Ark. Mat., **46** (2008), 197–217
DOI: 10.1007/s11512-008-0077-x

A direct approach to Bergman kernel asymptotics for positive line bundles

Robert Berman, Bo Berndtsson and Johannes Sjöstrand

Abstract. We give an elementary proof of the existence of an asymptotic expansion in powers of k of the Bergman kernel associated to L^k, where L is a positive line bundle over a compact complex manifold. We also give an algorithm for computing the coefficients in the expansion.

1. Introduction

Let L be a holomorphic line bundle with a positively curved Hermitian metric ϕ, over a complex manifold X. Then $i/2$ times the curvature form $\partial\bar{\partial}\phi$ of L defines a Kähler metric on X, that induces a scalar product on the space of global sections with values in L. The orthogonal projection P from $L^2(X,L)$ onto $H^0(X,L)$, the subspace of holomorphic sections, is the *Bergman projection*. Its kernel with respect to the scalar product is the Bergman kernel K of $H^0(X,L)$; it is a section of $\bar{L}\otimes L$ over $X\times X$. It can also be characterized as a reproducing kernel for the Hilbert space $H^0(X,L)$, i.e.

$$\alpha(x)=(\alpha,K_x) \tag{1.1}$$

for any element α of $H^0(X,L)$, where $K_x=K(\,\cdot\,,x)$ is identified with a holomorphic section of $L\otimes\bar{L}_x$, where L_x denotes the fiber of L over x. The restriction of K to the diagonal is a section of $\bar{L}\otimes L$ and we let $B(x)=|K(x,x)|$ be its pointwise norm.

Even though the Bergman kernel is of course impossible to compute in general, quite precise asymptotic formulas, when we replace L by L^k and ϕ by $k\phi$, are known, see Zelditch [15], Catlin [5] and Tian [14]. Namely

$$K_x(y)e^{-k\psi(y,x)}=\frac{k^n}{\pi^n}\left(1+\frac{b_1(x,y)}{k}+\frac{b_2(x,y)}{k^2}+\dots\right) \tag{1.2}$$

as $k\to\infty$.

Here ψ is an (almost) holomorphic extension of ϕ and the b_j's are certain Hermitian functions. In particular, on the diagonal $y=x$ we have an asymptotic series expansion for

$$K_x(x)e^{-k\phi(x)} = \frac{k^n}{\pi^n}\left(1+\frac{b_1(x,x)}{k}+\frac{b_2(x,x)}{k^2}+\dots\right).$$

The functions $b_j(x,x)$ contain interesting geometric information of X with the Kähler metric $i\partial\bar{\partial}\phi/2$, see Lu [11].

In [15] and [5] the existence of an asymptotic expansion is proved using a formula, due to Boutet de Monvel and Sjöstrand, for the boundary behaviour of the Bergman–Szegő kernel for a strictly pseudoconvex domain, [4], extending an earlier result of C. Fefferman, [6], to include also the off-diagonal behaviour. The purpose of this paper is to give a direct proof of the existence of this expansion, the main point being that it is actually simpler to construct an asymptotic formula directly. Our method also gives an effective way of computing the terms b_j in the asymptotic expansion. Even though the inspiration for the construction comes from the calculus of Fourier integral operators with complex phase, the arguments in this paper are elementary.

The method of proof uses localization near an arbitrary point of X. Local holomorphic sections of L^k in a small coordinate neighbourhood U are just holomorphic functions on U, and the local norm is a weighted L^2-norm over U with weight function $e^{-k\phi}$, where ϕ is a strictly plurisubharmonic function. Using ideas from [13] we then compute *local asymptotic Bergman kernels* on U. These are holomorphic kernel functions, and the scalar product with such a kernel function reproduces the values of holomorphic functions on U up to an error that is small as $k\to\infty$. Assuming that the bundle is globally positive it is then quite easy to see that the global Bergman kernel must be asymptotically equal to the local kernels.

Many essential ideas of our approach were already contained in the book [13] written by the third author. Here we use them in order to find a short derivation of the Bergman kernel asymptotics. For the closely related problem of finding the Bergman kernel for exponentially weighted spaces of holomorphic functions, this was done by Melin and the third author [12], but in the present paper we replace a square root procedure used in that paper by a more direct procedure, which we think is more convenient for the actual computations of the coefficients in the asymptotic expansions. There are also close relations to the subject of weighted integral formulas in complex analysis [3]. We have tried to make the presentation almost self-contained, hoping that it may serve as an elementary introduction to certain micro-local techniques with applications to complex analysis and differential geometry.

2. The local asymptotic Bergman kernel

The local situation is as follows. Fix a point in X. We may choose local holomorphic coordinates x centered at the point and a holomorphic trivialization s of L such that

$$|s|^2 = e^{-\phi(x)}, \tag{2.1}$$

where ϕ is a smooth real-valued function. L is positive if and only if all local functions ϕ arising this way are strictly plurisubharmonic. We will call $\phi_0(x)=|x|^2$ the *model fiber metric*, since it may be identified with the fiber metric of a line bundle of constant curvature on $\mathbb{C}^n$. The Kähler form ω of the metric on our base manifold X is given by $i/2$ times the curvature form of L,

$$\omega = \frac{i\partial\bar{\partial}\phi}{2}.$$

The induced volume form on X is equal to $\omega_n := \omega^n/n!$. Now any local holomorphic section u of L^k may be written as $us^{\otimes k}$, where u is a holomorphic function. The local expression of the norm of a section of L^k over U is then given by

$$\|u\|^2_{k\phi} := \int_U |u|^2 e^{-k\phi}\omega_n,$$

where u is a holomorphic function on U. The class of all such functions u with finite norm is denoted by $H_{k\phi}(U)$.

We now turn to the construction of local asymptotic Bergman kernels. The main idea is that since a posteriori Bergman kernels will be quite concentrated near the diagonal, we require a local Bergman kernel to satisfy the reproducing formula (1.1) locally, up to an error which is exponentially small in k. In the sequel we fix our coordinate neighbourhood to be the unit ball of $\mathbb{C}^n$. Let χ be a smooth function supported in the unit ball B and equal to one on the ball of radius $\frac{1}{2}$. We will say that K_k is a *reproducing kernel* $\operatorname{mod} O(e^{-\delta k})$ for $H_{k\phi}$ if for any fixed x in some neighbourhood of the origin we have that for any local holomorphic function u_k,

$$u_k(x) = (\chi u_k, K_{k,x})_{k\phi} + O(e^{k(\phi(x)/2-\delta)})\|u\|_{k\phi}, \tag{2.2}$$

uniformly in some neighbourhood of the origin. Furthermore, if $K_{k,x}(y)$ is holomorphic in y we say that $K_{k,x}$ is a *Bergman kernel* $\operatorname{mod} O(e^{-\delta k})$.

