rho(f\nu)(z)>\gamma t\})\\ &\qquad\le\sum_j \mu(\{z\in B_j^*: M_\mu^\rho(f\nu)(z)>\gamma t\})\\ &\qquad\le C\mu(\{z\in\Omega: M_\mu^\rho(f\nu)(z)>\gamma t\}),\end{aligned}$$

by the bounded overlap property.

So it remains to prove (17).

Set $B=B(z^B,K_1\varrho)=B_j^*$ and $B'=B(z^B,K_2\varrho)=B_j^{**}$.

Suppose without loss of generality that there exists $\xi^0\in B$ such that

$$M_\mu^\rho(f\nu)(\xi^0)\le\gamma t. \tag{18}$$

Let z^0 be a point in $E_t^c \cap B(z^B, K_3\varrho)$. Let $\overline{B}$ be a ball centered at z^0 whose radius is equal to $R = \max(2(1-|z^0|), C\varrho)$, where C is a constant greater than or equal to K_3 to be made precise later. Then $\overline{B}$ touches the boundary of $\mathbf{D}$.

Let $f_1 = f\chi_{\overline{B}}$ and $f_2 = f - f_1 = f\chi_{\overline{B}^c}$. There exists a constant A_1 such that

$$T^*(f\nu)(z) \le T^*(f_1\nu)(z) + (1+A_1\gamma)t, \quad z \in B. \tag{19}$$

To prove (19), let $z \in B$. Then

$$\begin{aligned} T^*(f_2\nu)(z) &= \int_{\overline{B}^c} \frac{|f(w)|\,d\nu(w)}{|1-z\cdot\overline{w}|} \\ &\le \int_{\overline{B}^c} \frac{|f(w)|\,d\nu(w)}{|1-z^0\overline{w}|} + \int_{\overline{B}^c} |f(w)| \left| \frac{1}{1-z\cdot\overline{w}} - \frac{1}{1-z^0\overline{w}} \right| d\nu(w) \\ &\le T^*(f\nu)(z^0) + \int_{\overline{B}^c} |f(w)| \left| \frac{1}{1-z\cdot\overline{w}} - \frac{1}{1-z^0\overline{w}} \right| d\nu(w) \\ &\le t + C_2 \int_{\overline{B}^c} |f(w)| \frac{d(z,z^0)^{1/2}}{d(w,z^0)^{3/2}}\, d\nu(w) \end{aligned}$$

provided that C is chosen large enough so that we can use (2) of Proposition 2.13. Hence by Lemma 4.5 and (18), we have that

$$T^*(f_2\nu)(z) \le (1+A_1\gamma)t,$$

as $d(z,z^0) \le R$ and $\xi^0 \in B(z^0, R)$ provided C is large enough. This proves (19).

Set $\widetilde{B} = B(\xi^0, C_1K_1\varrho)$ and observe that $B \subset \widetilde{B} \subset B' \subset E_t$.

Now, if $C\varrho \ge 2(1-|z^0|)$, there exists a constant $A_2 > 0$ such that for $z \in B$,

$$T^*(f_1\nu)(z) \le T^*(f\nu\chi_{\widetilde{B}})(z) + A_2\gamma t. \tag{20}$$

To prove (20), we have that

$$T^*(f_1\nu)(z) \le T^*(f\nu\chi_{\widetilde{B}})(z) + \int_{\overline{B}\setminus\widetilde{B}} \frac{|f(w)|\,d\nu(w)}{|1-z\cdot\overline{w}|} = T^*(f\nu\chi_{\widetilde{B}})(z) + I,$$

where $I = \int_{\overline{B}\setminus\widetilde{B}} (|f(w)|/|1-z\cdot\overline{w}|)\, d\nu(w)$.

By (6), $1/|1-z\cdot\overline{w}| \le C/d(z,w)$, and on the other hand, for $w \in \overline{B}\setminus\widetilde{B}$,

$$C_1K_1\varrho \le d(\xi^0, w) \le K(d(z^B, \xi^0) + d(z^B, w)) < K(K_1\varrho + K(d(z^B, z) + d(z,w))).$$

Thus

$$K^2 d(z,w) > C_1K_1\varrho - KK_1\varrho(K+1) = K_1K^2\varrho.$$

Therefore

$$I \leq CM(f\nu)(\xi^0) \leq A_2\gamma t.$$

This proves (20). Since $\widetilde{B}\subset\Omega$, we have that $T^*(f\nu\chi_{\widetilde{B}})(z)=0$.

