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Higher order Riesz transforms associated with Bessel operators

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Abstract. In this paper we investigate Riesz transforms $R_\mu^{(k)}$ of order $k\geq 1$ related to the Bessel operator $\Delta_\mu f(x)=-f''(x)-((2\mu+1)/x)f'(x)$ and extend the results of Muckenhoupt and Stein for the conjugate Hankel transform (a Riesz transform of order one). We obtain that for every $k\geq 1$, $R_\mu^{(k)}$ is a principal value operator of strong type (p,p), $p\in(1,\infty)$, and weak type $(1,1)$ with respect to the measure $d\lambda(x)=x^{2\mu+1}\,dx$ in $(0,\infty)$. We also characterize the class of weights ω on $(0,\infty)$ for which $R_\mu^{(k)}$ maps $L^p(\omega)$ into itself and $L^1(\omega)$ into $L^{1,\infty}(\omega)$ boundedly. This class of weights is wider than the Muckenhoupt class $\mathcal{A}_p^\mu$ of weights for the doubling measure $d\lambda$. These weighted results extend the ones obtained by Andersen and Kerman.

1. Introduction

A theory parallel to the classical Fourier analysis was developed by Muckenhoupt and Stein, in the descriptive and deep paper [13], in the context of orthogonal expansions (ultraspherical expansions) and their continuous analogues (associated with Hankel transforms), which are the objects treated in this paper. We consider the (positive) Laplacian of Bessel-type

$$\Delta_\mu=-\frac{\partial^2}{\partial x^2}-\frac{2\mu+1}{x}\frac{\partial}{\partial x}=D^*D,\quad \mu>-\frac{1}{2}, \tag{1}$$

where $D=\partial/\partial x$ and $D^*=-x^{-2\mu-1}Dx^{2\mu+1}$ denotes the adjoint operator of D in $L^2(x^{2\mu+1}\,dx)$. Our aim, inspired by classical investigations about higher order Riesz transforms (see [14]), is to define and study the appropriate higher order Riesz transforms for this context. Following the ideas in [15] (see also [7]), we define

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formally the Riesz transform of order k for any $k\in\mathbb{N}$ as

$$R_\mu^{(k)} = D^k \Delta_\mu^{-k/2}. \tag{2}$$

In order to give sense to the Riesz transforms (2), the first step is to define properly the "fractional integrals" $\Delta_\mu^{-k/2}$ for a general k (see Section 2). We see that, for a $C_c^\infty(0,\infty)$-function f, $\Delta_\mu^{-k/2} f$ is k times differentiable for x outside the support of f and $k-1$ times differentiable inside the support of f (see Proposition 14). Thus, for $f\in\mathcal{C}_c^\infty(0,\infty)$ and x outside the support of f, (2) makes perfect sense, and it is given by the integral against a kernel

$$R_\mu^{(k)} f(x) = \int_0^\infty R_\mu^{(k)}(x,y) f(y)\, y^{2\mu+1}\, dy, \quad x\notin \operatorname{supp} f.$$

A precise definition of the kernel $R_\mu^{(k)}(x,y)$, $x,y\in(0,\infty)$, in terms of the Poisson kernel associated with the operator Δ_μ, appears in (23). Moreover, if $f\in C_c^\infty(0,\infty)$,

$$R_\mu^{(k)} f(x) = \omega_k f(x) + \lim_{\varepsilon\to 0} \int_{|x-y|>\varepsilon} f(y) R_\mu^{(k)}(x,y) y^{2\mu+1}\, dy, \quad x\in(0,\infty),$$

where $\omega_k\in\mathbb{R}$ (see Lemma 31). The next step is to extend this definition to a general function in $L^p(x^{2\mu+1}dx)$, $1\le p<\infty$. In fact this is one of the main results of this paper.

Theorem 3. *For every $k\in\mathbb{N}$ and $f\in L^p(x^{2\mu+1}\,dx)$, $1\le p<\infty$, the limit*

$$\lim_{\varepsilon\to 0} \int_{|x-y|>\varepsilon} f(y) R_\mu^{(k)}(x,y) y^{2\mu+1}\, dy \quad \textit{exists for a.e. } x\in(0,\infty).$$

Moreover the Riesz transform $R_\mu^{(k)}$ can be extended to $L^p(x^{2\mu+1}\,dx)$, by defining

$$R_\mu^{(k)} f(x) = \omega_k f(x) + \lim_{\varepsilon\to 0} \int_{|x-y|>\varepsilon} f(y) R_\mu^{(k)}(x,y) y^{2\mu+1}\, dy \quad \textit{for a.e. } x\in(0,\infty), \tag{4}$$

as a bounded operator from $L^p(x^{2\mu+1}\,dx)$ into itself, for every $1<p<\infty$, and as a bounded operator from $L^1(x^{2\mu+1}\,dx)$ into $L^{1,\infty}(x^{2\mu+1}\,dx)$. Here $\omega_k=0$, when k is odd, and $\omega_k=(-1)^{k/2}\pi/(2\mu+1)$, when k is even.

The key ingredient in the proof of this theorem is a careful study of the kernel $R_\mu^{(k)}(x,y)$. Outside of the diagonal this kernel is bounded above by a kernel defining a bounded operator in $L^p(x^{2\mu+1}\,dx)$ while near the diagonal it is essentially a modification of the Hilbert transform, see Proposition 24.

Muckenhoupt and Stein, see [13], define and study a "Riesz transform" for the operator defined in (1), following the classical model of the conjugate function in the torus. More concretely, they define the "harmonic extension", or Poisson integral, $P_\mu f(t,x)$ of $f(x)$, and the appropriate conjugate extension $Q_\mu f(t,x)$ of $P_\mu f(t,x)$. The conjugate function $R_\mu f(x)$ is then defined as the boundary-value function of $Q_\mu f(t,x)$. They got L^p-boundedness of the conjugate function for p in the range $1<p<\infty$ and some substitutive inequality in the case $p=1$. The conjugate function defined by them coincides with our Riesz transform of order one. Therefore from our results it follows that Muckenhoupt and Stein's conjugate function is a principal value and that it is of weak type $(1,1)$.

Weighted inequalities for $R_\mu^{(k)}$, $k\geq 1$, are also studied here and we obtain that $R_\mu^{(k)}$, $k\geq 1$, are bounded operators from $L^p(\omega(x)x^{2\mu+1}\,dx)$, $1<p<\infty$, into itself and from $L^1(\omega(x)x^{2\mu+1}\,dx)$ into $L^{1,\infty}(\omega(x)x^{2\mu+1}\,dx)$, when ω is a weight in the usual Muckhenhoupt class $\mathcal{A}_p^\mu$ of weights in $(0,\infty)$ with respect to the doubling measure $x^{2\mu+1}\,dx$. The weights in $\mathcal{A}_p^\mu$ are not optimal for these operators. There exists a wider class of weights such that the former weighted L^p-boundedness properties still hold for $R_\mu^{(k)}$ (see Theorem 34). This wider class coincides with the one given by Andersen and Kerman in [2], who characterized the weights ω on $(0,\infty)$ such that $R_\mu^{(1)}$ maps $L^p(\omega(x)\,dx)$ into itself, $1<p<\infty$, and $L^1(\omega(x)\,dx)$ into $L^{1,\infty}(\omega(x)\,dx)$ boundedly.

These weighted inequalities allow us to get boundedness of operators associated with other Laplacians as follows. In [4] a Riesz transform associated with the Bessel-type operator

$$S_\mu=-\frac{\partial^2}{\partial x^2}+\frac{\mu^2-\frac14}{x^2}=-x^{-\mu-1/2}Dx^{2\mu+1}Dx^{-\mu-1/2}$$

is described. If we let $\mathcal{R}_\mu=x^{\mu+1/2}Dx^{-\mu-1/2}S_\mu^{-1/2}$ be the Riesz transform introduced in [4], it can be shown that,

$$\mathcal{R}_\mu(f)(x)=x^{\mu+1/2}R_\mu^{(1)}(y^{-\mu-1/2}f)(x).$$

Also we can define $\mathcal{R}_\mu^{(k)}$, $k\in\mathbb{N}$, (in the context of [4]) related to the operator S_μ following our procedure. Then,

$$\mathcal{R}_\mu^{(k)}(f)(x)=x^{\mu+1/2}R_\mu^{(k)}(y^{-\mu-1/2}f)(x).$$

Hence, Theorem 34 allows us to get the weights ω for which $\mathcal{R}_\mu^{(k)}$ is bounded from $L^p(\omega(x)\,dx)$ into itself for $1<p<\infty$, and from $L^1(\omega(x)\,dx)$ into $L^{1,\infty}(\omega(x)\,dx)$. The class of weights obtained in this way is wider than the one got by using Calderón–Zygmund theory in [4] for the first-order Riesz transforms. The result for higher-order Riesz transforms associated with S_μ are new, even in the unweighted case.

The organization of the paper is as follows. In Section 2 we give an appropriate definition of the fractional integrals $\Delta_\mu^{-\alpha/2}$, $\alpha>0$. We establish the main properties of these fractional integrals that will be useful in the sequel. Theorem 3 is proved in Section 3. In Section 4 we analyze the boundedness of the Riesz transform $\mathcal{R}_\mu^{(k)}$ on weighted L^p-spaces.

Throughout this paper, the letter C denotes a positive constant, not necessarily the same in each occurrence. Here, as usual, by $\mathcal{C}_c^\infty(0,\infty)$ we represent the space of smooth functions on $(0,\infty)$ having compact support on $(0,\infty)$.

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2. Fractional integrals

The usual way to define $\Delta_\mu^{-\alpha/2}$, $\alpha>0$, is

$$(5)\qquad \begin{aligned}\Delta_\mu^{-\alpha/2}f(x) &= \frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}e^{-t\sqrt{\Delta_\mu}}f(x)\,dt\\ &= \frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}\int_0^\infty P_\mu(t,x,y)f(y)y^{2\mu+1}\,dy\,dt,\end{aligned}$$

where $e^{-t\sqrt{\Delta_\mu}}f(x)=P_\mu(f)(t,x)=\int_0^\infty P_\mu(t,x,y)f(y)y^{2\mu+1}\,dy$ stays for the corresponding Poisson integral. But in the present case this formula has sense for every x only when $0<\alpha<2\mu+2$, see Proposition 10. If $\alpha\geq 2\mu+2$ we shall use a modification of the Poisson kernel which extends formula (5) and preserves the definition (2) of the Riesz transforms, see Proposition 14.

In [13] and [17], the Poisson kernel associated with Δ_μ, is found to be

$$(6)\qquad \begin{aligned}P_\mu(t,x,y) &= \int_0^\infty e^{-zt}\varphi_x(z)\varphi_y(z)z^{2\mu+1}\,dz\\ &= \frac{2\mu+1}{\pi}t\int_0^\pi \frac{\sin^{2\mu}\theta\,d\theta}{((x-y)^2+t^2+2xy(1-\cos\theta))^{\mu+3/2}},\quad t,x,y\in(0,\infty),\end{aligned}$$

where $\varphi_x(z)=(xz)^{-\mu}J_\mu(xz)$, $x,z\in(0,\infty)$, and J_μ denotes the Bessel function of the first kind of order μ. By the results in [13], P_μ defines a semigroup of contractions $P_\mu(f)(t,x)=e^{-t\sqrt{\Delta_\mu}}f(x)$, for $t>0$, in $L^p(x^{2\mu+1}\,dx)$, $1\leq p\leq\infty$.