Given a positive integer N, Bergman and reproducing kernels mod $O(k^{-N})$ are similarly defined.

2.1. Local reproducing kernels mod $e^{-\delta k}$

Let ϕ be a strictly plurisubharmonic function in the unit ball and let u be a holomorphic function in the ball such that

$$\|u\|^2 := \int_B |u|^2 e^{-k\phi}\omega_n < \infty.$$

We shall first show that (cf. [13]) integrals of the form

$$c_n\left(\frac{k}{2\pi}\right)^{-n}\int_\Lambda e^{k\theta\cdot(x-y)}u(y)\,d\theta\wedge dy, \tag{2.3}$$

define reproducing kernels mod $e^{-\delta k}$ for suitably chosen *contours*

$$\Lambda = \{(y,\theta)\,;\theta=\theta(x,y)\}.$$

Here we think of x as being fixed (close to the origin) and let y range over the unit ball, so that Λ is a $2n$-dimensional submanifold of $B_y\times\mathbb{C}^n_\theta$, and $c_n = i^n(-1)^{n(n+1)/2} = i^{-n^2}$ is a constant of modulus 1 chosen so that $c_n\, d\bar{y}\wedge dy$ is a positive form. Let us say that such a contour is *good* if uniformly on Λ for x in some neighbourhood of the origin and $|y|\le 1$,

$$2\operatorname{Re}\theta\cdot(x-y) \le -\delta|x-y|^2-\phi(y)+\phi(x).$$

Note that, by Taylor's formula

$$\phi(x)=\phi(y)+2\operatorname{Re}\sum_{j=1}^n Q_j(x,y)(x_j-y_j)+\sum_{j,l=1}^n \phi_{j\bar{l}}(x_j-y_j)\overline{(x_l-y_l)}+o(x-y)^2,$$

where $\sum_{j=1}^n Q_j(x,y)(x_j-y_j)$ is the part of the second order Taylor expansion which is holomorphic in x. Hence, if ϕ is strictly plurisubharmonic,

$$\theta(x,y)=Q(x,y)$$

is a good contour, depending on x in a holomorphic way. In particular, $\theta=\bar{y}$ defines a good contour for $\phi(x)=|x|^2$.

Proposition 2.1. *For any good contour,*

$$u(x)=\left(\frac{k}{2\pi}\right)^n c_n\int_\Lambda e^{k\theta\cdot(x-y)}u(y)\chi(y)\,d\theta\wedge dy+O(e^{k(\phi(x)/2-\delta)})\|u\|_{k\phi},$$

for x in some fixed neighbourhood of 0 if u is an element of $H_{k\phi}(B)$.

Proof. For a real variable s between 0 and ∞, we let

$$\Lambda_s = \{(x, y, \theta)\,;\theta - s(\bar{x} - \bar{y}) \in \Lambda\},$$

and denote by $\eta = \eta_k$ the differential form

$$\eta = c_n\left(\frac{k}{2\pi}\right)^n e^{k\theta\cdot(x-y)} u(y)\chi(y)\, d\theta \wedge dy.$$

Our presumptive reproducing formula is the integral, I_0, of η over Λ_0 and it is easy to see that the limit of

$$I_s := \int_{\Lambda_s} \eta,$$

as $s \to \infty$, equals $u(x)$. (This is because $c_n(s/2\pi)^n e^{-s|x-y|^2}\, d\bar{y} \wedge dy$ tends to a Dirac measure at x as $s \to \infty$.) The difference between I_0 and I_s is by Stokes' formula

$$I_0 - I_s = \int_{B\times[0,s]} dh^*(\eta),$$

where h is the homotopy map

$$h(y, \lambda) = (y, \theta(x, y) - \lambda(\bar{x} - \bar{y})).$$

Now,

$$d\eta = c_n\left(\frac{k}{2\pi}\right)^n e^{k\theta\cdot(x-y)} u\, d\chi \wedge d\theta \wedge dy.$$

This equals 0 if $|y| < \frac{1}{2}$, and since θ is good we have the estimate

$$|dh^*(\eta)| \le Ck^n e^{k(-(\delta/2+\lambda)|x-y|^2 - \phi(y)/2 + \phi(x)/2)}(1+\lambda)^n |u(y)|.$$

If $|x|$ is, say smaller than $\frac{1}{4}$, $|x-y| \ge \frac{1}{4}$ when $d\eta$ is different from 0, so we get

$$\left|\int_{B\times[0,s]} dh^*(\eta)\right| \le Ck^n e^{k(\phi(x)/2-\delta)} \int_{|y|>1/2} |u(y)| e^{-k\phi(y)/2} \int_0^s (1+\lambda)^n e^{-k\lambda}\, d\lambda$$

with a smaller δ. By the Cauchy inequality the first integral in the right-hand side is dominated by

$$\|u\|_{k\phi}.$$

Since the last integral is bounded by a constant independent of k we get the desired estimate. $\square$

Thus we have a family of reproducing kernels mod $e^{-\delta k}$. When $\phi=|y|^2$ and $\theta=\bar{y}$, the kernel $e^{\bar{x}\cdot y}$ in the representation is also holomorphic in y so we even have an asymptotic Bergman kernel mod $e^{-\delta k}$. To achieve the same thing for general weights we need to introduce a bit more flexibility in the construction by allowing for a more general class of amplitudes in the integral. We will replace the function χ in the integral by $\chi(1+a)$ for a suitably chosen function a, where a has to be chosen to give an exponentially small contribution to the integral.

For this we consider differential forms

$$A=A(x,y,\theta,k)=\sum_{l=1}^{n} A_l(x,y,\theta,k)\,\widehat{d\theta_l}$$

of bidegree $(n-1,0)$. By $\widehat{d\theta_l}$ we mean the wedge product of all the differentials $d\theta_j$ except $d\theta_l$, with sign chosen so that $d\theta_l\wedge\widehat{d\theta_l}=d\theta$. We assume that A has an asymptotic expansion of order 0,

$$A\sim A_0+\frac{A_1}{k}+...$$

By this we mean that for any $N\geq 0$,

$$A-\sum_{m=0}^{N}\frac{A_m}{k^m}=O\left(\frac{1}{k^{N+1}}\right)$$

uniformly as $k\to\infty$.