For the case $C\varrho<2(1-|z^0|)$, we have for $z\in B$ and $w\in\overline{B}$,

$$|1-z\cdot\overline{w}| > 1-|z| > C'(1-|z^0|).$$

The second inequality in the previous line follows from the fact that

$$|z|-|z^0| \leq d(z,z^0) \leq K(K_3+K_1)\varrho < \frac{2K(K_3+K_1)}{C}(1-|z^0|) < \frac{1}{2}(1-|z^0|),$$

provided C is sufficiently large. Hence

$$T^*(f\nu\chi_{\overline{B}})(z) \leq \frac{C}{1-|z^0|}\int_{\overline{B}} |f|\,d\nu \leq CM_\mu^\varrho(f\nu)(\xi^0) \leq C''\gamma t.$$

So we finally conclude that there exists a constant $A>0$ such that

$$T^*(f\nu)(z) \leq (1+A\gamma)t, \quad z\in B.$$

From this we have that

$$\begin{aligned}\mu(\{z\in B: T^*(f\nu)(z) > (1+\eta)t \text{ and } M_\mu^\varrho(f\nu)(z) \leq \gamma t\}) \\ \leq \mu(\{z\in B: (1+A\gamma)t > (1+\eta)t \text{ and } M_\mu^\varrho(f\nu)(z)\leq\gamma t\});\end{aligned}$$

so if we choose $0<\gamma\leq\eta/2A$, we obtain (17). This ends the proof of the lemma. □

4.3. Curvature in the unit ball

Definition 4.7. Given three points $z_1, z_2, z_3\in\mathbf{D}$, we define their curvature $c(z_1,z_2,z_3)$ by

$$c^2(z_1,z_2,z_3) = \sum_\sigma K(z_{\sigma(2)}, z_{\sigma(1)})K(z_{\sigma(3)}, z_{\sigma(1)}),$$

where the sum is taken over the six permutations of 1, 2 and 3.

For a positive Borel measure ν the quantity

$$c^2(\nu) = \iiint_{\mathbb{C}^{3n}} c^2(z_1,z_2,z_3)\,d\nu(z_1)\,d\nu(z_2)\,d\nu(z_3)$$

is called the total curvature of ν. One important fact about this curvature is that $c^2(z_1,z_2,z_3)>0$. Indeed for z and w in $\mathbf{D}$

$$K(z,w)=\frac{\operatorname{Re}(1-z\cdot\overline{w})}{|1-z\cdot\overline{w}|^2}=\frac{1-\operatorname{Re}(z\cdot\overline{w})}{|1-z\cdot\overline{w}|^2}>0$$

since $\operatorname{Re}(z\cdot\overline{w})<1$.

The next lemma gives a relation between this curvature and our operator T.

Lemma 4.8. *Let ν_j, $j=1,2,3$, be three Borel measures. Then*

$$\sum_\sigma \int_{\mathbf{D}} T(\nu_{\sigma(1)})T(\nu_{\sigma(2)})\,d\nu_{\sigma(3)}=\iiint_{\mathbf{D}^3} c^2(z_1,z_2,z_3)\,d\nu_1(z_1)\,d\nu_2(z_2)\,d\nu_3(z_3).$$

Proof. We have

$$\begin{aligned}\int_{\mathbf{D}} T(\nu_{\sigma(1)})&T(\nu_{\sigma(2)})\,d\nu_{\sigma(3)}\\ &=\iiint_{\mathbf{D}^3} K(z_{\sigma(3)},z_{\sigma(1)})K(z_{\sigma(3)},z_{\sigma(2)})\,d\nu_{\sigma(1)}(z_{\sigma(1)})\,d\nu_{\sigma(2)}(z_{\sigma(2)})\,d\nu_{\sigma(3)}(z_{\sigma(3)}).\end{aligned}$$

Since for each σ,

$$d\nu_{\sigma(1)}(z_{\sigma(1)})\,d\nu_{\sigma(2)}(z_{\sigma(2)})\,d\nu_{\sigma(3)}(z_{\sigma(3)})=d\nu_1(z_1)\,d\nu_2(z_2)\,d\nu_3(z_3),$$

summing over the six permutations we obtain

$$\sum_\sigma \int_{\mathbf{D}} T(\nu_{\sigma(1)})T(\nu_{\sigma(2)})\,d\nu_{\sigma(3)}=\iiint_{\mathbf{D}^3} c^2(z_1,z_2,z_3)\,d\nu_1(z_1)\,d\nu_2(z_2)\,d\nu_3(z_3).\qquad\square$$