Lemma 7. *Let* $0<\alpha<2\mu+2$. *If* $f\in\mathcal{C}_c^\infty(0,\infty)$ *the integral in* (5) *defining* $\Delta_\mu^{-\alpha/2}f(x)$ *is absolutely convergent for every* $x\in(0,\infty)$ *and*

$$\Delta_\mu^{-\alpha/2}f(x)=\frac{1}{\Gamma(\alpha)}\int_0^\infty f(y)y^{2\mu+1}\left(\int_0^\infty t^{\alpha-1}P_\mu(t,x,y)\,dt\right)dy,\quad x\in(0,\infty).$$

Proof. Let $f\in\mathcal{C}_c^\infty(0,\infty)$. According to [13, p. 86] we have that

$$P_\mu(t,x,y)\le Ct\min\left\{\frac{(xy)^{-\mu-1/2}}{|x-y|^2+t^2},\frac{1}{(|x-y|^2+t^2)^{\mu+3/2}}\right\},\quad t,x,y\in(0,\infty).$$

Hence, for every $x\in(0,\infty)$,

$$\begin{aligned}\int_0^\infty\int_0^\infty & t^{\alpha-1}P_\mu(t,x,y)|f(y)|y^{2\mu+1}\,dt\,dy\\ &\le C\int_0^\infty\left(\int_0^1\frac{t^\alpha|f(y)|(xy)^{-\mu-1/2}y^{2\mu+1}}{|x-y|^2+t^2}\,dt+\int_1^\infty\frac{t^\alpha|f(y)|y^{2\mu+1}\,dt}{(|x-y|^2+t^2)^{\mu+3/2}}\right)dy\\ &=I_1(x)+I_2(x).\end{aligned}$$

For the first term we have for each $x\in(0,\infty)$,

$$\begin{aligned}I_1(x)&\le Cx^{-\mu-1/2}\bigg(\int_{|x-y|\ge1}|f(y)|y^{\mu+1/2}\int_0^1\frac{t^\alpha\,dt\,dy}{|x-y|^2+t^2}\\ &\qquad+\int_{|x-y|<1}|f(y)|y^{\mu+1/2}\left(\int_0^{|x-y|}+\int_{|x-y|}^1\right)\frac{t^\alpha\,dt\,dy}{|x-y|^2+t^2}\bigg)\\ &\le Cx^{-\mu-1/2}\bigg(\int_0^\infty|f(y)|y^{\mu+1/2}\,dy\int_0^1\frac{t^\alpha\,dt}{1+t^2}+\int_0^\infty\frac{|f(y)|y^{\mu+1/2}}{|x-y|^{1-\alpha}}\,dy\int_0^1\frac{u^\alpha\,du}{1+u^2}\\ &\qquad+\int_0^\infty|f(y)|y^{\mu+1/2}\max\{1,|x-y|^{\alpha-1}\}\log\frac{|x-y|^2+1}{|x-y|^2}\,dy\bigg)\\ &<\infty.\end{aligned}$$

The second term is also finite for each $x\in(0,\infty)$, since

$$I_2(x)\le C\int_0^\infty|f(y)|y^{2\mu+1}\,dy\int_1^\infty t^{\alpha-2\mu-3}\,dt<\infty,\quad x\in(0,\infty).\quad\square$$

Moreover, $\Delta_\mu^{-\alpha/2}$ for $0<\alpha<2\mu+2$, turn out to be the fractional integrals, I_μ^α, defined by Muckenhoupt and Stein in [13, p. 89]. They introduced fractional integrals in the Hankel setting by using Hankel convolutions. Convolution operations associated with Hankel transforms were studied by Hirschman [9] and Haimo [8]. Let $f,g\in L^1(x^{2\mu+1}\,dx)$. The μ-Hankel convolution $f\#_\mu g$ of f and g is defined by

$$(f\#_\mu g)(x)=\int_0^\infty f(y)\,{}_\mu\tau_x(g)(y)\frac{y^{2\mu+1}}{2^\mu\Gamma(\mu+1)}\,dy,$$

where the μ-Hankel translation ${}_\mu\tau_x(g)$ of g is given by

$${}_\mu\tau_x(g)(y)=\int_0^\infty D_\mu(x,y,z)g(z)\frac{z^{2\mu+1}}{2^\mu\Gamma(\mu+1)}\,dz,$$

and D_μ denotes the Delsarte kernel

$$D_\mu(x,y,z)=(2^\mu\Gamma(\mu+1))^2\int_0^\infty \varphi_x(t)\varphi_y(t)\varphi_z(t)t^{2\mu+1}\,dt$$

for $x,y,z\in(0,\infty)$. The μ-fractional potential $I_\mu^\alpha(f)$ of f is defined in [13, p. 89] by $I_\mu^\alpha(f)=f\#_\mu K_\alpha$, where

$$K_\alpha(y)=y^{\alpha-2\mu-2}\frac{2^\mu\Gamma((\alpha+1)/2)\Gamma(\mu-\alpha/2+1)}{\sqrt{\pi}\Gamma(\alpha)}$$

for $y\in(0,\infty)$.

Lemma 8. *Let $0<\alpha<2\mu+2$. Then $\Delta_\mu^{-\alpha/2}f=I_\mu^\alpha f$ for every $f\in\mathcal{C}_c^\infty$.*

Proof. Let $f\in\mathcal{C}_c^\infty(0,\infty)$. According to Lemma 7 to see that $\Delta_\mu^{-\alpha/2}f=I_\mu^\alpha f$ it is sufficient to show that

$$\int_0^\infty t^{\alpha-1}P_\mu(t,x,y)\,dt=\Gamma(\alpha)\frac{{}_\mu\tau_x(K_\alpha)(y)}{2^\mu\Gamma(\mu+1)}. \tag{9}$$

By using [16, pp. 22–23] we obtain that $P_\mu(t,x,y)={}_\mu\tau_x(P_t)(y)$ for any $t,x,y\in(0,\infty)$, where $P_t(u)=2\Gamma(\mu+\frac{3}{2})t/\sqrt{\pi}\Gamma(\mu+1)(t^2+u^2)^{\mu+3/2}$, $t,u\in(0,\infty)$. Since all the functions involved are positive, we can interchange the order of integration and obtain (9) as follows

$$\begin{aligned}\int_0^\infty t^{\alpha-1}P_\mu(t,x,y)\,dt&=\frac{1}{2^\mu\Gamma(\mu+1)}\int_0^\infty t^{\alpha-1}\int_0^\infty D_\mu(x,y,z)P_t(z)z^{2\mu+1}\,dz\,dt\\&=\frac{\Gamma\left(\mu+\frac{3}{2}\right)}{\sqrt{\pi}2^{\mu-1}\Gamma(\mu+1)^2}\int_0^\infty D_\mu(x,y,z)z^{\alpha-2\mu-2}z^{2\mu+1}\,dz\\&\quad\times\int_0^\infty\frac{u^\alpha\,du}{(1+u^2)^{\mu+3/2}}\\&=\Gamma(\alpha)\frac{{}_\mu\tau_x(K_\alpha)(y)}{2^\mu\Gamma(\mu+1)}\end{aligned}$$

for $x,y\in(0,\infty)$. □

As a consequence of [13, p. 89] we have the following result.

Proposition 10. *In the case* $0<\alpha<2\mu+2$, *the operator* $\Delta_\mu^{-\alpha/2}$ *can be extended to* $L^p(x^{2\mu+1}\,dx)$ *as a bounded operator from* $L^p(x^{2\mu+1}\,dx)$ *into* $L^q(x^{2\mu+1}\,dx)$ *provided that* $1<p<q<\infty$ *and* $1/q=1/p-\alpha/(2\mu+2)$.

We now establish a formula relating the Hankel transform defined by

$$h_\mu(f)(x)=\int_0^\infty (xy)^{-\mu}J_\mu(xy)f(y)y^{2\mu+1}\,dy$$

and the operator $\Delta_\mu^{-\alpha/2}$. The behavior of h_μ on $\Delta_\mu^{-\alpha/2}$ corresponds to the one of the fractional integrals of the usual Laplacian with respect to the Fourier transform. The mapping h_μ is an automorphism on the space $\mathcal{S}_e$ of the even functions in the Schwartz space $\mathcal{S}$ ([1, Satz 5]). The Hankel transform is defined on the dual space $\mathcal{S}_e'$ of $\mathcal{S}_e$ by transposition and is then denoted by h_μ'.

Proposition 11. *Let* $0<\alpha<2\mu+2$ *and* $f\in\mathcal{C}_c^\infty(0,\infty)$. *Then* $h_\mu'(\Delta_\mu^{-\alpha/2}f)(y)=(1/y^\alpha)h_\mu(f)(y)$.

Proof. We only give some hints for the proof of this proposition. For every $s\in(0,1)$, we define the operator

$$G_s(f)(x)=\frac{1}{\Gamma(\alpha)}\int_s^{1/s} t^{\alpha-1}\int_0^\infty P_\mu(t,x,y)f(y)y^{2\mu+1}\,dy\,dt,\quad x\in(0,\infty).$$

It follows that $\lim_{s\to 0}G_sf=\Delta_\mu^{-\alpha/2}f$, in the weak-$*$ topology of S_e'. Thus we can conclude the result. $\square$

We extend the definition of the operator $\Delta_\mu^{-\alpha/2}$ to all $\alpha>0$. Choose $l_\alpha=\min\{l\in\mathbb{N}:2\mu+2+2l>\alpha\}$. If $h(z)=(1+z)^{-\mu-3/2}$, $z\in(0,\infty)$, we introduce the extended Poisson kernel by

$$P_\mu^{(\alpha)}(t,x,y)=P_\mu(t,x,y)-\chi_{(1,\infty)}(t)\frac{2\mu+1}{\pi t^{2\mu+2}} \times\sum_{j=0}^{l_\alpha-1}\frac{h^{(j)}(0)}{j!t^{2j}}\int_0^\pi((x-y)^2+2xy(1-\cos\theta))^j\sin^{2\mu}\theta\,d\theta \tag{12}$$

for $t,x,y\in(0,\infty)$. The operator $\Delta_\mu^{-\alpha/2}$ is defined by

$$\Delta_\mu^{-\alpha/2}f(x)=\frac{1}{\Gamma(\alpha)}\int_0^\infty f(y)y^{2\mu+1}\int_0^\infty P_\mu^{(\alpha)}(t,x,y)t^{\alpha-1}\,dt\,dy,\quad f\in\mathcal{C}_c^\infty(0,\infty). \tag{13}$$

By using the mean-value theorem and proceeding as in the proof of Lemma 7 we can prove that the integral in (13) is absolutely convergent. Note that definition (13) reduces to definition (5) provided that $0<\alpha<2\mu+2$ or $\int_0^\infty f(x)x^j x^{2\mu+1}\,dx=0$, $j=0,...,2l_\alpha-2$. The definition (12) of $P_\mu^{(\alpha)}(t,x,y)$ improves the behavior of the original Poisson kernel $P_\mu(t,x,y)$ when t is large. This allows us to define $\Delta_\mu^{-\alpha/2}$ in (13) for $f\in C_c^\infty(0,\infty)$ although f has not any zero moment. Note also that $(d^k/dx^k)P_\mu(t,x,y)=(d^k/dx^k)P_\mu^{(k)}(t,x,y)$, $t,x,y\in(0,\infty)$. Then, the modification of the Poisson kernel will not change the definition of the Riesz transforms $R_\mu^{(k)}$ in any case (see Proposition 14).

Proposition 14. *Let $k\in\mathbb{N}$ and let $f\in\mathcal{C}_c^\infty(0,\infty)$. Then $\Delta_\mu^{-k/2}f$ is k-times differentiable on $(0,\infty)\setminus\operatorname{supp} f$ and $(k-1)$-times differentiable on $(0,\infty)$. Moreover,*

$$\frac{d^m}{dx^m}\Delta_\mu^{-k/2}f(x)=\frac{1}{\Gamma(k)}\int_0^\infty f(y)y^{2\mu+1}\left(\int_0^\infty \frac{d^m}{dx^m}P_\mu^{(k)}(t,x,y)t^{k-1}\,dt\right)dy$$

for every $m=0,...,k-1$ and $x\in(0,\infty)$, and for $m=k$ and $x\notin\operatorname{supp} f$.

Proof. Let $f\in\mathcal{C}_c^\infty(0,\infty)$, $k\in\mathbb{N}$, and $u=((x-y)^2+2xy(1-\cos\theta))/t^2$. Note that

$$P_\mu^{(k)}(t,x,y)=\frac{2\mu+1}{\pi t^{2\mu+2}}\int_0^\pi\left(h(u)-\sum_{j=0}^{l_k-1}\frac{h^{(j)}(0)}{j!}u^j\chi_{(1,\infty)}(t)\right)\sin^{2\mu}\theta\,d\theta,$$

where $t,x,y\in(0,\infty)$ and $h(u)=(1+u)^{-\mu-3/2}$, $u\in(0,\infty)$.

It is not hard to see that for every $m\in\mathbb{N}$ there exist $c_{j,m}$, $j=[(m+1)/2],...,m$, such that

$$\frac{d^m}{dx^m}=\sum_{j=[(m+1)/2]}^{m}c_{j,m}\left(\frac{du}{dx}\right)^{2j-m}\frac{1}{t^{2(m-j)}}\frac{d^j}{du^j}. \tag{15}$$

Also note that if $x,y\in(0,\infty)$ then

$$\frac{d^j}{du^j}\left(h(u)-\sum_{l=0}^{l_k-1}\frac{h^{(l)}(0)}{l!}u^l\chi_{(1,\infty)}(t)\right)=\frac{d^j}{du^j}h(u),$$

provided that $t\in(0,\infty)$ and $j\ge l_k$, or $t\in(0,1)$ and $j\in\mathbb{N}$. Hence, if $x,y\in(0,\infty)$ we have that

$$\left|\frac{d^j}{du^j}\left(h(u)-\sum_{l=0}^{l_k-1}\frac{h^{(l)}(0)}{l!}u^l\chi_{(1,\infty)}(t)\right)\right|\le\frac{C}{(1+u)^{\mu+3/2+j}},$$

when $t\in(0,\infty)$ and $j\geq l_k$, or $t\in(0,1)$ and $j\in\mathbb{N}$. Moreover by using the mean-value theorem we obtain that

$$\left|\frac{d^j}{du^j}\left(h(u)-\sum_{l=0}^{l_k-1}\frac{h^{(l)}(0)}{l!}u^l\chi_{(1,\infty)}(t)\right)\right|\leq Cu^{l_k-j}$$

for $t\in(1,\infty)$, $x,y\in(0,\infty)$ and $j\in\mathbb{N}$, $j\leq l_k-1$.