We assume also that the coefficients are holomorphic (in the smooth case almost holomorphic) for x, y and θ of norm smaller than 2. Let

$$a\,d\theta=e^{-k\theta\cdot(x-y)}\,d_\theta e^{k\theta\cdot(x-y)}A,$$

so that

$$a=D_\theta\cdot A+k(x-y)\cdot A=:\nabla A, \tag{2.4}$$

where $D_\theta=\partial/\partial\theta$. We will say that a function a arising in this way is a *negligible amplitude*. In the applications we will also need to consider finite order approximations to amplitude functions. Let

$$A^{(N)}=\sum_{m=0}^{N}\frac{A_m}{k^m}$$

and similarly

$$a^{(N)}=\sum_{m=0}^{N}\frac{a_m}{k^m}.$$

Then

$$a^{(N)} = \nabla A^{(N+1)} - D_\theta \cdot \frac{A_{N+1}}{k^{N+1}},$$

so $a^{(N)}$ is a negligible amplitude modulo an error term which is $O(k^{-N-1})$.

Proposition 2.2. *For any good contour Λ and any negligible amplitude a,*

$$u(x) = c_n \left(\frac{k}{2\pi}\right)^n \int_\Lambda e^{k\theta\cdot(x-y)} u(y)\chi(y)(1+a)\, d\theta\wedge dy + O(e^{k(\phi(x)/2-\delta)})\|u\|_{k\phi},$$

for all x in a sufficiently small neighbourhood of the origin, if u is an element of $H_{k\phi}(B)$. Moreover

$$u(x) = c_n \left(\frac{k}{2\pi}\right)^n \int_\Lambda e^{k\theta\cdot(x-y)} u(y)\chi(y)(1+a^{(N)})\, d\theta\wedge dy + O\left(\frac{e^{k\phi(x)/2}}{k^{N+1-n}}\right)\|u\|_{k\phi}.$$

Proof. For the first statement we need to verify that the contribution from a is exponentially small as $k\to\infty$. But

$$\begin{aligned}\int_\Lambda e^{k\theta\cdot(x-y)} u(y)\chi(y) a\, d\theta\wedge dy &= \int_\Lambda u(y)\chi(y)\, d_\theta(e^{k\theta\cdot(x-y)} A)\wedge dy \\ &= \int_\Lambda \chi\, d(u(y) e^{k\theta\cdot(x-y)} A\wedge dy) \\ &= -\int_\Lambda d\chi \wedge u(y) e^{k\theta\cdot(x-y)} A\wedge dy.\end{aligned}$$

Again, the last integrand vanishes for $|y|<\frac{1}{2}$ and is, since Λ is good, dominated by a constant times

$$|u(y)| e^{k(-\delta|x-y|^2-\phi(y)/2+\phi(x)/2)}$$

The last integral is therefore smaller than

$$\|u\| O(e^{k(\phi(x)/2-\delta)})$$

so the first formula is proved. The second formula follows since by the remark immediately preceding the proposition, $a^{(N)}$ is a good amplitude modulo an error of order k^{-N-1}. $\square$

The condition that an amplitude function a can be written in the form (2.4) can be given in an equivalent very useful way. For this we will use the infinite order differential operator

$$Sa=\sum_{m=0}^{\infty}\frac{1}{k^m(m!)}(D_\theta\cdot D_y)^m.$$

This is basically the classical operator that appears in the theory of pseudodifferential operators when we want to replace an amplitude $a(x,y,\theta)$ by an amplitude $b(x,\theta)$ independent of y, see [8]. We let S act on $(n-1)$-forms A as above componentwise. We say that $Sa=b$ for a and b admitting asymptotic expansions if all the coefficients of the powers k^{-m} in the expansion obtained by applying S to a formally equal the corresponding coefficients in the expansion of b. No convergence of any kind is implied. That Sa equals b to order N means that the same thing holds for $m\leq N$. Note also that since formally

$$S=e^{D_\theta\cdot D_y/k},$$

we have that

$$S^{-1}=e^{-D_\theta\cdot D_y/k}=\sum_{m=0}^{\infty}\frac{1}{(-k)^m(m!)}(D_\theta\cdot D_y)^m.$$

Lemma 2.3. *Let*

$$a\sim\sum_{m=0}^{\infty}\frac{a_m(x,y,\theta)}{k^m}$$

be given. Then there exists an A satisfying (2.4) *asymptotically if and only if*

$$Sa|_{x=y}=0.$$

Moreover the last equation holds to order N if and only if $a^{(N)}$ can be written as

$$a^{(N)}=\nabla A^{(N+1)}+O\left(\frac{1}{k^{N+1}}\right). \tag{2.5}$$

Proof. Note first that S commutes with D_θ and that

$$S((x-y)\cdot A)=(x-y)\cdot SA-\frac{1}{k}D_\theta\cdot SA.$$

Moreover

$$\nabla A=D_\theta\cdot A+k(x-y)\cdot A,$$

so it follows that

$$S\nabla A = SD_\theta \cdot A + kS(x-y)\cdot A \tag{2.6}$$
$$= D_\theta SA + k(x-y)\cdot SA - D_\theta S\cdot A = k(x-y)\cdot SA.$$

Thus, if a admits a representation $a=\nabla A$, then Sa must vanish for $x=y$. Similarly, if

$$a^{(N)} = \nabla A^{(N+1)} + O\left(\frac{1}{k^{N+1}}\right).$$

it follows that $Sa^{(N)}|_{y=x}=0$ to order N.