We apply Lemma 4.8 to $\nu_1=\nu_2=f\mu$ with f (a real function) in $L^2(\mu)$ and $\nu_3=\chi_B\mu$ with B a fixed pseudoball which touches the boundary. We then have

$$\begin{aligned}(21)\quad 2\int_B |T(f\mu)|^2\,d\mu+4\int_{\mathbf{D}} T(f\mu)T(\chi_B\mu)f\,d\mu&\\ =\iiint_{\mathbf{D}^3} c^2(z,w,\zeta)f(z)f(w)\chi_B(\zeta)\,d\mu(z)\,d\mu(w)\,d\mu(\zeta).&\end{aligned}$$

In particular, if we replace f by χ_B in the integral, one gets that

$$6\int_B |T(f\mu)|^2\,d\mu=\iiint_{B^3} c^2(z,w,\zeta)\,d\mu(z)\,d\mu(w)\,d\mu(\zeta),$$

and thus

$$\iiint_{B^3} c^2(z,w,\zeta)\,d\mu(z)\,d\mu(w)\,d\mu(\zeta)\le C\mu(B),\tag{22}$$

provided μ satisfies condition (3)(ii) in Theorem 4.1 (the case $k=1$).

We are now ready to produce a subset inside a given pseudoball B which touches the boundary. As in [V], set

$$c_B^2(z)=\iint_{B^2} c^2(z,w,\zeta)\,d\mu(w)\,d\mu(\zeta), \quad z\in B.$$

By Chebyshev's inequality, condition (3)(ii) in Theorem 4.1 and (22) we have that

$$\begin{aligned}(23)\quad \mu(\{z\in B: c_B(z)>t \text{ or } |T(\chi_B\mu)(z)|>t\}) & \\ &\le \frac{1}{t^2}\left(\int_B c_B^2(z)\,d\mu(z)+\int_B |T(\chi_B\mu)(z)|^2\,d\mu(z)\right)\\ &\le C\frac{\mu(B)}{t^2}.\end{aligned}$$

From this we have the following lemma.

Lemma 4.9. *Given $0<\theta<1$, there exists a set $E\subset B$ such that*

$$c_B^2(z)\le\frac{C}{\theta} \text{ and } |T(\chi_B\mu)(z)|^2\le\frac{C}{\theta}, \quad z\in E,$$

and

$$\mu(B\setminus E)\le\theta\mu(B).$$

Proof. Fix $0<\theta<1$. If $\mu(B)=0$, there is nothing to do. If $\mu(B)\neq 0$, set

$$E=\left\{z\in B: c_B^2(z)\le\frac{C}{\theta} \text{ and } |T(\chi_B\mu)(z)|^2\le\frac{C}{\theta}\right\}.$$

It is then easy to verify that this set satisfies our requirements. □

We set $k(z,w)=\int_B c^2(z,w,\zeta)\,d\mu(\zeta)$ so that

$$(24)\qquad \int_E k(z,w)\,d\mu(w)=c_B^2(z)\le\frac{C}{\theta}, \quad z\in E.$$

Since $k(z,w)=k(w,z)$ we obtain the following lemma.

Lemma 4.10. *There exists a constant $C=C(\theta)$ which does not depend on B such that*

$$\iiint_{E^2\times\mathbf{D}} c^2(z,w,\zeta)f(z)f(w)\chi_B(\zeta)\,d\mu(z)\,d\mu(w)\,d\mu(\zeta)\le C\int_E f^2\,d\mu,$$

where $f\in L^2(E)=L^2(E,\mu)$, with f real.