Let $m\in\mathbb{N}$, $m\leq k$. We write, for $x,y\in(0,\infty)$, $x\neq y$, $|x-y|<1$, $0\leq j\leq m$,

$$\begin{aligned}\int_0^{|x-y|}&\frac{t^{k-1}}{t^{2(m-j)}}\int_0^\pi\left|\left(\frac{du}{dx}\right)^{2j-m}\frac{d^jh(u)}{du^j}\frac{\sin^{2\mu}\theta}{t^{2\mu+2}}\right|d\theta\,dt\\ &\leq C\int_0^{|x-y|}\frac{t^{k-2\mu-3-2(m-j)}}{t^{4j-2m}}\int_0^\pi\frac{\sin^{2\mu}\theta(|x-y|+y(1-\cos\theta))^{2j-m}\,d\theta\,dt}{(1+((x-y)^2+2xy(1-\cos\theta))/t^2)^{\mu+3/2+j}}\\ &\leq C\sum_{i=0}^{2j-m}\int_0^{|x-y|}t^k|x-y|^{2j-m-i}y^i\\ &\quad\times\left(\int_0^{\pi/2}+\int_{\pi/2}^\pi\right)\frac{\sin^{2\mu}\theta(1-\cos\theta)^i\,d\theta\,dt}{((x-y)^2+2xy(1-\cos\theta))^{\mu+3/2+j}}.\end{aligned}$$

In the inner integral for $\theta\in(0,\pi/2)$ we use that $\sin\theta\sim\theta$ and $1-\cos\theta\sim\theta^2/2$ and in the inner integral for $\theta\in(\pi/2,\pi)$ we apply the change of variables $\eta=\pi-\theta$, obtaining that this sum of integrals is bounded from above by

$$\begin{aligned}\int_0^{\pi/2}&\frac{\theta^{2\mu+2i}\,d\theta}{((x-y)^2+xy\theta^2)^{\mu+3/2+j}}+\int_{\pi/2}^\pi\frac{\sin^{2\mu}\theta\,d\theta}{(|x-y|^2+2xy)^{\mu+3/2+j}}\\ &\leq\frac{C(xy)^{-\mu-i-1/2}}{|x-y|^{2+2j-2i}}\int_0^{\pi\sqrt{xy}/2|x-y|}\frac{v^{2\mu+2i}\,dv}{(1+v^2)^{\mu+3/2+j}}+\frac{C}{(|x-y|^2+2xy)^{\mu+3/2+j}}\\ &\leq\frac{C(xy)^{-\mu-i-1/2}}{|x-y|^{2(1+j-i)}},\end{aligned}\tag{16}$$

where in the penultimate inequality we have performed the change of variables $v^2=xy\theta^2/(x-y)^2$. With this estimate, we get that

$$\begin{aligned}\int_0^{|x-y|}\frac{t^{k-1}}{t^{2(m-j)}}\int_0^\pi\left|\left(\frac{du}{dx}\right)^{2j-m}\frac{d^jh(u)}{du^j}\frac{\sin^{2\mu}\theta}{t^{2\mu+2}}\right|d\theta\,dt&\\ \leq C\sum_{i=0}^{2j-m}|x-y|^{i+k-m-1}x^{-\mu-i-1/2}y^{-\mu-1/2}.&\end{aligned}\tag{17}$$

Also, for $x,y\in(0,\infty)$, $x\neq y$, $|x-y|<1$, $0\leq j\leq m$, we obtain by similar arguments that

$$\begin{aligned}
&\int_{|x-y|}^{1}\frac{t^{k-1}}{t^{2(m-j)}}\int_0^{\pi}\left|\left(\frac{du}{dx}\right)^{2j-m}\frac{d^jh(u)}{du^j}\frac{\sin^{2\mu}\theta}{t^{2\mu+2}}\right|d\theta\,dt\\
&\quad\leq C\sum_{i=0}^{2j-m}|x-y|^{2j-m-i}y^i\int_{|x-y|}^{1}t^k\biggl(\frac{(xy)^{-\mu-i-1/2}}{((x-y)^2+t^2)^{1+j-i}}\\
&\qquad\times\int_0^{(\pi/2)\sqrt{xy/((x-y)^2+t^2)}}\frac{v^{2\mu+2i}\,dv}{(1+v^2)^{\mu+3/2+j}}+\frac{1}{((x-y)^2+t^2+2xy)^{\mu+3/2+j}}\biggr)\,dt\\
&\quad\leq C\sum_{i=0}^{2j-m}|x-y|^{2j-m-i}y^i(xy)^{-\mu-i-1/2}\int_{|x-y|}^{1}t^{k-1-2j+2i}\frac{t\,dt}{(x-y)^2+t^2}\\
&\quad\leq C\sum_{i=0}^{2j-m}|x-y|^{2j-m-i}y^i\max\{1,|x-y|^{k-1-2j+2i}\}(xy)^{-\mu-i-1/2}\log\frac{(x-y)^2+1}{2(x-y)^2}.
\end{aligned}\tag{18}$$

On the other hand, by proceeding as above we obtain, for every $x,y\in(0,\infty)$ with $|x-y|>1$, that

$$\begin{aligned}
&\int_0^{1}\frac{t^{k-1}}{t^{2(m-j)}}\int_0^{\pi}\left|\left(\frac{du}{dx}\right)^{2j-m}\frac{d^jh(u)}{du^j}\frac{\sin^{2\mu}\theta}{t^{2\mu+2}}\right|d\theta\,dt\\
&\qquad\qquad\leq C\sum_{i=0}^{2j-m}|x-y|^{i-m-2}x^{-\mu-i-1/2}y^{-\mu-1/2}.
\end{aligned}\tag{19}$$

By using the same procedure as in (16) and by [13, p. 60], if $j\geq l_k$ we get, for every $x,y\in(0,\infty)$, that

$$\begin{aligned}
&\int_1^{\infty}\frac{t^{k-1}}{t^{2(m-j)}}\int_0^{\pi}\left|\left(\frac{du}{dx}\right)^{2j-m}\frac{d^jh(u)}{du^j}\frac{\sin^{2\mu}\theta}{t^{2\mu+2}}\right|d\theta\,dt\\
&\quad\leq C\sum_{i=0}^{2j-m}|x-y|^{2j-m-i}y^i\int_1^{\infty}t^k\biggl(\frac{(xy)^{-\mu-i-1/2}}{((x-y)^2+t^2)^{1+j-i}}\\
&\qquad\times\int_0^{(\pi/2)\sqrt{xy/((x-y)^2+t^2)}}\frac{v^{2\mu+2i}\,dv}{(1+v^2)^{\mu+3/2+j}}+\frac{1}{((x-y)^2+t^2+2xy)^{\mu+3/2+j}}\biggr)\,dt\\
&\quad\leq C\sum_{i=0}^{2j-m}|x-y|^{2j-m-i}y^i\int_1^{\infty}t^{k-2\mu-3-2j}\,dt\\
&\quad\leq C\sum_{i=0}^{2j-m}|x-y|^{2j-m-i}y^i.
\end{aligned}\tag{20}$$

In the case $j<l_k$ the following estimate holds

$$\int_1^\infty \frac{t^{k-1}}{t^{2(m-j)}}\int_0^\pi \left|\left(\frac{du}{dx}\right)^{2j-m}\frac{d^j}{du^j}\left(h(u)-\sum_{l=0}^{l_k-1}\frac{h^{(l)}(0)}{l!}u^l\right)\frac{\sin^{2\mu}\theta}{t^{2\mu+2}}\right| d\theta\,dt$$
$$\le C\int_1^\infty \frac{t^{k-1-2(m-j)}}{t^{2\mu+2}}$$
$$\times\int_0^\pi \frac{((x-y)+y(1-\cos\theta))^{2j-m}}{t^{2(2j-m)}}\left(\frac{(x-y)^2+2xy(1-\cos\theta)}{t^2}\right)^{l_k-j}\sin^{2\mu}\theta\,d\theta\,dt$$
$$\le C\int_1^\infty \frac{dt}{t^{2\mu+3-k+2l_k}}$$
$$\times\int_0^\pi ((x-y)+y(1-\cos\theta))^{2j-m}((x-y)^2+2xy(1-\cos\theta))^{l_k-j}\sin^{2\mu}\theta\,d\theta. \tag{21}$$

Now, from (15) and the estimates (17)–(21), we deduce that

$$\int_0^\infty |f(y)|y^{2\mu+1}\int_0^\infty \left|t^{k-1}\frac{d^m}{dx^m}P_\mu^{(k)}(t,x,y)\right| dt\,dy<\infty,$$

provided that $m=0,1,2,...,k-1$ and $x\in(0,\infty)$, or $m=k$ and $x\notin\operatorname{supp} f$. Thus we establish the smoothness of $\Delta_\mu^{-k/2}f$. □

3. Proof of Theorem 3

As a consequence of Proposition 14, for $k\in\mathbb{N}$ and $f\in\mathcal{C}_c^\infty(0,\infty)$, we have that

$$R_\mu^{(k)}f(x)=D^k\Delta_\mu^{-k/2}f(x)=\int_0^\infty R_\mu^{(k)}(x,y)f(y)y^{2\mu+1}\,dy,\quad x\notin\operatorname{supp} f, \tag{22}$$

where, in view of our choice of l_k,

$$R_\mu^{(k)}(x,y)=\frac{1}{\Gamma(k)}\int_0^\infty t^{k-1}\frac{d^k}{dx^k}P_\mu(t,x,y)\,dt,\quad x,y\in(0,\infty),\ x\ne y. \tag{23}$$

Proposition 24. *Let $k\in\mathbb{N}$. There exist $b>1$ and $C>0$ (depending only on μ and k) such that*

$$\left|R_\mu^{(k)}(x,y)-a_k(x,y)\frac{(xy)^{-\mu-1/2}}{x-y}\right|\le CH_\mu(x,y),\quad x,y\in(0,\infty),\ x\ne y,$$

where

$$H_\mu(x,y)=\begin{cases} x^{-2\mu-2}, & 0<y<x/b,\\ y^{-2\mu-2}(1+\log(1+xy/|x-y|^2)), & x/b\le y\le bx,\\ y^{-2\mu-2}, & y>bx,\end{cases}$$

and $a_k(x,y)=0$ *if* k *is even,* $a_k(x,y)=(1/\iota^{k+1}\pi)\chi_{\{x/b\le y\le bx\}}(x,y)$ *for odd* k*, and* ι *denotes the imaginary unit.*

Proof. According to (15) we have that

$$\begin{aligned}\frac{d^k}{dx^k}&P_\mu(t,x,y)\\ &=\sum_{j=[(k+1)/2]}^{k}\frac{c_{j,k}}{t^{2\mu+2+2j}}\int_0^\pi\frac{\sin^{2\mu}\theta((x-y)+y(1-\cos\theta))^{2j-k}\,d\theta}{(1+((x-y)^2+2xy(1-\cos\theta))/t^2)^{\mu+3/2+j}}\\ &=\sum_{j=[(k+1)/2]}^{k}\sum_{i=0}^{2j-k}c_{i,j,k}ty^i(x-y)^{2j-k-i}\int_0^\pi\frac{\sin^{2\mu}\theta(1-\cos\theta)^i\,d\theta}{((x-y)^2+t^2+2xy(1-\cos\theta))^{\mu+3/2+j}}\end{aligned}$$

for certain $c_{j,k},c_{i,j,k}\in\mathbb{R}$. We split the kernel of $R_\mu^{(k)}$ as

$$\Gamma(k)R_\mu^{(k)}(x,y)=\sum_{j=[(k+1)/2]}^{k}\sum_{i=0}^{2j-k}c_{i,j,k}(S_{i,j}^1(x,y)+S_{i,j}^2(x,y)), \tag{25}$$

where

$$S_{i,j}^1(x,y)=(x-y)^{2j-k-i}y^i\int_0^\infty t^k\int_0^{\pi/2}\frac{\sin^{2\mu}\theta(1-\cos\theta)^i\,d\theta\,dt}{((x-y)^2+t^2+2xy(1-\cos\theta))^{\mu+3/2+j}}$$

and $S_{i,j}^2(x,y)$ has the same expression as that for $S_{i,j}^1(x,y)$; the only difference being that we take the inner integral over $\theta\in(\pi/2,\pi)$.