Conversely, assume that $Sa|_{y=x}=0$. Then $Sa=(x-y)\cdot B$ for some form B. But (2.6) implies that

$$\nabla S^{-1} = kS^{-1}(x-y)\cdot$$

so

$$a = S^{-1}((x-y)\cdot B) = \frac{1}{k}\nabla S^{-1}B$$

and (2.4) holds with $A=k^{-1}S^{-1}B$. If the equation $Sa|_{y=x}=0$ only holds to order N, then

$$Sa^{(N)} = (x-y)\cdot B^{(N)}$$

to order N. Hence

$$a^{(N)} = S^{-1}((x-y)\cdot B^{(N)}) = \frac{1}{k}\nabla S^{-1}B^{(N)} = \frac{1}{k}\nabla (S^{-1}B)^{(N)}$$

to order N, so (2.5) holds with $A^{(N+1)}=k^{-1}(S^{-1}B)^{(N)}$. □

2.2. The phase

Let us now see how to choose the contour Λ to get the phase function $\psi(x,\bar{y})$ appearing in the expression

$$e^{\theta\cdot(x-y)}.$$

In this section we still assume that the plurisubharmonic function ϕ is real-analytic and let $\psi(x,y)$ be the unique holomorphic function of $2n$ variables such that

$$\psi(x,\bar{x}) = \phi(x).$$

By looking at the Taylor expansions of ψ and ϕ one can verify that

$$2\,\mathrm{Re}\,\psi(x,\bar{y}) - \phi(x) - \phi(y) \le -\delta|x-y|^2 \tag{2.7}$$

for x and y sufficiently small. Following an idea of Kuranishi, see [9] and [7], we now find a holomorphic function of $3n$ variables $\theta(x,y,z)$ that solves the division problem

$$\theta\cdot(x-y)=\psi(x,z)-\psi(y,z). \tag{2.8}$$

This can be done in many ways, but any choice of θ satisfies

$$\theta(x,x,z)=\psi_x(x,z).$$

To fix ideas, we take

$$\theta(x,y,z)=\int_0^1 \partial\psi(tx+(1-t)y,z)\,dt$$

with ∂ denoting the differential of ψ with respect to the first n variables.

Since $\theta(x,x,z)=\psi_x(x,z)$ it follows that

$$\theta_z(0,0,0)=\psi_{xz}(0,0)=\phi_{x\bar{x}}(0,0)$$

is a nonsingular matrix. Therefore

$$(x,y,z)\longmapsto(x,y,\theta)$$

defines a biholomorphic change of coordinates near the origin. After rescaling we may assume that ψ is defined and satisfies (2.7) and that the above change of coordinates is well defined when $|x|$, $|y|$ and $|z|$ are all smaller than 2. We now define Λ by

$$\Lambda=\{(y,\theta)\,;z=\bar{y}\}.$$

Thus, on Λ, θ is a holomorphic function of x, y and $\bar{y}$. The point of this choice is that by (2.8),

$$\theta\cdot(x-y)=\psi(x,\bar{y})-\psi(y,\bar{y}) \text{ on } \Lambda.$$

Therefore we get the right phase function in our kernel and by (2.7),

$$2\operatorname{Re}\theta\cdot(x-y)=2\operatorname{Re}\psi(x,\bar{y})-2\phi(y)\le\phi(x)-\phi(y)-\delta|x-y|^2,$$

which means that Λ is a good contour in the sense of the previous section. By Proposition 2.2 we therefore get the following proposition, where we use the notation β for the standard Kähler form in $\mathbb{C}^n$,

$$\beta=\frac{i}{2}\sum_{j=1}^n dy_j\wedge d\bar{y}_j.$$

Proposition 2.4. *Suppose that u is in $H_{k\phi}$. If $a(x,y,\theta,1/k)$ is a negligible amplitude, we have that*

(2.9)

$$u(x)=\left(\frac{k}{\pi}\right)^n\int_{\mathbb{C}^n}\chi_x e^{k(\psi(x,\bar{y})-\psi(y,\bar{y}))}(\det\theta_{\bar{y}})u(y)(1+a)\beta_n+O(e^{k(\phi(x)/2+\delta)})\|u\|_{k\phi},$$

with $a = a(x, y, \theta(x, y, \bar{y}), 1/k)$. *Moreover*

$$u(x) = \left(\frac{k}{\pi}\right)^n \int_{\mathbb{C}^n} \chi_x e^{k(\psi(x,\bar{y}) - \psi(y,\bar{y}))} (\det \theta_{\bar{y}}) u(y)(1 + a^{(N)})\beta_n + O(e^{k\phi(x)/2} k^{n-N-1}) \|u\|_{k\phi}.$$

2.3. The amplitude

In order to get an asymptotic Bergman kernel from (2.9) we need to choose the amplitude a so that

$$\det \theta_{\bar{y}}(1+a) = B(x, \bar{y}) \det \psi_{y\bar{y}},$$

with B analytic. Polarizing in the y-variable, i.e. replacing $\bar{y}$ by z, this means that

$$\left(1 + a\left(x, y, \theta(x, y, z), \frac{1}{k}\right)\right) = B\left(x, z, \frac{1}{k}\right) \frac{\det \psi_{yz}(y, z)}{\det \theta_z(x, y, z)},$$

where B is analytic and independent of y. Consider this as an equation between functions of the variables x, y and θ. Let

$$\Delta_0(x, y, \theta) = \frac{\det \psi_{yz}(y, z)}{\det \theta_z(x, y, z)} = \det \partial_\theta \psi_y.$$

Since $\psi_y = \theta$ when $y = x$ we have that $\Delta_0 = 1$ for $y = x$. We need a to be representable in the form (2.4) which by Lemma 2.3 means that $Sa = 0$ for $y = x$. Equivalently, $S(1+a) = 1$ for $y = x$, so we must solve

$$S\left(B\left(x, z(x, y, \theta), \frac{1}{k}\right)\Delta_0(x, y, \theta)\right) = 1 \tag{2.10}$$

for $y = x$. This equation should hold in the sense of formal power series which means that the coefficient of k^0 must equal 1, whereas the coefficient of each power k^{-m} must vanish for $m > 0$. In the computations, x is held fixed and $z = z(y, \theta)$. The first equation is

$$b_0(x, z(x, y, \theta))\Delta_0(x, x, \theta) = 1. \tag{2.11}$$

This means that $b_0(x, z(x, \theta)) = 1$ for all θ, which implies that b_0 is identically equal to 1.

The second condition is

$$(D_\theta \cdot D_y)(b_0 \Delta_0) + b_1 \Delta_0 = 0 \tag{2.12}$$

for $y = x$. Since we already know that $b_0 = 1$ this means that

$$b_1(z(x, \theta)) = -(D_\theta \cdot D_y)(\Delta_0)|_{y=x},$$

which again determines b_1 uniquely. Continuing in this way, using the recursive formula

$$\sum_{l=0}^{m} \frac{(D_\theta \cdot D_y)^l}{l!}(b_{m-l}\Delta_0)|_{x=y} = 0 \tag{2.13}$$

for $m>0$, we can determine all the coefficients b_m, and hence a. Then $Sa|_{y=x}=0$ so $Sa^{(N)}|_{y=x}=0$ to order N, and the next proposition follows from Proposition 2.4 and Lemma 2.3.