Proof. The result follows from Schur's test since

$$\int_E k(z,w)\,d\mu(w) \le \frac{C}{\theta}, \quad z\in E. \quad \square$$

Therefore from (21), Lemma 4.9 and Lemma 4.10, for any $f\in L^2(E,\mu)$, we get that

$$\int_B |T(f\mu)|^2\,d\mu \le C\left(\int_B |T(f\mu)|^2\,d\mu\right)^{1/2}\left(\int_E f^2\,d\mu\right)^{1/2} + C\int_E f^2\,d\mu$$

and consequently

$$\int_B |T(f\mu)|^2\,d\mu \le C\int_E f^2\,d\mu, \quad f\in L^2(E).$$

By duality this implies that

$$\int_E |T(g\mu)|^2\,d\mu \le C\int_B g^2\,d\mu, \quad g\in L^2(B).$$

So by Chebyshev's inequality

$$\mu(\{z\in E: |T(g\mu)(z)|>t\}) \le \frac{C}{t^2}\int_B g^2\,d\mu, \quad g\in L^2(B). \tag{25}$$

Now, for every $h\in L^2(\mathbf{D},\mu)$, Lemma 4.6 and (25) give

$$\begin{aligned}
\mu(\{z\in E: |T(h\mu)(z)|>t\}) &\le \mu\left(\left\{z\in E: |T(h\chi_B\mu)(z)|>\frac{t}{2}\right\}\right) \\
&\quad + \mu\left(\left\{z\in E: |T(h\chi_{B^c}\mu)(z)|>\frac{t}{2}\right\}\right) \\
&\le \frac{C}{t^2}\int_B h^2\,d\mu + \frac{C}{t^2}\int_B |T(h\chi_{B^c}\mu)(z)|^2\,d\mu \\
&\le \frac{C}{t^2}\int_B h^2\,d\mu + \frac{C}{t^2}\int_{B^c} h^2\,d\mu \\
&\le \frac{C}{t^2}\int_{\mathbf{D}} h^2\,d\mu.
\end{aligned} \tag{26}$$

4.4. A good λ inequality

In this subsection we will establish the next crucial argument in the proof of the implication (3)⇒(2) of Theorem 4.1. The result is the following theorem.

Theorem 4.11. *Let μ be a positive Borel measure on* $\mathbf{D}$ *with satisfies* (i) *and* (ii) *of Theorem* 4.1. *Then, for each $\eta>0$ there exists $\gamma=\gamma(\eta)>0$ small enough so that*

$$\mu(\{z\in\mathbf{D}:|T(f\mu)(z)|>(1+\eta)t \text{ and } M_\mu^\rho(f^2\mu)^{1/2}(z)\le\gamma t\}) \le \tfrac{1}{2}\mu(\{z\in\mathbf{D}:|T(f\mu)(z)|>t\}).$$

Proof. Let $\Omega=\{z\in\mathbf{D}:|T(f\mu)(z)|>t\}$. The set Ω is open. By Lemma 2.4 applied to this set, the theorem will follow if we can prove the following lemma.

Lemma 4.12. *Let $\eta>0$ and $0<\alpha<1$. There exists $\gamma=\gamma(\eta,\alpha)>0$ such that*

$$\mu(\{z\in B_j^*:|T(f\mu)(z)|>(1+\eta)t \text{ and } M_\mu^\rho(f^2\mu)^{1/2}(z)\le\gamma t\})\le\alpha\mu(B_j^{**}), \tag{27}$$

where B_j^* and B_j^{**} are the first and the second decompositions of the open set Ω, respectively, with respect to Lemma 2.4.

Indeed, if the lemma is true, then

$$\begin{aligned}\mu(\{z\in\mathbf{D}:|T(f\mu)(z)|>(1+\eta)t \text{ and } M_\mu^\rho(f^2\mu)^{1/2}(z)\le\gamma t\})\\ &=\mu(\{z\in\Omega:|T(f\mu)(z)|>(1+\eta)t \text{ and } M_\mu^\rho(f^2\mu)^{1/2}(z)\le\gamma t\})\\ &\le\sum_j\mu(\{z\in B_j^*:|T(f\mu)(z)|>(1+\eta)t \text{ and } M_\mu^\rho(f^2\mu)^{1/2}(z)\le\gamma t\})\\ &\le\alpha\sum_j\mu(B_j^{**})\\ &\le\alpha M\mu(\Omega),\end{aligned}$$

by the bounded overlap property.

We then have to choose α so that $\alpha M=\frac{1}{2}$ to obtain the result. □

Proof of Lemma 4.12. Set $B=B(z^B,K_1\varrho)=B_j^*$ and $B'=B(z^B,K_2\varrho)=B_j^{**}$. We follow with a little change, the proof of the Lemma 4.6.

Suppose without loss of generality that there exists $\xi^0\in B$ such that

$$M_\mu^\rho(f^2\mu)^{1/2}(\xi^0)\le\gamma t.$$

Let z^0 be a point in $\Omega^c\cap B(z^B,K_3\varrho)$. Let $\overline{B}$ be a ball centered at z^0 whose radius is equal to $\max(2(1-|z^0|),C\varrho)$, where C is a constant greater than or equal to K_3 to be made precise later. Then $\overline{B}$ touches the boundary of $\mathbf{D}$.