Assume that $b>1$ is a constant whose precise value will be fixed later (see (30)). As $1-\cos\theta\ge1$, $\theta\in[\pi/2,\pi]$, for (x,y) in the local region $x/b<y<bx$, we can write

$$\begin{aligned}|S_{i,j}^2(x,y)|&\le C|x-y|^{2j-k-i}y^i\left(\int_0^y+\int_y^\infty\right)\frac{t^k\,dt}{((x-y)^2+t^2+2xy)^{\mu+3/2+j}}\\ &\le C|x-y|^{2j-k-i}y^i\left(\frac{1}{((x-y)^2+2xy)^{\mu+3/2+j}}\int_0^y t^k\,dt+\int_y^\infty\frac{dt}{t^{2j-k+2\mu+3}}\right)\\ &\le C|x-y|^{2j-k-i}y^i\left(\frac{y^{k+1}}{(xy)^{\mu+3/2+j}}+y^{-(2j-k+2\mu+2)}\right)\\ &\le\frac{C}{y^{2\mu+2}},\end{aligned}$$

since in this region $|x-y|\le Cx$. Outside the local region, for $y>bx$ or $y<x/b$, $|x-y|\ge C\max\{x,y\}$ and therefore

$$|S_{i,j}^2(x,y)|\le C|x-y|^{2j-k-i}y^i\int_0^\infty\frac{t^k\,dt}{((x-y)^2+t^2)^{\mu+3/2+j}}$$

$$\le C\int_0^\infty\frac{t\,dt}{((x-y)^2+t^2)^{\mu+2}}\le\frac{C}{|x-y|^{2\mu+2}}\le C\begin{cases}x^{-2\mu-2}, & y<x/b,\\ y^{-2\mu-2}, & y>bx.\end{cases} \tag{26}$$

From these estimates we get that $|S^2_{i,j}(x,y)|\le CH_\mu(x,y)$. In order to analyze $S^1_{i,j}$, note that by proceeding as in (26) we can see that

$$|S^1_{i,j}(x,y)|\le C\begin{cases} x^{-2\mu-2}, & y<x/b,\\ y^{-2\mu-2}, & y>bx.\end{cases}$$

In the region where $x/b\le y\le bx$, let us split $S^1_{i,j}(x,y)$ as

$$\begin{aligned} S^1_{i,j}(x,y) &= S^{1,0}_{i,j}(x,y)+S^{1,\infty}_{i,j}(x,y)\\ &= (x-y)^{2j-k-i}y^i\left(\int_0^{\delta\sqrt{xy}}+\int_{\delta\sqrt{xy}}^\infty\right)t^k\\ &\quad\times\int_0^{\pi/2}\frac{\sin^{2\mu}\theta(1-\cos\theta)^i\,d\theta\,dt}{((x-y)^2+t^2+2xy(1-\cos\theta))^{\mu+3/2+j}}, \end{aligned}$$

where δ is a positive number whose precise value will be specified later (see (30)). Applying the change of variables $u^2=xy\theta^2/((x-y)^2+t^2)$ we obtain that

$$\begin{aligned} |S^{1,\infty}_{i,j}(x,y)| &\le C|x-y|^{2j-k-i}y^i\int_{\delta\sqrt{xy}}^\infty\frac{t^k}{((x-y)^2+t^2)^{j+1-i}(xy)^{\mu+i+1/2}}\\ &\quad\times\int_0^{(\pi/2)\sqrt{xy}/\sqrt{(x-y)^2+t^2}}\frac{u^{2\mu+2i}\,du\,dt}{(1+u^2)^{\mu+3/2+j}}. \end{aligned}$$

By using the estimate [13, p. 60] we have that

$$\int_0^{(\pi/2)\sqrt{xy}/\sqrt{(x-y)^2+t^2}}\frac{u^{2\mu+2i}\,du}{(1+u^2)^{\mu+3/2+j}}$$

is bounded above by a constant times $(xy)^{i/2}((x-y)^2+t^2)^{-i/2}$. In these circumstances, and taking into account that $x/b<y<bx$, we obtain that

$$\begin{aligned} |S^{1,\infty}_{i,j}(x,y)| &\le C|x-y|^{2j-k-i}y^i\int_{\delta\sqrt{xy}}^\infty\frac{1}{t^2}\frac{dt}{((x-y)^2+t^2)^{j-i/2-k/2}(xy)^{\mu+1/2+i/2}}\\ &\le C\frac{1}{y^{2\mu+1}}\int_{\delta\sqrt{xy}}^\infty\frac{dt}{t^2}\le C\frac{1}{y^{2\mu+2}}. \end{aligned}$$

The study of $S^{1,0}_{i,j}(x,y)$ is more involved. We consider the following kernels

$$A_{i,j}(x,y)=\int_0^{\delta\sqrt{xy}}\frac{t^k}{2^i}(x-y)^{2j-k-i}y^i\int_0^{\pi/2}\frac{\theta^{2\mu+2i}\,d\theta\,dt}{((x-y)^2+t^2+2xy(1-\cos\theta))^{\mu+3/2+j}}$$

and

$$B_{i,j}(x,y)=\int_0^{\delta\sqrt{xy}}\frac{t^k}{2^i}(x-y)^{2j-k-i}y^i\int_0^{\pi/2}\frac{\theta^{2\mu+2i}\,d\theta\,dt}{((x-y)^2+t^2+xy\theta^2)^{\mu+3/2+j}},$$

and we write

$$S^{1,0}_{i,j}(x,y)=(S^{1,0}_{i,j}(x,y)-A_{i,j}(x,y))+(A_{i,j}(x,y)-B_{i,j}(x,y))+B_{i,j}(x,y). \tag{27}$$

For the first difference, since $\sin\theta\sim\theta$ and $1-\cos\theta\sim\theta^2/2$ for $\theta\in[0,\pi/2]$, by using the mean-value theorem we get that

$$\left|\sin^{2\mu}\theta(1-\cos\theta)^i-\frac{\theta^{2\mu+2i}}{2^i}\right|\le C\theta^{2\mu+2i+2},\quad \theta\in\left[0,\frac{\pi}{2}\right].$$

Hence,

$$\begin{aligned}
&|S^{1,0}_{i,j}(x,y)-A_{i,j}(x,y)|\\
&\quad\le C|x-y|^{2j-k-i}y^i\int_0^{\delta\sqrt{xy}}t^k\int_0^{\pi/2}\frac{\theta^{2\mu+2i+2}\,d\theta\,dt}{((x-y)^2+t^2+\theta^2xy)^{\mu+3/2+j}}\\
&\quad\le C|x-y|^{2j-k-i}y^i\int_0^{\delta\sqrt{xy}}\frac{t^k(xy)^{-\mu-3/2-i}}{((x-y)^2+t^2)^{j-i}}\int_0^{(\pi/2)\sqrt{xy}/\sqrt{(x-y)^2+t^2}}\frac{u^{2\mu+2i+2}\,du\,dt}{(1+u^2)^{\mu+3/2+j}},
\end{aligned}$$

where in the last inequality we have performed the usual change of variables $u^2=xy\theta^2/((x-y)^2+t^2)$. Then, from [13, p. 60] we deduce that, when $i\neq k$ or $j\neq k$,

$$\begin{aligned}
&|S^{1,0}_{i,j}(x,y)-A_{i,j}(x,y)|\\
&\qquad\le C|x-y|^{2j-k-i}y^i\int_0^{\delta\sqrt{xy}}\frac{t^k(xy)^{-\mu-3/2-i}}{((x-y)^2+t^2)^{j-i}}\left(\frac{\sqrt{xy}}{\sqrt{(x-y)^2+t^2}+\sqrt{xy}}\right)^{2\mu+3+2i}dt\\
&\qquad\le C|x-y|^{2j-k-i}y^i\int_0^{\delta\sqrt{xy}}\frac{t^k\,dt}{((x-y)^2+t^2)^{j-i}(xy)^{\mu+1+i/2}((x-y)^2+t^2)^{1/2+i/2}}\\
&\qquad\le C\frac{1}{y^{2\mu+2}}\int_0^{\delta\sqrt{xy}}\frac{t\,dt}{(x-y)^2+t^2}\\
&\qquad\le C\frac{1}{y^{2\mu+2}}\log\left(1+\frac{\delta^2xy}{(x-y)^2}\right).
\end{aligned}$$

On the other hand, we have that

$$\begin{aligned}
|S^{1,0}_{k,k}(x,y)-A_{k,k}(x,y)|&\le Cy^k\int_0^{\delta\sqrt{xy}}t^k\int_0^{\pi/2}\frac{\theta^{2\mu+2k+2}\,d\theta\,dt}{((x-y)^2+t^2+\theta^2xy)^{\mu+3/2+k}}\\
&\le C\int_0^{\delta\sqrt{xy}}\int_0^{\pi/2}\frac{\theta}{(x-y)^2+\theta^2xy}\frac{\theta^{2\mu+1+2k}t^ky^k\,d\theta\,dt}{t^k(\theta^2xy)^{\mu+1/2+k/2}}\\
&\le\frac{C}{y^{2\mu+2}}\log\left(1+\frac{\pi^2}{4}\frac{xy}{(x-y)^2}\right).
\end{aligned}$$

By using again the mean-value theorem one obtains that, for every $\theta\in[0,\pi/2]$,

$$\left|\frac{1}{((x-y)^2+y^2+2xy(1-\cos\theta))^{\mu+3/2+j}}-\frac{1}{((x-y)^2+y^2+xy\theta^2)^{\mu+3/2+j}}\right| \leq C\frac{\theta^4 xy}{((x-y)^2+y^2+xy\theta^2)^{\mu+5/2+j}}.$$

Then, by proceeding as for $|S_{i,j}^{1,0}-A_{i,j}|$, we get that

$$\begin{aligned}|A_{i,j}(x,y)-B_{i,j}(x,y)| &\leq C\int_0^{\delta\sqrt{xy}} t^k \int_0^{\pi/2} \frac{\theta^{2\mu+2i+4}xy^{i+1}|x-y|^{2j-k-i}}{((x-y)^2+y^2+xy\theta^2)^{\mu+5/2+j}}\,d\theta\,dt \\ &\leq C\int_0^{\delta\sqrt{xy}} t^k \int_0^{\pi/2} \frac{\theta^{2\mu+2i+2}y^{i}|x-y|^{2j-k-i}}{((x-y)^2+t^2+xy\theta^2)^{\mu+3/2+j}}\,d\theta\,dt \\ &\leq \frac{C}{y^{2\mu+2}}\begin{cases}\log\left(1+\delta^2\dfrac{xy}{(x-y)^2}\right), & i\neq k \text{ or } j\neq k,\\ \log\left(1+\dfrac{\pi^2}{4}\dfrac{xy}{(x-y)^2}\right), & i=j=k.\end{cases}\end{aligned}$$

To analyze $B_{i,j}$ we need to proceed in a different way. If we substitute (27) in (25), it turns out that the term that is left to study is given by

$$B(x,y)=\sum_{j=[(k+1)/2]}^{k}\sum_{i=0}^{2j-k} c_{i,j,k}B_{i,j}(x,y).$$

It turns out that this kernel behaves like that of the Hilbert transform in the case when k is odd, and that it is an integrable kernel in the case when k is even. More concretely, the following lemma holds, concluding the proof of Proposition 24. □

Lemma 28. *For every $k\in\mathbb{N}$, and $0<x/b<y<bx$,*

$$B(x,y)=c_k\frac{(xy)^{-\mu-1/2}}{x-y}+O\left(\frac{1}{x^{2\mu+2}}\log\left(1+\frac{xy}{(x-y)^2}\right)\right),$$

where $c_k=0$ if k is even and $c_k=(k-1)!/\iota^{k+1}\pi$ for odd k.