Proposition 2.5. *Suppose that ϕ is analytic. Then there are analytic functions $b_m(x,z)$ defined in a fixed neighbourhood of x so that for each N*

$$\left(\frac{k}{\pi}\right)^n \left(1+\frac{b_1(x,\bar{y})}{k}+\ldots+\frac{b_N(x,\bar{y})}{k^N}\right)e^{k\psi(x,\bar{y})}, \tag{2.14}$$

is an asymptotic Bergman kernel mod $O(k^{-N-1})$.

2.4. Computing b_1

Let us first recall how to express some Riemannian curvature notions in Hermitian geometry. The Hermitian metric two-form $\omega := \frac{1}{2} i H_{ij}\, dy^i \wedge \overline{dy^j}$ determines a connection η on the complex tangent bundle TX with connection matrix (with respect to a holomorphic frame)

$$\eta = H^{-1}\partial H =: \sum_{j=1}^{n} \eta_j dy_j. \tag{2.15}$$

The curvature is the matrix-valued two-form $\bar{\partial}\eta$ and the scalar curvature s is $\Lambda \operatorname{Tr} \bar{\partial}\eta$, where Λ is contraction with the metric form ω. Hence, in coordinates centered at x, where $H(0)=I$, the scalar curvature s at 0 is given by

$$s(0) = -\operatorname{Tr}\left(\sum_{j=1}^{n} \frac{\partial}{\partial \bar{y}_j}\eta_j\right), \tag{2.16}$$

considering η as matrix. We now turn to the computation of the coefficient b_1 in the expansion (2.14). By the definition of θ we have that

$$\theta_j(x,y,z) = \psi_{y_j}(y,z) + \frac{1}{2}\sum_{k=1}^{n}\left(\frac{\partial}{\partial y_k}\psi_{y_j}\right)(y,z)(x^k - y^k) + \ldots. \tag{2.17}$$

Differentiating with respect to z gives

$$\theta_z = H + \tfrac{1}{2}\partial_y H(x-y) + \ldots,$$

where $H = H(y, z)$. Multiplying both sides by H^{-1} and inverting the relation we get

$$\theta_z^{-1} H = I - \tfrac{1}{2}(H^{-1}\partial H)(x-y) + \dots. \tag{2.18}$$

Taking the determinant of both sides in formula (2.18) gives

$$\Delta_0 = 1 - \tfrac{1}{2} \operatorname{Tr} \eta(x-y) + \dots. \tag{2.19}$$

Hence, equation (2.12) now gives, since $-(\partial/\partial y)(x-y) = 1$, that

$$b_1(0,0) = \frac{\partial}{\partial \theta} \cdot \left(-\frac{1}{2} \operatorname{Tr} \eta\right)\bigg|_{x=y} = -\frac{1}{2} \frac{\partial}{\partial \bar{y}} \cdot \operatorname{Tr} \eta$$

showing that $b_1(x, \bar{x}) = s/2$, according to (2.16).

2.5. Twisting with a vector bundle E

We here indicate how to extend the previous calculation to the case of sections with values in $L^k \otimes E$, where E is a holomorphic vector bundle with a Hermitian metric G (see also [10]). First observe that $u(x)$ is now, locally, a holomorphic vector and the Bergman kernel may be identified with a matrix $K(x, \bar{y})$ such that

$$u(x) = \int_{\mathbf{C}^n} K(x, \bar{y}) G(y, \bar{y}) u(y) \psi_{y\bar{y}} e^{-k\psi(y,\bar{y})} \, d\bar{y} \wedge dy.$$

To determine K one now uses the ansatz

$$K(x, \bar{y}) = c_n \left(\frac{k}{2\pi}\right)^n e^{k\psi(x,\bar{y})} B(x, \bar{y}, k^{-1}) G(x, \bar{y})^{-1}.$$

Then the condition on the amplitude function becomes

$$\begin{aligned} \left(1 + a\left(x, y, \theta(x,y,z), \frac{1}{k}\right)\right) \det\left(\frac{\partial \theta}{\partial z}(x,y,z)\right) \\ = B\left(x, z, \frac{1}{k}\right) G(x,z)^{-1} G(y,z) \det(\psi_{yz}), \end{aligned} \tag{2.20}$$

where a now is a matrix-valued form, i.e. Δ_0 in Section 2.3 is replaced by the matrix $\Delta_G := \Delta_0 G(x,z)^{-1} G(y,z)$. Note that

$$G(x,z)^{-1} G(y,z) = I - G^{-1}(y,z) \frac{\partial}{\partial y} G(y,z)(x-y) + \dots =: I - \eta_E(y,z)(x-y) + \dots,$$

where $\eta_E := G^{-1}(\partial/\partial y) G$ is the connection matrix of E. Hence, the equation (2.19) is replaced by

$$\Delta_G = 1 - \left(\frac{\operatorname{Tr} \eta}{2} \otimes I + \eta_E\right)(x-y) + \dots.$$

The same calculation as before then shows that the matrix $b_1(0,0)$ is given by

$$b_1(0,0) = -\frac{1}{2}\frac{\partial}{\partial\bar{y}}\operatorname{Tr}\eta\otimes I - \frac{\partial}{\partial\bar{y}}\cdot\eta_E = \frac{s}{2}\otimes I + \Lambda\Theta_E,$$

where $\Theta_E := \bar{\partial}\eta_E$ is the curvature matrix of E and Λ denotes contraction with the metric two-form ω.

Remark. Let K_k be the Bergman kernel of $H^0(X, L^k)$, defined with respect a general volume form μ_n. Then the function $G := \mu_n/\omega_n$ defines a Hermitian metric on the trivial line bundle E and the asymptotics of K_k can then be obtained as above.