Let $f_1=f\chi_{\overline{B}}$ and $f_2=f-f_1=f\chi_{\overline{B}^c}$. As in the proof of (19) there exists a constant A_1 such that

$$|T(f\mu)(z)|\le|T(f_1\mu)(z)|+(1+A_1\gamma)t,\quad z\in B. \tag{28}$$

On the other hand, if we set $\widetilde{B}=B(\xi^0, C_1K_1\varrho)$, we observe that

$$B \subset \rho\widetilde{B} \subset B' \subset \Omega$$

for some $\rho>1$.

Now, if $C\varrho\geq 2(1-|z^0|)$, there exists a constant $A_2>0$ such that for $z\in B$,

$$|T(f_1\mu)(z)| \leq |T(f\mu\chi_{\widetilde{B}})(z)|+A_2\gamma t. \tag{29}$$

We obtain (29) as in the proof of (20).

From (28), we have that

$$\mu(\{z\in B:|T(f\mu)(z)|>(1+\eta)t \text{ and } M_\mu^\rho(f^2\mu)^{1/2}(z)\leq\gamma t\}) \\ \leq \mu(\{z\in B:|T(f_1\mu)(z)|>(\eta-A_1\gamma)t \text{ and } M_\mu^\rho(f^2\mu)^{1/2}(z)\leq\gamma t\}).$$

If $C\varrho<2(1-|z^0|)$, then as in the proof of the Lemma 4.6 we have that

$$|T(f\mu\chi_{\overline{B}})(z)| \leq C''\gamma t$$

so that for γ small enough

$$\{z\in B:|T(f_1\mu)(z)|>(\eta-A_1\gamma)t \text{ and } M_\mu^\rho(f^2\mu)^{1/2}(z)\leq\gamma t\}=\varnothing.$$

Thus (27) is satisfied in this case. If $C\varrho\geq 2(1-|z^0|)$, then from (29), we have that

$$\mu(\{z\in B:|T(f\mu)(z)|>(1+\eta)t \text{ and } M_\mu^\rho(f^2\mu)^{1/2}(z)\leq\gamma t\}) \\ \leq \mu(\{z\in B:|T(f_1\mu)(z)|>(\eta-(A_1+A_2)\gamma)t \text{ and } M_\mu^\rho(f^2\mu)^{1/2}(z)\leq\gamma t\}).$$

If we choose γ small enough ($0<\gamma\leq\eta/2(A_1+A_2)$ will do), we finally have that

$$\mu(\{z\in B:|T(f\mu)(z)|>(1+\eta)t \text{ and } M_\mu^\rho(f^2\mu)^{1/2}(z)\leq\gamma t\}) \\ \leq \mu\left(\left\{z\in B:|T(f\mu\chi_{\widetilde{B}})(z)|>\frac{\eta}{2}t \text{ and } M_\mu^\rho(f^2\mu)^{1/2}(z)\leq\gamma t\right\}\right). \tag{30}$$

We distinguish two cases.

If B does not touch the boundary then we easily obtain that

$$|T(f\mu\chi_{\widetilde{B}})(z)| \leq C^*Mf(\xi^0) \leq C\gamma t,$$

such that for γ small enough ($0<\gamma\leq\eta/4C$), (27) is satisfied.

Finally, suppose that B touches the boundary. Let E be a subset associated with the ball B and the number θ as in Lemma 4.9 . From (30) and (26) we have that

$$\begin{aligned}
\mu(\{z\in B:|T(f\mu)(z)|>(1+\eta)t \text{ and } M_\mu^\rho(f^2\mu)^{1/2}(z)\le\gamma t\})\\
&\le\mu\left(\left\{z\in B:|T(f\mu\chi_{\widetilde{B}})(z)|>\frac{\eta}{2}t \text{ and } M_\mu^\rho(f^2\mu)^{1/2}(z)\le\gamma t\right\}\right)\\
&\le\mu(B\backslash E)+\mu\left(\left\{z\in E:|T(f\mu\chi_{\widetilde{B}})(z)|>\frac{\eta}{2}t\right\}\right)\\
&\le\theta\mu(B)+\frac{C}{\eta^2t^2}\int_{\widetilde{B}}|f|^2\,d\mu\\
&\le\theta\mu(B)+\frac{C}{\eta^2t^2}\mu(\rho\widetilde{B})M_\mu^\rho(f^2\mu)(\xi^0)\\
&\le\theta\mu(B)+\frac{C}{\eta^2t^2}\mu(\rho\widetilde{B})\gamma^2t^2\\
&\le(\theta+C\eta^{-2}\gamma^2)\mu(\rho\widetilde{B})\\
&\le\alpha\mu(B'),
\end{aligned}$$