Proof. Note firstly that, from (27) and (25), and by using (15),

$$\begin{aligned}B(x,y)&=\sum_{j=[(k+1)/2]}^{k}\sum_{i=0}^{2j-k} c_{i,j,k}B_{i,j}(x,y)\\ &=\frac{2\mu+1}{\pi}\int_0^{\delta\sqrt{xy}} t^k\frac{d^k}{dx^k}\int_0^{\pi/2}\frac{\theta^{2\mu}\,d\theta\,dt}{((x-y)^2+t^2+xy\theta^2)^{\mu+3/2}}.\end{aligned}$$

By the usual change of variables $z^2=xy\theta^2/((x-y)^2+t^2)$, we obtain that

$$\int_0^{\pi/2} \frac{\theta^{2\mu}\,d\theta}{((x-y)^2+t^2+xy\theta^2)^{\mu+3/2}} = -\int_{\pi/2}^{\infty} \frac{\theta^{2\mu}\,d\theta}{((x-y)^2+t^2+xy\theta^2)^{\mu+3/2}} + \frac{1}{2\mu+1}\frac{(xy)^{-\mu-1/2}}{(x-y)^2+t^2}.$$

Therefore, we get that

(29)

$$B(x,y) = \sum_{j=[(k+1)/2]}^{k} \sum_{i=0}^{2j-k} c_{i,j} H_{i,j}(x,y) + \sum_{l=0}^{k-1} \sum_{s=[(l+1)/2]}^{l} d_{l,s} D_{l,s}(x,y) + d_k D_k(x,y),$$

where for every $j=[(k+1)/2],...,k$, $i=0,...,2j-k$, $c_{i,j}\in\mathbb{R}$ and

$$H_{i,j}(x,y) = (x-y)^{2j-k-i} y^i \int_0^{\delta\sqrt{xy}} t^k \int_{\pi/2}^{\infty} \frac{\theta^{2\mu+2i}\,d\theta}{((x-y)^2+t^2+xy\theta^2)^{\mu+3/2+j}},$$

$d_k=1/\pi$ and, for $l=0,...,k-1$, $s=[(l+1)/2],...,l$, $d_{l,s}\in\mathbb{R}$ and

$$D_{l,s}(x,y) = y^{-\mu-1/2} x^{-\mu-1/2-k+l} (x-y)^{2s-l} \int_0^{\delta\sqrt{xy}} \frac{t^k\,dt}{((x-y)^2+t^2)^{1+s}},$$

$$D_k(x,y) = (xy)^{-\mu-1/2} \int_0^{\delta\sqrt{xy}} \frac{d^k}{dx^k}\left(\frac{1}{(x-y)^2+t^2}\right) t^k\,dt.$$

Let us start with the term $H_{i,j}$. We first need to make some observations. Let us consider the function $g_{i,j}(u)=u^{2\mu+2i}/(1+u^2)^{\mu+1/2+j}$, $u\in(0,\infty)$. It is not hard to see that if $\mu+i\le 0$, then $g_{i,j}$ is decreasing on $(0,\infty)$, and if $\mu+i>0$, then $g_{i,j}$ is decreasing on $(\sqrt{(2\mu+2i)/(1+2j-2i)},\infty)$. Note that $1+2j-2i>0$. One can prove (see [4] for the details) that there exist $b>1$ and $\delta>0$ such that

(30)
$$\frac{\pi}{2}\sqrt{\frac{xy}{(x-y)^2+t^2}} \ge \sqrt{\frac{2\mu+2i}{1+2j-2i}}$$

for $x/b\le y\le bx$, $x\in(0,\infty)$ and $t\in(0,\delta\sqrt{xy})$. Observe that δ and b can be taken independently of j and i. For these values of the variables we have

$$\int_A^{\infty} \frac{u^{2\mu+2i}\,du}{(1+u^2)^{\mu+3/2+j}} \le \left.\frac{Cu^{2\mu+2i}}{(1+u^2)^{\mu+1/2+j}}\right|_{u=A} \le \frac{C(xy)^{\mu+i}((x-y)^2+t^2)^{j-i+1/2}}{((x-y)^2+t^2+xy)^{\mu+1/2+j}}$$

in the particular case when $A=(\pi/2)\sqrt{xy/((x-y)^2+t^2)}$. This estimate, together with the usual change of variables $u^2=xy\theta^2/((x-y)^2+t^2)$, leads to

$$\begin{aligned}|H_{i,j}(x,y)| &\le C\frac{|x-y|^{2j-k-i}y^i}{(xy)^{\mu+i+1/2}}\int_0^{\delta\sqrt{xy}}\frac{t^k}{((x-y)^2+t^2)^{j+1-i}}\\ &\quad\times\int_{(\pi/2)\sqrt{xy/((x-y)^2+t^2)}}^{\infty}\frac{u^{2\mu+2i}\,du\,dt}{(1+u^2)^{\mu+3/2+j}}\\ &\le C\frac{|x-y|^{2j-k-i}y^i}{(xy)^{1/2}}\int_0^{\delta\sqrt{xy}}\frac{t}{(x-y)^2+t^2}\frac{t^{k-1}((x-y)^2+t^2)^{1/2}\,dt}{(xy)^{\mu+1/2+i/2}((x-y)^2+t^2)^{j-i/2}}\\ &\le C\frac{1}{x^{2\mu+2}}\log\biggl(1+\frac{\delta^2xy}{(x-y)^2}\biggr).\end{aligned}$$

For the terms $D_{l,s}$ in (29), we have that

$$\begin{aligned}&|D_{l,s}(x,y)|\\ &\quad\le Cy^{-\mu-1/2}x^{-\mu-1/2-k+l}|x-y|^{k-l-1}\int_0^{\delta\sqrt{xy}/|x-y|}\frac{u^k\,du}{(1+u^2)^{1+s}}\\ &\quad\le Cy^{-\mu-1/2}x^{-\mu-1/2-k+l}|x-y|^{k-l-1}\biggl(\frac{\sqrt{xy}}{|x-y|}\biggr)^{k-l-1}\int_0^{\delta\sqrt{xy}/|x-y|}\frac{u^{l+1}\,du}{(1+u^2)^{1+s}}.\end{aligned}$$

By using [13, pp. 60–61], since $l+1-2(1+s)=-1+l-2s\le -1$, we get that

$$\int_0^{\delta\sqrt{xy}/|x-y|}\frac{u^{l+1}\,du}{(1+u^2)^{1+s}}\le C\biggl(\frac{\delta\sqrt{xy}}{\delta\sqrt{xy}+|x-y|}\biggr)^{l+2}\biggl(1+\log\biggl(1+\frac{\delta\sqrt{xy}}{|x-y|}\biggr)\biggr).$$

Hence we conclude that

$$|D_{l,s}(x,y)|\le Cy^{-2\mu-2}\biggl(1+\log\biggl(1+\frac{\delta\sqrt{xy}}{|x-y|}\biggr)\biggr).$$

We now write $D_k(x,y)=-D_k^1(x,y)+D_k^2(x,y)$, where

$$D_k^1(x,y)=(xy)^{-\mu-1/2}\int_{\delta\sqrt{xy}}^{\infty}t^k\frac{d^k}{dx^k}\biggl(\frac{1}{(x-y)^2+t^2}\biggr)\,dt$$

and

$$D_k^2(x,y)=(xy)^{-\mu-1/2}\int_0^{\infty}t^k\frac{d^k}{dx^k}\biggl(\frac{1}{(x-y)^2+t^2}\biggr)\,dt.$$

Note that, for certain $c_j \in \mathbb{R}$, $j=[(k+1)/2],...,k$,

$$D_k^1(x,y)=(xy)^{-\mu-1/2}\sum_{j=[(k+1)/2]}^{k} c_j \int_{\delta\sqrt{xy}}^{\infty}\frac{t^k(x-y)^{2j-k}\,dt}{((x-y)^2+t^2)^{1+j}}.$$

Then

$$|D_k^1(x,y)| \le \frac{C}{(xy)^{\mu+1/2}}\sum_{j=[(k+1)/2]}^{k}\int_{\delta\sqrt{xy}}^{\infty}\frac{1}{t^2}\frac{t^{k+2}}{((x-y)^2+t^2)^{k/2+1}}\frac{|x-y|^{2j-k}\,dt}{((x-y)^2+t^2)^{j-k/2}}$$
$$\le \frac{C}{(xy)^{\mu+1}} \le Cx^{-2\mu-2}.$$

On the other hand, a straightforward manipulation allows us to write

$$\frac{d^k}{du^k}\left(\frac{1}{u^2+t^2}\right)=-\frac{(-1)^k k!}{2\iota t}\left(\frac{1}{(u+\iota t)^{k+1}}-\frac{1}{(u-\iota t)^{k+1}}\right).$$

By partial integration $k-1$ times we obtain that

$$\int_0^{\infty}\frac{t^{k-1}\,dt}{(u+\iota t)^{k+1}}-\int_0^{\infty}\frac{t^{k-1}\,dt}{(u-\iota t)^{k+1}}=\frac{1}{ku}\left(\frac{1}{\iota^k}-\frac{1}{(-\iota)^k}\right).$$

Hence, we conclude that

$$D_k^2(x,y)=c_k\frac{(xy)^{-\mu-1/2}}{x-y}, \quad \text{where } c_k=\begin{cases} 0, & \text{if } k \text{ is even},\\ \dfrac{(k-1)!}{\iota^{k+1}}, & \text{if } k \text{ is odd}. \end{cases} \quad \square$$

By using the procedure developed in Proposition 24 we can obtain the following result.

Lemma 31. *Let $f\in\mathcal{C}_c^{\infty}(0,\infty)$ and $k\in\mathbb{N}$. Then, for all $x\in(0,\infty)$,*

$$R_\mu^{(k)}f(x)=\omega_k f(x)+\lim_{\varepsilon\to 0}\int_{|x-y|>\varepsilon} f(y)R_\mu^{(k)}(x,y)y^{2\mu+1}\,dy,$$

where $\omega_k=0$, when k is odd, and $\omega_k=(-1)^{k/2}\pi/(2\mu+1)$, when k is even.

Proof. According to Proposition 14 and taking into account that

$$l_k=\min\{l\in\mathbb{N}: 2\mu+2+2l>k\}\le(k-1)/2,$$

we get that

$$\frac{d^{k-1}}{dx^{k-1}}\Delta_\mu^{-k/2}f(x)=\frac{1}{\Gamma(k)}\int_0^{\infty}f(y)y^{2\mu+1}\int_0^{\infty}t^{k-1}\frac{d^{k-1}}{dx^{k-1}}P_\mu(t,x,y)\,dt\,dy$$

for $x\in(0,\infty)$. We now write, for every $x\in(0,\infty)$,

$$\begin{aligned}\frac{d^{k-1}}{dx^{k-1}}\Delta_\mu^{-k/2}f(x)=&\frac{1}{\Gamma(k)}\int_0^\infty f(y)y^{2\mu+1}\bigg(\int_0^\infty t^{k-1}\frac{d^{k-1}}{dx^{k-1}}P_\mu(t,x,y)\,dt\\&-\frac{1}{2\mu+1}(xy)^{-\mu-1/2}\int_0^1\frac{d^{k-1}}{dx^{k-1}}\bigg(\frac{1}{(x-y)^2+t^2}\bigg)t^k\,dt\bigg)\,dy\\&+\frac{1}{2\mu+1}\frac{1}{\Gamma(k)}\int_0^\infty f(y)y^{2\mu+1}(xy)^{-\mu-1/2}\\&\times\int_0^1\frac{d^{k-1}}{dx^{k-1}}\bigg(\frac{1}{(x-y)^2+t^2}\bigg)t^k\,dt\,dy.\end{aligned}$$

By proceeding as in the proof of Proposition 24 we can see that for $x\in(0,\infty)$,

$$\begin{aligned}&\frac{d}{dx}\bigg(\frac{1}{\Gamma(k)}\int_0^\infty f(y)y^{2\mu+1}\bigg(\int_0^\infty t^{k-1}\frac{d^{k-1}}{dx^{k-1}}P_\mu(t,x,y)\,dt\\&\qquad\qquad-\frac{1}{2\mu+1}(xy)^{-\mu-1/2}\int_0^1\frac{d^{k-1}}{dx^{k-1}}\bigg(\frac{1}{(x-y)^2+t^2}\bigg)t^k\,dt\bigg)\,dy\bigg)\\&=\frac{1}{\Gamma(k)}\int_0^\infty f(y)y^{2\mu+1}\int_1^\infty t^{k-1}\frac{d^k}{dx^k}P_\mu(t,x,y)\,dt\,dy\\&\quad+\frac{1}{\Gamma(k)}\int_0^\infty f(y)y^{2\mu+1}\int_0^1\frac{d}{dx}\bigg(t^{k-1}\frac{d^{k-1}}{dx^{k-1}}P_\mu(t,x,y)\\&\qquad\qquad-\frac{1}{2\mu+1}(xy)^{-\mu-1/2}\frac{d^{k-1}}{dx^{k-1}}\bigg(\frac{1}{(x-y)^2+t^2}\bigg)t^k\bigg)\,dt\,dy\\&=\frac{1}{\Gamma(k)}\int_0^\infty f(y)y^{2\mu+1}\int_1^\infty t^{k-1}\frac{d^k}{dx^k}P_\mu(t,x,y)\,dt\,dy\\&\quad+\frac{1}{\Gamma(k)}\int_0^\infty f(y)y^{2\mu+1}\int_0^1\bigg(t^{k-1}\frac{d^k}{dx^k}P_\mu(t,x,y)\\&\qquad\qquad-\frac{1}{2\mu+1}(xy)^{-\mu-1/2}\frac{d^k}{dx^k}\bigg(\frac{1}{(x-y)^2+t^2}\bigg)t^k\bigg)\,dt\,dy\\&\quad+\frac{1}{2}x^{-\mu-3/2}\frac{1}{\Gamma(k)}\int_0^\infty f(y)y^{\mu+1/2}\int_0^1\frac{d^{k-1}}{dx^{k-1}}\bigg(\frac{1}{(x-y)^2+t^2}\bigg)t^k\,dt\,dy.\end{aligned}$$

The integrals are absolutely convergent.