2.6. Smooth metrics

Denote by $\psi(y,z)$ any almost holomorphic extension of ϕ from $\bar{\Delta} = \{(z,y); z = \bar{y}\}$, i.e. an extension such that the anti-holomorphic derivatives vanish to infinite order on $\bar{\Delta}$. We may also assume that $\overline{\psi(y,\bar{z})} = \psi(z,\bar{y})$. That ψ is almost holomorphic means that for any multi-index α,

$$D^\alpha(\bar{\partial}\psi) = 0, \tag{2.21}$$

(where D^α is the local real derivative of order α) when evaluated at a point in $\bar{\Delta}$, i.e. when $z = \bar{y}$. Let now

$$\theta = \int_0^1 (\partial_y\psi)(tx + (1-t)y, z)\, dt, \quad \theta^* = \int_0^1 (\bar{\partial}_y\psi)(tx + (1-t)y, z)\, dt, \tag{2.22}$$

(where $\partial_y\psi$ denotes the vector of partial holomorphic derivatives with respect to the first argument of ψ) so that

$$(x-y)\theta + \overline{(x-y)}\theta^* = \psi(x,z) - \psi(y,z). \tag{2.23}$$

Then the smooth map corresponding to $(x,y,z) \mapsto (x,y,\theta)$ is locally smoothly invertible for the same reason as in the analytic case, since $\bar{\partial}_z\theta = 0$ when $x = y = \bar{z}$. Define the algebra $\mathcal{A}$ of all functions almost holomorphic when $x = y = \bar{z}$ as the set of smooth functions f of x, y and z, such that

$$D^\alpha\bar{\partial}f = 0$$

for all multi-indices α, when $x = y = \bar{z}$. For a vector-valued function we will say that it is in $\mathcal{A}$, if its components are in $\mathcal{A}$. We also define the vanishing ideal $\mathcal{I}^\infty$ as the set of smooth functions f such that

$$D^\alpha f = 0$$

for all multi-indices α, when $x = y = \bar{z}$. Hence, if f belongs to $\mathcal{A}$ then (the coefficients of) $\bar{\partial}f$ will belong to $\mathcal{I}^\infty$.

Note that $\psi(tx+(1-t)y, z)$ is in $\mathcal{A}$ for each fixed t. Hence θ is in $\mathcal{A}$ and θ^* is in $\mathcal{I}^\infty$, so that (2.23) gives that (with obvious abuse of notation)

$$(x-y)\theta = \psi(x,\bar{y}) - \psi(y,\bar{y}) + O(|x-y|^\infty) \tag{2.24}$$

when $\bar{z}=y$. The next simple lemma is used to show that the contribution of elements in $\mathcal{I}^\infty$ to the phase function and the amplitude is negligible.

Lemma 2.6. *Let f_i be elements of the vanishing ideal $\mathcal{I}^\infty$ and let $b(x,y)$ be a local smooth function. Then*

$$\begin{aligned}\int_{\mathbf{C}^n} \chi_x(y) e^{k(\psi(x,\bar{y})-\psi(y,\bar{y})+f_1(x,y,\bar{y}))}(b(x,y)+f_2(x,y,\bar{y}))u(y)\beta_n(y) \\ = \int \chi_x(y) e^{k(\psi(x,\bar{y})-\psi(y,\bar{y}))} b(x,y)u(y)\beta_n(y) + O\left(\frac{1}{k^\infty}\right)\|u\|_{k\phi}.\end{aligned}$$

Proof. First observe that if f_j is an element of $\mathcal{I}^\infty$, then $f_j(x,y,\bar{y}) = O(|x-y|^\infty)$. Moreover, we have that

$$\left||x-y|^{2N} e^{k(\psi(x,\bar{y})-\phi(x)/2-\phi(y)/2)}\right| \le \frac{C}{k^{2N}}(k|x-y|)^{2N} e^{-k\delta|x-y|^2} = O\left(\frac{1}{k^{2N}}\right), \tag{2.25}$$

where we have used (2.7). Combining this bound with the Cauchy–Schwarz inequality proves the lemma when $f_1=0$. Now write

$$e^{k(\psi(x,\bar{y})-\psi(y,\bar{y})+f_1(x,y,\bar{y}))} = e^{k(\psi(x,\bar{y})-\psi(y,\bar{y}))} + \int_0^1 \partial_t\big(e^{(\psi(x,\bar{y})-\psi(y,\bar{y})+tf_1(x,y,\bar{y}))}\big)\,dt.$$

By (2.25) the second term gives a contribution which is of the order $O(k^{-\infty})$. Hence the general case follows. □

Proposition 2.7. *Suppose that L is smooth. Then there exists an asymptotic reproducing kernel $K_k^{(N)}$* mod $O(k^{n-N-1})$ *for $H_{k\phi}$, such that*

$$K_k^{(N)}(x,\bar{y}) = e^{k\psi(x,\bar{y})}\left(b_0 + \frac{b_1}{k} + \ldots + \frac{b_N}{k^N}\right), \tag{2.26}$$

where b_j is a polynomial in the derivatives $\partial_x^\alpha \bar{\partial}_y^\beta \psi(x,\bar{y})$ of the almost holomorphic extension ψ of ϕ. In particular,

$$e^{-k(\phi(x)/2+\phi(y)/2)}(D_{x,y}^\alpha(\bar{\partial}_x, \partial_y))K(x,y) = O\left(\frac{1}{k^\infty}\right) \tag{2.27}$$

uniformly in x and y for any given α.

Proof. We go through the steps in the proof of the analytic case and indicate the necessary modifications.

First we determine the coefficients $b_m(x,z)$ in the same way as in the analytic case, i.e. by fixing x and solving

$$S(B(z)\Delta_0)|_{y=x} = 1.$$

Here S has the same meaning as before and in particular contains only derivatives with respect to θ and no derivatives with respect to $\bar{\theta}$. The difference is that Δ_0 is no longer analytic so B will not be holomorphic, but it will still belong to $\mathcal{A}$ since Δ_0 does.

We next need to consider Lemma 2.3 with $a \in \mathcal{A}$. Then we get that

$$(Sa)_{y=x} = O\left(\frac{1}{k^{N+1}}\right), \quad a \in \mathcal{A}$$

if and only if there exists an $A \in \mathcal{A}$ such that

$$a = \nabla A + O\left(\frac{1}{k^{N+1}}\right) \quad \mod \mathcal{I}^\infty.$$

Indeed, this follows from the argument in the analytic case and the fact that if $c(=c(x,y,z)) \in \mathcal{A}$, then

$$c(x,x,z) = 0$$

if and only if there exists a $d \in \mathcal{A}$ such that

$$c = (x-y)d \quad \mod \mathcal{I}^\infty,$$

as can be seen by defining d by

$$d = \int_0^1 (\partial_y c)(x, x+(1-t)y, z)\, dt.$$

Here ∇ also has the same meaning as before and contains only a derivative with respect to θ and no derivative with respect to $\bar{\theta}$. Then Proposition 2.2 holds as before except that there will be one extra contribution in the application of Stokes' theorem coming from $\bar{\partial}_\theta A$ (when $z=\bar{y}$). Since $\bar{\partial}_\theta A$ vanishes to infinite order when $x=y=\bar{z}$, it gives a contribution to the integral which is $O(k^{-N})$ for any N by Lemma 2.6.