provided θ and γ are chosen small enough so that $(\theta+C\eta^{-2}\gamma^2)\le\alpha$. This completes the proof of Lemma 4.12 and consequently the proof of Theorem 4.11. □

4.5. Proof of the implication (3)⇒(2) in Theorem 4.1

Let $2<p<\infty$ and $f\in L^p(\mu)$. We have that

$$\begin{aligned}
\int|T(f\mu)|^p\,d\mu\\
&=\int_0^\infty\mu(\{z\in\mathbf{D}:|T(f\mu)(z)|>t\})\,dt^p\\
&=(1+\eta)^p\int_0^\infty\mu(\{z\in\mathbf{D}:|T(f\mu)(z)|>(1+\eta)t\})\,dt^p\\
&\le(1+\eta)^p\int_0^\infty\mu(\{z\in\mathbf{D}:|T(f\mu)(z)|>(1+\eta)t,\ M_\mu^\rho(f^2\mu)^{1/2}(z)\le\gamma t\})\,dt^p\\
&\quad+(1+\eta)^p\int_0^\infty\mu(\{z\in\mathbf{D}:M_\mu^\rho(f^2\mu)^{1/2}(z)\ge\gamma t\})\,dt^p\\
&\le\frac{(1+\eta)^p}{2}\int_0^\infty\mu(\{z\in\mathbf{D}:|T(f\mu)(z)|>t\})\,dt^p\\
&\quad+\frac{(1+\eta)^p}{\gamma^p}\int(M_\mu^\rho(f^2\mu)^{1/2}(z))^p\,d\mu
\end{aligned}$$

by Theorem 4.11. We then choose η small and use Proposition 4.3 to obtain that

$$\int |T(f\mu)|^p \, d\mu \le C \int |f|^p \, d\mu. \tag{31}$$

4.6. Proof of the implication (2)⇒(1) in Theorem 4.1

Since T is self-adjoint, it follows by duality that (31) holds also for $1<p<2$. Hence, by interpolation it also holds for $p=2$. This finishes the proof.

We have proved the result for the case $\alpha=-n$. The same argument holds with minor changes for $-n-1<\alpha<-n$. In fact one has to use the following maximal operator

$$M_k\nu(z) = \sup_{r>1-|z|} \frac{|\nu|(B(z,r))}{r^k}$$

in the place of M, where $k=n+1+\alpha$, and the following curvature

$$c^2(z_1,z_2,z_3) = \sum_\sigma K_\alpha(z_{\sigma(2)},z_{\sigma(1)})K_\alpha(z_{\sigma(3)},z_{\sigma(1)}).$$

Indeed, in Lemma 4.5, one should replace $d(z^0,w)^{1+\beta}$ by $d(z^0,w)^{k+\beta}$. The growth condition $(\mu(B(z,r))\le Cr^k)$ ensures the boundedness of M_k in $L^p(\mu)$ $(1<p<\infty)$. The rest of the proof works the same way with T replaced by T_α and K by K_α. Proposition 2.13 ensures that the estimates used are still valid in this case.

5. Comments

One reason why we could not carry out our argument in the remaining range $-n<\alpha<-1$ is that the kernel's real part is signed and we do not have immediate information on the sign of the curvature we defined. Nevertheless, we conjecture that condition (3) in Theorem 4.1 is sufficient for boundedness in the remaining range.

Since T_α is a self-adjoint Calderón–Zygmund operator, one classical result in the Calderón–Zygmund theory is that a Calderón–Zygmund operator which is bounded in $L^2(\mu)$ is weakly bounded. So a natural question comes: Is it true that condition (3) in Theorem 4.1 implies that T_α is weakly bounded?, that is

$$\mu(\{z\in\mathbf{D}:|T_\alpha f(z)|>\lambda\}) \le C\frac{\|f\|_{L^1(\mu)}}{\lambda} \quad \text{for all } f\in L^1(\mu).$$

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