We define

$$\Phi(x)=\int_0^1\frac{d^{k-1}}{dx^{k-1}}\bigg(\frac{1}{x^2+t^2}\bigg)t^k\,dt,\quad x\in\mathbb{R}.$$

Then $\Phi\in L^1(\mathbb{R},dx)$ and $\Phi\in C^\infty(\mathbb{R}\setminus\{0\})$. Moreover, by defining the function g as $g(y)=f(y)y^{\mu+1/2}$, $y\geq 0$, and $g(y)=0$, $y<0$, it has, for every $x\in(0,\infty)$,

$$\begin{aligned}\frac{d}{dx}\int_0^\infty &\Phi(x-y)f(y)y^{\mu+1/2}\,dy\\ &=\frac{d}{dx}\int_{-\infty}^\infty \Phi(y)g(x-y)\,dy\\ &=-\int_{-\infty}^\infty \Phi(y)\frac{d}{dy}g(x-y)\,dy=-\lim_{\varepsilon\to 0}\int_{|y|>\varepsilon}\Phi(y)\frac{d}{dy}g(x-y)dy\\ &=\lim_{\varepsilon\to 0}\left(\int_{|x-y|>\varepsilon}\Phi'(x-y)g(y)\,dy-(\Phi(-\varepsilon)g(x+\varepsilon)-\Phi(\varepsilon)g(x-\varepsilon))\right).\end{aligned}$$

Let $x\in(0,\infty)$. Note that if k is odd, then Φ is even and

$$\lim_{\varepsilon\to 0}(\Phi(-\varepsilon)g(x+\varepsilon)-\Phi(\varepsilon)g(x-\varepsilon))=\lim_{\varepsilon\to 0}\Phi(\varepsilon)(g(x+\varepsilon)-g(x-\varepsilon))=0,$$

because

$$|\Phi(\varepsilon)(g(x+\varepsilon)-g(x-\varepsilon))|\leq C\varepsilon|\Phi(\varepsilon)|\to 0,\quad \text{as } \varepsilon\to 0.$$

On the other hand, if k is even, then Φ is odd and, according to [5, Lemma 4.3, (4.6)], for every $\varepsilon>0$, we have

$$\begin{aligned}\Phi(-\varepsilon)g(x+\varepsilon)-\Phi(\varepsilon)g(x-\varepsilon)&=-\Phi(\varepsilon)(g(x+\varepsilon)+g(x-\varepsilon))\\ &=-(g(x+\varepsilon)+g(x-\varepsilon))\int_0^1\frac{d^{k-1}}{dx^{k-1}}\left(\frac{1}{x^2+t^2}\right)t^k\,dt\Big|_{x=\varepsilon}\\ &=(g(x+\varepsilon)+g(x-\varepsilon))\sum_{j=0}^{k/2-1}2^{k-1-2j}\frac{\Gamma(k)\Gamma(k-j)}{\Gamma(j+1)\Gamma(k-2j)}\\ &\qquad\times(-1)^{k-1-j}\int_0^{1/\varepsilon}\frac{u^k\,du}{(1+u^2)^{k-j}}.\end{aligned}$$

Hence

$$\begin{aligned}\lim_{\varepsilon\to 0}&(\Phi(-\varepsilon)g(x+\varepsilon)-\Phi(\varepsilon)g(x-\varepsilon))\\ &=2\sum_{j=0}^{k/2-1}2^{k-1-2j}\frac{\Gamma(k)\Gamma(k-j)}{\Gamma(j+1)\Gamma(k-2j)}(-1)^{k-1-j}f(x)x^{\mu+1/2}\int_0^\infty\frac{u^k\,du}{(1+u^2)^{k-j}}.\end{aligned}$$

Using the duplication formula for the gamma function we get that

$$\sum_{j=0}^{k/2-1}2^{k-1-2j}\frac{\Gamma(k)\Gamma(k-j)}{\Gamma(j+1)\Gamma(k-2j)}(-1)^{k-1-j}\int_0^\infty\frac{u^k}{(1+u^2)^{k-j}}\,du=\frac{\pi}{2}(-1)^{k/2}\Gamma(k).$$

We obtain that

$$\begin{aligned}&\frac{d}{dx}\bigg(\frac{1}{\Gamma(k)}\int_0^\infty f(y)y^{2\mu+1}(xy)^{-\mu-1/2}\int_0^1\frac{d^{k-1}}{dx^{k-1}}\bigg(\frac{t^k}{(x-y)^2+t^2}\bigg)\,dt\,dy\bigg)\\&=-\bigg(\mu+\frac{1}{2}\bigg)x^{-\mu-3/2}\frac{1}{\Gamma(k)}\int_0^\infty f(y)y^{\mu+1/2}\int_0^1\frac{d^{k-1}}{dx^{k-1}}\bigg(\frac{t^k}{(x-y)^2+t^2}\bigg)\,dt\,dy\\&\quad+\lim_{\varepsilon\to0}\frac{1}{\Gamma(k)}\int_{\substack{0<y<\infty\\|x-y|>\varepsilon}}f(y)y^{2\mu+1}(xy)^{-\mu-1/2}\int_0^1\frac{d^{k}}{dx^{k}}\bigg(\frac{t^k}{(x-y)^2+t^2}\bigg)\,dt\,dy+c_kf(x),\end{aligned}$$

where $c_k=0$, for k odd, and $c_k=(-1)^{k/2}\pi$, when k is even.

Then,

$$\frac{d^k}{dx^k}\Delta_\mu^{-k/2}f(x)=\omega_kf(x)+\lim_{\varepsilon\to0}\int_{|x-y|>\varepsilon}f(y)R_\mu^{(k)}(x,y)y^{2\mu+1}\,dy,\quad x\in(0,\infty),$$

where $\omega_k=0$, for k odd, and $\omega_k=(-1)^{k/2}\pi/(2\mu+1)$, when k is even.

Thus the proof is finished. □

L^p-boundedness of $R_\mu^{(k)}$ is also a consequence of Proposition 24 and the corresponding properties of the Hilbert transform and the Hardy-type operators H_μ and H_μ^* defined by

$$H_\mu f(x)=\frac{1}{x^{2\mu+2}}\int_0^x f(y)y^{2\mu+1}\,dy\quad\text{and}\quad H_\mu^*f(x)=\int_x^\infty\frac{f(y)}{y}\,dy,\quad x>0.$$

Note that H_μ^* is the adjoint operator in $L^2((0,\infty),x^{2\mu+1}\,dx)$ of H_μ. Let $1\le p<\infty$. According to Proposition 24, we can write for every $f\in L^p(x^{2\mu+1}\,dx)$,

$$|R_{\mu,\varepsilon}^{(k)}f(x)|\le C(H_\mu(|f|)(x)+H_\mu^*(|f|)(x)+|\mathcal{H}_{\mathrm{loc},\mu}^\varepsilon f(x)|+T_\mu(|f|)(x)),\tag{32}$$

where

$$\begin{aligned}R_{\mu,\varepsilon}^{(k)}f(x)&=\int_{|x-y|>\varepsilon}f(y)R_\mu^{(k)}(x,y)y^{2\mu+1}\,dy,\\\mathcal{H}_{\mathrm{loc},\mu}^\varepsilon f(x)&=\int_{\substack{x/2<y<2x\\|x-y|>\varepsilon}}\frac{(xy)^{-\mu-1/2}}{x-y}f(y)y^{2\mu+1}\,dy,\\T_\mu f(x)&=\int_{x/2}^{2x}\frac{f(y)}{y}\log\bigg(1+\frac{xy}{(x-y)^2}\bigg)\,dy.\end{aligned}$$

By [12, Theorems 1 and 2] and [3, Theorems 1 and 2], both H_μ and H_μ^* map $L^p(x^{2\mu+1}\,dx)$ into itself, $1<p<\infty$, and $L^1(x^{2\mu+1}\,dx)$ into $L^{1,\infty}(x^{2\mu+1}\,dx)$ boundedly.

The same boundedness properties hold for the maximal operator $\mathcal{H}^*_{\mathrm{loc},\mu}$ defined by

$$\mathcal{H}^*_{\mathrm{loc},\mu} f = \sup_{\varepsilon>0} |\mathcal{H}^\varepsilon_{\mathrm{loc},\mu} f|.$$

For T_μ, proceeding as in [13] and [2], the same boundedness properties are obtained. Thus we see that the maximal operator

$$R^{(k)}_{\mu,*} f = \sup_{\varepsilon>0} |R^{(k)}_{\mu,\varepsilon} f|$$

is bounded from $L^p(x^{2\mu+1}\,dx)$ into itself, for $1<p<\infty$, and from $L^1(x^{2\mu+1}\,dx)$ into $L^{1,\infty}(x^{2\mu+1}\,dx)$. Hence, the existence of the principal value in (4) for $f\in L^p(x^{2\mu+1}\,dx)$, $1\le p<\infty$, follows from Lemma 31 in a standard way by using density arguments.

L^p-boundedness of the principal-value operator $R^{(k)}_\mu$ can be obtained by using again the corresponding properties for the Hilbert transform and the Hardy-type operators.

4. Weighted inequalities

Further, we analyze the boundedness of the Riesz transforms $R^{(k)}_\mu$ on weighted L^p-spaces. Our next objective is to obtain the class of weights having good L^p-behavior for $R^{(k)}_\mu$. Let us consider the weights introduced in [2]. A nonnegative measurable function ω on $(0,\infty)$ is in $A_{p,\mu}$, where $1<p<\infty$, provided that there exists $C>0$ such that, for every $0<a<b<\infty$,

$$\int_a^b \omega(t) t^p\,dt \biggl(\int_a^b \omega(t)^{-1/(p-1)} t^{p(2\mu+1)/(p-1)}\,dt\biggr)^{p-1} \le C(b^{2\mu+3}-a^{2\mu+3})^p.$$

In the case $p=1$, we say that a nonnegative measurable function ω on $(0,\infty)$ is in $A_{1,\mu}$, when for some (equivalently, for all) $\varepsilon>0$, there exists $C_\varepsilon>0$ such that, for every $0<a<b<\infty$,

$$\sup_{t\in(a,b)} \biggl(\frac{t^{\mu-1/2}}{\omega(t)}\biggr) \int_a^b \Bigl(\frac{a}{s}+\frac{s}{b}\Bigr)^{\mu+3/2+\varepsilon} \omega(s) s^{-\mu-1/2}\,ds \le C_\varepsilon \frac{b^{2\mu+3}-a^{2\mu+3}}{(ab)^{\mu+3/2}}.$$

The measure $x^{2\mu+1}\,dx$ has the doubling property on $(0,\infty)$ with respect to the usual Euclidean metric $d(x,y)=|x-y|$, $x,y\in(0,\infty)$. We denote by $\mathcal{A}^\mu_p$, $1\le p<\infty$, the Muckenhoupt class of weights associated with the measure $x^{2\mu+1}\,dx$ on $(0,\infty)$, i.e. the class of nonnegative measurable functions ω on $(0,\infty)$ such that there exists

$C>0$ satisfying, for every $0\leq a<b<\infty$,

$$\int_a^b \omega(t)t^{2\mu+1}\,dt\biggl(\int_a^b \omega(t)^{-1/(p-1)}t^{2\mu+1}\,dt\biggr)^{p-1} \leq C(b^{2(\mu+1)}-a^{2(\mu+1)})^p,$$

in the case $1<p<\infty$, and

$$\int_a^b \omega(t)t^{2\mu+1}\,dt \leq C(b^{2(\mu+1)}-a^{2(\mu+1)}) \inf_{a<t<b} \omega(t)$$

for $p=1$.

In the following we prove that if $\omega\in\mathcal{A}_p^\mu$ then $x^{2\mu+1}\omega\in A_{p,\mu}$.

Proposition 33. *Let* $1\leq p<\infty$ *and let* $\tilde{\mathcal{A}}_p^\mu=\{\widetilde{\omega}(x)=x^{2\mu+1}\omega(x):\omega\in\mathcal{A}_p^\mu\}$. *Then,* $\tilde{\mathcal{A}}_p^\mu\subset A_{p,\mu}$.

Proof. Assume that ω belongs to $\mathcal{A}_p^\mu$. Then the measures $\omega(t)t^{2\mu+1}\,dt$ and $\omega(t)^{-1/(p-1)}t^{2\mu+1}\,dt$ satisfy the doubling condition with respect to d.

Suppose that $1<p<\infty$. Then, if $0\leq a<b<\infty$,

$$\begin{aligned}\int_a^b \omega(t)t^p t^{2\mu+1}\,dt&\biggl(\int_a^b (\omega(t)t^{2\mu+1})^{-1/(p-1)}t^{p(2\mu+1)/(p-1)}\,dt\biggr)^{p-1}\\ &\leq Cb^p\int_a^b \omega(t)t^{2\mu+1}\,dt\biggl(\int_a^b \omega(t)^{-1/(p-1)}t^{2\mu+1}\,dt\biggr)^{p-1} \leq Cb^p(b^{2\mu+2}-a^{2\mu+2})^p,\end{aligned}$$

where in the last inequality we have used the Muckenhoupt $\mathcal{A}_p^\mu$-condition. It is clear that $b(b^{2\mu+2}-a^{2\mu+2})\leq b^{2\mu+3}-a^{2\mu+3}$ and the proof finishes in this case.