We therefore get from Proposition 2.2 a reproducing kernel of the form claimed in (2.26) except that the phase function equals

$$k\theta \cdot (x-y) = k(\psi(x,\bar{y}) - \phi(y) + f)$$

with f in $\mathcal{A}$. Again by Lemma 2.6 we may remove f at the expense of adding a contribution to the integral which is negligible, i.e. which is $O(k^{-N})$ for any N. $\square$

3. The global Bergman kernel

In this section we will show that, if the curvature of L is positive everywhere on X, then the global Bergman kernel K_k of $H^0(X, L^k)$ is asymptotically equal to the local Bergman kernel $K_k^{(N)}$ of $H_{k\phi}$ (constructed in Section 2).

Recall (Section 1) that the Bergman kernel K associated to L is a section of $\bar{L}\boxtimes L$ over $X\times X$. By restriction, K_x is identified with a holomorphic section of $\bar{L}_x\otimes L$, where L_x is the fiber of L over x. Given any two vector spaces E and F, the scalar product on L extends uniquely to a pairing

$$(\cdot,\cdot)\colon L\otimes E\times L\otimes F\longrightarrow E\otimes F, \tag{3.1}$$

linear over E and anti-linear over F. In terms of this pairing, K_y has the global reproducing property

$$\alpha(y)=(\alpha, K_y) \tag{3.2}$$

for any element α of $H^0(X,L)$. By taking $\alpha{=}K_x$ (so that $E{=}L_x$ and $F{=}\bar{L}_y$ in (3.1)) one gets that

$$K(y,x):=K_x(y)=(K_x,K_y). \tag{3.3}$$

This also implies that $\overline{K(x,y)}{=}K(y,x)$ and that

$$K(x,x)=(K_x,K_x)=\|K_x\|^2. \tag{3.4}$$

$K(x,x)$ is a section of $\bar{L}\otimes L$. Its norm as a section to this bundle is the Bergman function, which, in a local frame with respect to which the metric on L is given by $e^{-\phi}$, equals

$$B(x)=K(x,x)e^{-\phi(x)}.$$

Notice also that by the Cauchy inequality we have an extremal characterization of the Bergman function:

$$B(x)=\sup|s(x)|^2,$$

where the supremum is taken over all holomorphic sections to L of norm not greater than 1.

We now denote by K_k the Bergman kernel associated to L^k, and write B_k for the associated Bergman function. It follows from the extremal characterization of the Bergman function and the submeanvalue inequality for a holomorphic section s over a small ball with radius roughly $1/k^{1/2}$ that

$$B_k\le Ck^n,$$

uniformly on X (see e.g. [1]).

Let now $K_x^{(N)}(y)$ be the local Bergman kernel as in Proposition 2.5 and Lemma 2.6, where the coefficients b_m are given by (2.13),

$$\overline{K_x^{(N)}(y)} = \left(\frac{k}{\pi}\right)^n \left(1+\frac{b_1(x,\bar{y})}{k}+\ldots+\frac{b_N(x,\bar{y})}{k^N}\right) e^{k\psi(x,\bar{y})}. \tag{3.5}$$

By construction, the coefficients $b_m(x,z)$ are holomorphic if the metric on L – locally represented by ϕ – is real-analytic. In case ϕ is only smooth the b_m's are almost holomorphic, meaning that

$$\bar{\partial}_{xz} b_m$$

vanishes to infinite order when $z=\bar{x}$.

Replacing K_y in the relation (3.3) with the local Bergman kernel $K_k^{(N)}$ will now show that $K_k=K_k^{(N)}$ up to a small error term.

Theorem 3.1. *Assume that the smooth line bundle L is globally positive. Let $K_k^{(N)}$ be defined by* (3.5)*, where the coefficients b_m are determined by the recursion* (2.13)*.*

If the distance $d(x,y)$ is sufficiently small, then

$$K_k(x,y) = K_k^{(N)}(x,y) + O(k^{n-N-1}) e^{k(\phi(x)/2+\phi(y)/2)}. \tag{3.6}$$

Moreover,

$$D^\alpha (K_k(x,y) - K_k^{(N)}(x,y)) = O(k^{m+n-N-1}) e^{k(\phi(x)/2+\phi(y)/2)}$$

if D^α is any differential operator with respect to x and y of order at most m.

Proof. Let us first show that

$$K_k(y,x) = (\chi K_{k,x}, K_{k,y}^{(N)}) + O(k^{n-N-1}) e^{k(\phi(x)/2+\phi(y)/2)}, \tag{3.7}$$

where χ is a cut-off function equal to 1 in a neighbourhood of x which is large enough to contain y. Fixing x and applying Proposition 2.5 to $u_k=K_{k,x}$ gives (3.7) with the error term

$$e^{\phi(y)/2} O\left(\frac{1}{k^{N+1}}\right) \|K_x\|.$$

Now, by (3.4) and the estimate for B_k,

$$\|K_{k,x}\|^2 = B_k(x) e^{k\phi(x)} \le C k^n e^{k\phi(x)}.$$

This proves (3.7) with uniform convergence.

Next we estimate the difference

$$u_{k,y}(x) := K_{k,y}^{(N)}(x) - (\chi K_y^{(N)}, K_{k,x}).$$

Since the scalar product in this expression is the Bergman projection,

$$P_k(\chi K_{k,y}^{(N)})(x),$$

$u_{k,y}$ is the L^2-minimal solution to the $\bar\partial$-equation

$$\bar\partial u_{k,y} = \bar\partial(\chi K_{k,y}^{(N)}).$$

The right-hand side equals

$$(\bar\partial\chi) K_{k,y}^{(N)} + \chi \bar\partial K_{k,y}^{(N)}.$$

Since χ equals 1 near y, it follows from (2.7) and the explicit form of $K_{k,y}^{(N)}$ that the first term is dominated by

$$e^{-\delta k} e^{k(\phi(\,\cdot\,)/2+\phi(y)/2)}.$$

The second term vanishes identically in the analytic case. In the smooth case $\bar\partial K_k^{(N)}$ can, by Proposition 2.7, be estimated by

$$O\biggl(\frac{1}{k^\infty}\biggr) e^{k(\phi(\,\cdot\,)/2+\phi(y)/2)}. \tag{3.8}$$

Altogether $\bar\partial u_{k,y}$ is therefore bounded by (3.7), so by the Hörmander L^2-estimate we get that

$$\|u_{k,y}\|^2 \le O\biggl(\frac{1}{k^\infty}\biggr) e^{k\phi(y)/2}.$$

But, since the estimate on $\bar\partial u_{k,y}$ is even uniform, we get by a standard argument involving the Cauchy integral formula in a ball around x of radius roughly $k^{-1/2}$ that $u_{k,y}$ satisfies a pointwise estimate

$$|u_{k,y}(x)|^2 \le O\biggl(\frac{1}{k^\infty}\biggr) e^{k(\phi(y)/2+\phi(x)/2)}.$$

Combining this estimate for $u_{k,y}(x)$ with (3.6) we finally get that

$$|K_{k,y}^{(N)}(x) - \overline{K_k(y,x)}| e^{-k\phi(x)/2 - k\phi(y)/2} \le O\biggl(\frac{1}{k^{N+1}}\biggr). \tag{3.9}$$

Since K_k is Hermitian (i.e. $K_k(x,y) = \overline{K_k(y,x)}$) this proves the proposition except for the statement on convergence of derivatives.

In the analytic case the convergence of derivatives is, by the Cauchy estimates, an automatic consequence of the uniform convergence, since the kernels are holomorphic in x and $\bar{y}$. In the smooth case, we have that

$$\bar{\partial} K_k^{(N)}(x,\bar{z}) = O\left(\frac{1}{k^\infty}\right) e^{k(\phi(\,\cdot\,)/2+\phi(y)/2)}.$$

This implies that the Cauchy estimates still hold for the difference between K_k and $K_k^{(N)}$, up to an error which is $O(k^{-\infty})$, and so we get the convergence of derivatives even in the smooth case. □

Remark 3.2. The proof above actually shows that the asymptotic expansion for the global Bergman kernel $K_k(x,y)$ holds close to any point x which is in $X(0)$ (the open subset of X where the curvature form of ϕ is positive) and is such that x satisfies the following global condition: for any given $\bar{\partial}$-closed $(0,1)$-form g_k with values in L^k supported in some fixed neighbourhood of x we may find sections u_k with values in L^k such that

$$\bar{\partial} u_k = g_k, \tag{3.10}$$

on X and

$$\|u_k\|_{k\phi} \le C\|g_k\|_{k\phi}. \tag{3.11}$$

After this paper was written this observation was used in [2] to obtain an asymptotic expansion of $K_k(x,y)$ on a certain subset of $X(0)$ for any Hermitian line bundle (L,ϕ) over a projective manifold X.

References

1. Berman, R., Bergman kernels and local holomorphic Morse inequalities, *Math. Z.* **248** (2004), 325–344.
2. Berman, R., Bergman kernels and equilibrium measures for line bundles over projective manifolds, *Preprint*, 2007. `arXiv:0710.4375`.
3. Berndtsson, B. and Andersson, M., Henkin–Ramirez formulas with weight factors, *Ann. Inst. Fourier* (*Grenoble*) **32**:3 (1982), v–vi, 91–110.
4. Boutet de Monvel, L. and Sjöstrand, J., Sur la singularité des noyaux de Bergman et de Szegő, in *Journées*: *Équations aux dérivées partielles de Rennes* (1975), Astérisque **34**–**35**, pp. 123–164, Soc. Math. France, Paris, 1976.
5. Catlin, D., The Bergman kernel and a theorem of Tian, in *Analysis and Geometry in Several Complex Variables* (*Katata, 1997*), Trends Math., pp. 1–23, Birkhäuser, Boston, MA, 1999.
6. Fefferman, C., The Bergman kernel and biholomorphic mappings of pseudoconvex domains, *Invent. Math.* **26** (1974), 1–65.

7. Grigis, A. and Sjöstrand, J., *Microlocal Analysis for Differential Operators*, London Mathematical Society Lecture Note Series **196**, Cambridge University Press, Cambridge, 1994.
8. Hörmander, L., Fourier integral operators. I, *Acta Math.* **127** (1971), 79–183.
9. Hörmander, L., *The Analysis of Linear Partial Differential Operators.* III, Grundlehren der Mathematischen Wissenschaften **274**, Springer, Berlin–Heidelberg, 1985.
10. Keller, J., Asymptotique du noyau de Bergman généralisé sur une varieté de Kähler, ouverte, *Preprint*, Toulouse, 2004.
11. Lu, Z., On the lower order terms of the asymptotic expansion of Tian–Yau–Zelditch, *Amer. J. Math.* **122** (2000), 235–273.
12. Melin, A. and Sjöstrand, J., Determinants of pseudodifferential operators and complex deformations of phase space, *Methods Appl. Anal.* **9** (2002), 177–237.
13. Sjöstrand, J., *Singularités Analytiques Microlocales*, Astérisque **95**, Soc. Math. France, Paris, 1982.
14. Tian, G., On a set of polarized Kähler metrics on algebraic manifolds, *J. Differential Geom.* **32** (1990), 99–130.
15. Zelditch, S., Szegő kernels and a theorem of Tian, *Int. Math. Res. Notices* 1998 (1998), 317–331.

Robert Berman
Department of Mathematics
Chalmers University of Technology
SE-412 96 Göteborg
Sweden
and
Department of Mathematics
University of Gothenburg
SE-412 96 Göteborg
Sweden
robertb@math.chalmers.se

Bo Berndtsson
Department of Mathematics
Chalmers University of Technology
SE-412 96 Göteborg
Sweden
and
Department of Mathematics
University of Gothenburg
SE-412 96 Göteborg
Sweden
bob@math.chalmers.se

Johannes Sjöstrand
Ecole Polytechnique
Centre de Mathématiques Laurent Schwartz
FR-91120 Palaiseau
France
johannes@math.polytechnique.fr

Received September 29, 2005
in revised form November 12, 2007
published online May 6, 2008