We now turn to the case $p=1$. Assume that $\omega\in\mathcal{A}_1^\mu$. Since the conditions are dilatation invariant, it suffices to prove the required inequality for $a=1$ and $b>1$. We shall consider two subcases.

Assume that $b\geq 2$. If $1<b<2$ we can proceed in a similar way. We need to show that

$$\sup_{t\in(1,b)}\biggl(\frac{t^{\mu-1/2}}{w(t)t^{2\mu+1}}\biggr)\int_1^b\biggl(\frac{1}{s}+\frac{s}{b}\biggr)^{\mu+3/2+\varepsilon}\omega(s)s^{2\mu+1}s^{-\mu-1/2}\,ds \leq C\frac{b^{2\mu+3}-1}{b^{\mu+3/2}}\sim b^{\mu+3/2},$$

with $\varepsilon>0$. We split the integral in the left-hand side into two integrals extended over $(1,\sqrt{b})$ and $(\sqrt{b},b)$, respectively. Then

$$\begin{aligned}\sup_{t\in(1,b)}\biggl(\frac{t^{\mu-1/2}}{w(t)t^{2\mu+1}}\biggr)&\int_{\sqrt{b}}^b\biggl(\frac{1}{s}+\frac{s}{b}\biggr)^{\mu+3/2+\varepsilon}\omega(s)s^{2\mu+1}s^{-\mu-1/2}\,ds\\ &\leq C\sup_{t\in(1,b)}\biggl(\frac{1}{w(t)t^{\mu+3/2}}\biggr)\int_{\sqrt{b}}^b\biggl(\frac{s}{b}\biggr)^{\mu+3/2+\varepsilon}\omega(s)s^{2\mu+1}s^{-\mu-1/2}\,ds\end{aligned}$$

$$\begin{aligned}
&\leq C \sup_{t\in(1,b)} \left(\frac{1}{w(t)}\right) \int_{\sqrt{b}}^{b} \omega(s) s^{2\mu+1} \frac{s^{1+\varepsilon}}{b^{1+\varepsilon}} b^{-\mu-1/2}\, ds \\
&\leq C \sup_{t\in(1,b)} \left(\frac{1}{w(t)}\right) \int_{\sqrt{b}}^{b} \omega(s) s^{2\mu+1} b^{-\mu-1/2}\, ds \\
&\leq C b^{\mu+3/2}.
\end{aligned}$$

In the last inequality we have used the $\mathcal{A}_1^\mu$-condition.

To estimate the part that consists of the integral over $(1,\sqrt{b})$ we proceed as follows. Let β be the positive integer such that $2^\beta \leq \sqrt{b} < 2^{\beta+1}$. We divide the interval $(1,b)$ into intervals $I_0=(1,\sqrt{b})$, $I_1=(\sqrt{b}, 2\sqrt{b})$, ..., $I_{\beta-1}=(2^{\beta-2}\sqrt{b}, 2^{\beta-1}\sqrt{b})$, $I_\beta=(2^{\beta-1}\sqrt{b}, b)$. Let $k\in\{0,...,\beta\}$ be such that

$$\sup_{t\in(1,b)} \frac{1}{\omega(t)t^{\mu+3/2}} = \sup_{t\in I_k} \frac{1}{\omega(t)t^{\mu+3/2}}.$$

We assume firstly that $k>0$. Then

$$\begin{aligned}
L &= \sup_{t\in(1,b)} \left(\frac{t^{\mu-1/2}}{w(t)t^{2\mu+1}}\right) \int_1^{\sqrt{b}} \left(\frac{1}{s}+\frac{s}{b}\right)^{\mu+3/2+\varepsilon} \omega(s) s^{2\mu+1} s^{-\mu-1/2}\, ds \\
&\leq C \sup_{t\in I_k} \left(\frac{1}{w(t)t^{\mu+3/2}}\right) \int_1^{\sqrt{b}} \left(\frac{1}{s}\right)^{\mu+3/2+\varepsilon} \omega(s) s^{2\mu+1} s^{-\mu-1/2}\, ds \\
&\leq C \sup_{t\in I_k} \left(\frac{1}{w(t)}\right) (2^k\sqrt{b})^{-\mu-3/2} \int_1^{\sqrt{b}} \omega(s) s^{-1-\varepsilon}\, ds.
\end{aligned}$$

By the doubling property of $\omega(t)t^{2\mu+1}\,dt$ with respect to d,

$$\int_1^{\sqrt{b}} \omega(s) s^{-1-\varepsilon}\, ds \leq \int_1^{\sqrt{b}} \omega(s) s^{2\mu+1}\, ds \leq C \int_{I_k} \omega(s) s^{2\mu+1}\, ds.$$

Hence

$$L \leq C \sup_{t\in I_k} \left(\frac{1}{w(t)}\right) (2^k\sqrt{b})^{-\mu-3/2} \int_{I_k} \omega(s) s^{2\mu+1}\, ds.$$

Recalling that $2^k \leq C\sqrt{b}$ and using the $\mathcal{A}_1^\mu$-condition we get that

$$L \leq C(2^k\sqrt{b})^{-\mu-3/2} (2^k\sqrt{b})^{2\mu+2} \leq C b^{\mu+3/2}.$$

If $k=0$ the proof of the needed inequality is analogous and simpler than in the previous case.

Finally, when $1<b<2$ the proof of the desired inequality can be proved in a simpler way following the same procedure that we have employed above. □

Note that the inclusion in Proposition 33 is strict. Indeed, assume that $p>1$. According to [2, p. 16], $\omega_\alpha(t)=t^\alpha$ is in $A_{p,\mu}$ when $-p-1<\alpha<(2\mu+2)p-1$. However, if $\alpha=-2\mu-\frac{5}{2}$, $\omega_\alpha\notin\mathcal{A}_p^\mu$ and $w_{\alpha+2\mu+1}\in A_{p,\mu}$.

In the next theorem, we show that the class $A_{p,\mu}$ of weights is in some sense the optimal one, since we find that $\omega\in A_{p,\mu}$ is also necessary in order to have weighted inequalities for $R_\mu^{(k)}$ for all k odd such that $k<2\mu+2$.

Theorem 34. *Let $k\in\mathbb{N}$ and $1\le p<\infty$.*

(i) *If $\omega\in A_{p,\mu}$ then $R_\mu^{(k)}$ defines a bounded operator from $L^p(\omega(x)\,dx)$ into itself, $1<p<\infty$, and from $L^1(\omega(x)\,dx)$ into $L^{1,\infty}(\omega(x)\,dx)$.*

(ii) *If k is odd, $k<2\mu+2$ and $R_\mu^{(k)}$ maps $L^p(\omega(x)\,dx)$ boundedly into itself, $1<p<\infty$ (respectively, $L^1(\omega(x)\,dx)$ into $L^{1,\infty}(\omega(x)\,dx)$), then $\omega\in A_{p,\mu}$ (respectively, $\omega\in A_{1,\mu}$).*

Proof. The proof of Theorem 34 is a consequence of Propositions 24 and 35 (stated and proved below), by proceeding as in the proof of [2, Theorems 1 and 2]. □

In the next proposition new estimates for the kernel $R_\mu^{(k)}(x,y)$ are given, extending the ones stated in Proposition 24. The technique we use in the proof of this result is different from the one employed in [10, Lemma 2.1] and [11, Theorem 2.1].

Proposition 35. *Let $k,l\in\mathbb{N}$. There exist $\alpha,b>1$ such that*

(i) *if $y>bx$, then $\alpha^{-1}\le y^{2\mu+3}R_\mu^{(k)}(x,y)/x\le\alpha$ if k is odd and $|y^{2\mu+2}R_\mu^{(k)}(x,y)|\le\alpha(x/y)^l$ if k is even.*

(ii) *If $0<y<x/b$ and $k<2\mu+2$, then $\alpha^{-1}\le(-1)^k x^{2\mu+2}R_\mu^{(k)}(x,y)\le\alpha$.*

Proof. We begin by expressing the kernel $R_\mu^{(k)}(x,y)$ in a suitable way. By applying the change of variables $x=uy$ and $t=vy$ in the formula (6) for $P_\mu(t,x,y)$ and recalling (23), we can write

$$\Gamma(k)R_\mu^{(k)}(x,y)=y^{-2\mu-2}\left(\int_0^\infty v^{k-1}\frac{d^k}{du^k}P_\mu(v,u,1)\,dv\right)\bigg|_{u=x/y}=y^{-2\mu-2}T_k(u)|_{u=x/y}.$$

Note that T_k is an infinitely differentiable function on $\mathbb{R}$. Moreover, since $z^{-\mu}J_\mu(z)$ is an even function, $P_\mu(v,u,1)$ is also an even function of u. Hence, T_k is an even function (respectively, odd) provided that k is even (respectively, odd). Thus, to prove (i) is equivalent to show that $T_k'(0)>0$, when k is odd, and $T_k^{(l)}(0)=0$, $l\in\mathbb{N}$, $l\ge1$, if k is even.

The following expression for $(d^n/du^n)T_k$, $n\in\mathbb{N}$, will be useful later. We can find $c_j>0$ such that, for every $u>0$

$$\frac{d^n}{du^n}T_k(u)=\sum_{j=[(k+n+1)/2]}^{k+n}(-1)^j c_j u^{2j-k-n}\int_0^\infty v^{k-1}\times\int_0^\infty e^{-vz}(uz)^{-\mu-j}J_{\mu+j}(uz)z^{-\mu}J_\mu(z)z^{2\mu+2j+1}\,dz\,dv. \tag{36}$$

Indeed, we only must take into account that for every $l\in\mathbb{N}$ we can write

$$\frac{d^l}{du^l}=\sum_{j=[(l+1)/2]}^{l}c_{j,l}u^{2j-l}\left(\frac{1}{u}\frac{d}{du}\right)^j, \tag{37}$$

for suitable $c_{j,l}\in\mathbb{R}$, with $c_{[(l+1)/2],l}>0$, and that by [18, §5.1 (7)]

$$\left(\frac{1}{u}\frac{d}{du}\right)^l[(uz)^{-\mu}J_\mu(uz)]=(-1)^l z^{2l}(uz)^{-\mu-l}J_{\mu+l}(uz). \tag{38}$$

Let us first prove (i) for odd k. As was mentioned, in this case T_k is an odd function and consequently $T_k(0)=0$. Hence, in order to prove that $T_k(u)\sim u$ near the origin we have to see that $(d/du)T_k(0)\neq 0$. By using (36) for $n=1$ and observing that the function $z^{-\sigma}J_\sigma(z)$ takes the value $A_\sigma=2^{-\sigma}\Gamma(\sigma+1)^{-1}$ when $z=0$, we get that

$$\begin{aligned}\frac{d}{du}T_k(0)&=B_\mu\int_0^\infty v^{k-1}\int_0^\infty e^{-vz}z^{-\mu}J_\mu(z)z^{2\mu+k+2}\,dz\,dv\\&=D_\mu\int_0^\infty v^{k-1}\frac{d^{k+1}}{dv^{k+1}}\frac{v}{(1+v^2)^{\mu+3/2}}\,dv,\end{aligned} \tag{39}$$

where $B_\mu=(-1)^{(k+1)/2}c_{(k+1)/2}A_{\mu+(k+1)/2}$ and $D_\mu=B_\mu 2^{\mu+1}\Gamma(\mu+3/2)\pi^{-1/2}$. In the last equality we have used that the integral with respect to z is $\mathcal{L}(z^{k+\mu+2}J_\mu(z))$, where $\mathcal{L}$ denotes the Laplace transform, and we have applied [6, 6.623(2)]. After integrating by parts $k-1$ times, we have that

$$\frac{d}{du}T_k(0)=D_\mu\left(\sum_{l=0}^{k-1}(-1)^{l+1}\frac{\Gamma(k)}{\Gamma(l+1)}v^l\frac{d^{l+1}}{du^{l+1}}\frac{v}{(1+v^2)^{\mu+3/2}}\bigg|_{v=0}^{\infty}\right).$$

Denote by g_l, $l=0,...,k-1$, the function

$$g_l(v)=v^l\frac{d^{l+1}}{du^{l+1}}\frac{v}{(1+v^2)^{\mu+3/2}}.$$

Straightforward manipulations allow us to see that $g_l(v)=O(v^{-2\mu-3})$, when $v\to\infty$, for each $l=0,...,k-1$, that $g_l(0)=0$, $l=1,...,k-1$, and that $g_0(0)=1$. Hence,

$$\frac{d}{du}T_k(0)=-\Gamma(k)D_\mu.$$

Therefore, if k is odd, there exists $b>1$ sufficiently large for which $y^{2\mu+2}R_\mu^{(k)}(x,y)\sim x/y$, when $bx<y$.

Let us consider the case of even k in (i). It suffices to prove in this case that $(d^n/du^n)T_k(0)=0$ for every $n\in\mathbb{N}$. Since T_k is now an even function, it follows that $(d^n/du^n)T_k(0)=0$, provided that n is odd. Suppose that n is even. By (36) and proceeding as to get (39), we find that

$$\begin{aligned}\frac{d^n}{du^n}T_k(0)&=B_{\mu,n}\int_0^\infty v^{k-1}\int_0^\infty e^{-vz}z^{-\mu}J_\mu(z)z^{2\mu+k+n+1}\,dz\,dv\\&=D_{\mu,n}\int_0^\infty v^{k-1}\frac{d^{k+n}}{dv^{k+n}}\frac{v}{(1+v^2)^{\mu+3/2}}\,dv,\end{aligned}$$

for a certain coefficient $B_{\mu,n}$, where $D_{\mu,n}=B_{\mu,n}2^{\mu+1}\Gamma\big(\mu+\frac{3}{2}\big)\pi^{-1/2}$ in the last equality. After integrating by parts $k-1$ times we get that

$$\frac{d^n}{du^n}T_k(0)=D_{\mu,n}\left(\sum_{l=0}^{k-1}(-1)^{l+1}\frac{\Gamma(k)}{\Gamma(l+1)}v^l\frac{d^{l+n}}{du^{l+n}}\frac{v}{(1+v^2)^{\mu+3/2}}\bigg|_{v=0}^{\infty}\right).$$

If

$$h_l(v)=v^l\frac{d^{l+n}}{du^{l+n}}\frac{v}{(1+v^2)^{\mu+3/2}},\quad l=0,...,k-1,$$

in the same way as before we can see that $h_l(v)=O(v^{-2\mu-2-n})$, when $v\to\infty$, and that $h_l(0)=0$, $l=1,...,k-1$. Moreover, since $H(v)=v/(1+v^2)^{\mu+3/2}$ is an odd function and n is even, $h_0(v)$ is an odd function and therefore $h_0(0)=0$. We conclude then that $(d^n/du^n)T_k(0)=0$.

Let us now prove (ii), thus in the sequel $k<2\mu+2$. We rewrite the kernel $R_\mu^{(k)}(x,y)$ in the following way

$$\begin{aligned}R_\mu^{(k)}(x,y)=&\sum_{j=[(k+1)/2]}^{k}(-1)^jc_{j,k}x^{2j-k}\int_0^\infty t^{k-1}\\&\times\int_0^\infty e^{-tw}(xw)^{-\mu-j}J_{\mu+j}(xw)(yw)^{-\mu}J_\mu(yw)w^{2\mu+2j+1}\,dw\,dt,\end{aligned}$$

where $c_j>0$. Here we have used (37) and (38). By performing the changes of variables $z=xw$ and $t=xv$ we get that

$$x^{2\mu+2}R_\mu^{(k)}(x,y)=\sum_{j=[(k+1)/2]}^{k}(-1)^j c_{j,k}\int_0^\infty v^{k-1} \times\int_0^\infty e^{-vz}z^{-\mu-j}J_{\mu+j}(z)\left(\frac{y}{x}z\right)^{-\mu}J_\mu\left(\frac{y}{x}z\right)z^{2\mu+2j+1}\,dz\,dv.$$

Let us denote by S_k the function

$$S_k(u)=\int_0^\infty v^{k-1}\int_0^\infty e^{-vz}\frac{d^k}{dz^k}[z^{-\mu}J_\mu(z)](uz)^{-\mu}J_\mu(uz)z^{2\mu+k+1}\,dz\,dv. \tag{40}$$

By taking into account (37) and (38) it is easy to see that $x^{2\mu+2}R_\mu^{(k)}(x,y)=S_k(y/x)$. To prove (ii) it is then sufficient to see that $S_k(0)>0$, if k is even, and $S_k(0)<0$, when k is odd. Taking $u=0$ in (40) we have

$$S_k(0)=\frac{1}{2^\mu\Gamma(\mu+1)}\int_0^\infty v^{k-1}\int_0^\infty e^{-vz}\frac{d^k}{dz^k}[z^{-\mu}J_\mu(z)]z^{2\mu+k+1}\,dz\,dv. \tag{41}$$

After integrating by parts k times in the last integral we get that

$$\begin{aligned}\int_0^\infty & e^{-vz}\frac{d^k}{dz^k}[z^{-\mu}J_\mu(z)]z^{2\mu+k+1}\,dz\\ &=\sum_{j=0}^{k-1}(-1)^j\frac{d^j}{dz^j}[e^{-vz}z^{2\mu+k+1}]\frac{d^{k-j-1}}{dz^{k-j-1}}[z^{-\mu}J_\mu(z)]\bigg|_{z=0}^{\infty}\\ &\quad+(-1)^k\int_0^\infty z^{-\mu}J_\mu(z)\frac{d^k}{dz^k}[e^{-vz}z^{2\mu+k+1}]\,dz, \qquad v\in(0,\infty).\end{aligned}$$

Since $z^{-\sigma}J_\sigma(z)$, $\sigma\geq-\frac{1}{2}$, is a bounded function on $(0,\infty)$, it is not difficult to see, by taking into account (38), that for $v\in(0,\infty)$,

$$\begin{aligned}\int_0^\infty & e^{-vz}\frac{d^k}{dz^k}[z^{-\mu}J_\mu(z)]z^{2\mu+k+1}\,dz\\ &=(-1)^k\int_0^\infty z^{-\mu}J_\mu(z)\frac{d^k}{dz^k}[e^{-vz}z^{2\mu+k+1}]\,dz\\ &=(-1)^k\sum_{j=0}^{k}(-1)^j\binom{k}{j}\frac{\Gamma(2\mu+k+2)}{\Gamma(2\mu+j+2)}v^j\int_0^\infty z^{-\mu}J_\mu(z)e^{-vz}z^{2\mu+j+1}\,dz.\end{aligned}$$

Let $b_j=(-1)^j\binom{k}{j}/\Gamma(2\mu+j+2)$, $j=0,...,k$. By using the Laplace transform $\mathcal{L}$ and [6, 6.623(2)] we can write for $v\in(0,\infty)$,

$$\begin{aligned}\int_0^\infty e^{-vz}\frac{d^k}{dz^k}[z^{-\mu}J_\mu(z)]z^{2\mu+k+1}\,dz\\ &=(-1)^k\Gamma(2\mu+k+2)\sum_{j=0}^k b_j v^j\mathcal{L}(z^{j+\mu+1}J_\mu)(v)\\ &=\frac{(-1)^k2^{\mu+1}\Gamma(2\mu+k+2)\Gamma\big(\mu+\frac{3}{2}\big)}{\sqrt{\pi}}\sum_{j=0}^k(-1)^j b_j v^j\frac{d^j}{dv^j}\bigg(\frac{v}{(1+v^2)^{\mu+3/2}}\bigg).\end{aligned}$$

By inserting this integral into (41) we infer that

$$S_k(0)=\frac{(-1)^k2\Gamma(2\mu+k+2)\Gamma\big(\mu+\frac{3}{2}\big)}{\sqrt{\pi}\Gamma(\mu+1)}\sum_{j=0}^k(-1)^j b_j\int_0^\infty v^{k+j-1}\frac{d^j}{dv^j}\bigg(\frac{v}{(1+v^2)^{\mu+3/2}}\bigg)\,dv.$$

For each $j=0,...,k$ we analyze the integral

$$I_j=\int_0^\infty v^{k+j-1}\frac{d^j}{dv^j}\bigg(\frac{v}{(1+v^2)^{\mu+3/2}}\bigg)\,dv.$$

Let $j=1,...,k$. If we integrate by parts j times we get that

$$\begin{aligned}I_j&=\sum_{l=1}^j(-1)^{j-l}a_{l,j}v^{k+l-1}\frac{d^{l-1}}{dv^{l-1}}\bigg(\frac{v}{(1+v^2)^{\mu+3/2}}\bigg)\bigg|_{v=0}^\infty\\ &\quad+(-1)^j a_{0,j}\int_0^\infty\frac{v^k}{(1+v^2)^{\mu+3/2}}\,dv.\end{aligned}$$

Here $a_{l,j}=\Gamma(k+j)/\Gamma(k+l)$, $0\le l\le j$. Since $k<2\mu+2$, for each $l=1,...,j$, the function

$$q_l(v)=v^{k+l-1}\frac{d^{l-1}}{dv^{l-1}}\bigg(\frac{v}{(1+v^2)^{\mu+3/2}}\bigg)$$

satisfies that $q_l(0)=0$ and $q_l(v)=O(v^{k-2\mu-2})$, as $v\to\infty$. Therefore,

$$I_j=(-1)^j a_{0,j}\int_0^\infty\frac{v^k}{(1+v^2)^{\mu+3/2}}\,dv,\quad j=0,...,k,$$

where $a_{0,j}=\Gamma(k+j)/\Gamma(k)$, $j=0,...,k$. Then, we can write

$$S_k(0)=\frac{(-1)^k2\Gamma(2\mu+k+2)\Gamma\big(\mu+\frac{3}{2}\big)}{\sqrt{\pi}\Gamma(\mu+1)}\int_0^\infty\frac{v^k}{(1+v^2)^{\mu+3/2}}\,dv\sum_{j=0}^k b_j a_{0,j}.$$

Since $\sum_{j=0}^k b_j a_{0,j}=\Gamma(k)^{-1}\sum_{j=0}^k(-1)^j\binom{k}{j}\Gamma(k+j)/\Gamma(2\mu+2+j)$ and $2\mu+2>k$, according to Lemma 42 below we conclude that $S_k(0)>0$, when k is even, and $S_k(0)<0$, if k is odd. □

Lemma 42. *Let $k \in \mathbb{N}$ and f_k the function defined by*

$$f_k(x) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} \frac{\Gamma(k+j)}{\Gamma(x+j)}, \quad x > 0.$$

Then, for every $l = 1, ..., k$, $f_k(l) = 0$ and $f_k(x) \neq 0$, when $x \notin \{1, ..., k\}$. Moreover, $f_k(x) > 0$, for $x > k$.

Proof. We can write

$$f_k(x) = \frac{\Gamma(2k)}{\Gamma(x+k)} \sum_{j=0}^{k} (-1)^j \binom{k}{j} \frac{(x+k-1)(x+k-2)...(x+j)}{(2k-1)(2k-2)...(k+j)} = \frac{\Gamma(2k)}{\Gamma(x+k)} p_k(x),$$

where p_k is a polynomial of degree k satisfying that $\lim_{x\to\infty} p_k(x) = \infty$. Hence, since $\Gamma(2k)/\Gamma(x+k) > 0$, for $x > 0$, and p_k has exactly k complex roots, the statement of the lemma will be established as soon as we prove that $p_k(l) = 0$, $l = 1, ..., k$.

Let $l \in \{1, ..., k\}$. We observe that

$$f_k(l) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} \frac{\Gamma(k+j)}{\Gamma(l+j)} = \sum_{i=0}^{k-l} a_l(i) \sum_{j=0}^{k} (-1)^j \binom{k}{j} j^i,$$

for certain $\{a_l(i)\}_{i=0}^{k-l} \subset \mathbb{N}$, where we use the convention that $0^0 = 1$. Therefore it is sufficient to prove that

$$H_k(i) := \sum_{j=0}^{k} (-1)^j \binom{k}{j} j^{\,i} = 0, \quad i = 0, ..., k-1. \tag{43}$$

We proceed by induction on k. Consider first $k = 1$. In this case it is clear that (43) is satisfied. Assume now that for a given $k \in \mathbb{N}$, $k \geq 1$, $H_k(i) = 0$, for each $i = 0, ..., k-1$, and let us show that $H_{k+1}(i) = 0$, $i = 0, ..., k$. It is known that

$$H_{k+1}(0) = \sum_{j=0}^{k+1} (-1)^j \binom{k+1}{j} = 0.$$

Take now $i \in \{1, ..., k\}$. Since $\binom{m}{n} = (m/n)\binom{m-1}{n-1}$, for $m \geq n \geq 1$, we get that

$$\begin{aligned} H_{k+1}(i) &= \sum_{j=0}^{k+1} (-1)^j \binom{k+1}{j} j^i \\ &= \sum_{j=1}^{k+1} (-1)^j \binom{k+1}{j} j^i \end{aligned}$$

$$
\begin{aligned}
&= (k+1)\sum_{j=1}^{k+1}(-1)^j\binom{k}{j-1}j^{i-1} \\
&= -(k+1)\sum_{j=0}^{k}(-1)^j\binom{k}{j}(j+1)^{i-1} \\
&= -(k+1)\sum_{r=0}^{i-1}\binom{i-1}{r}\sum_{j=0}^{k}(-1)^j\binom{k}{j}j^r \\
&= -(k+1)\sum_{r=0}^{i-1}\binom{i-1}{r}H_k(r).
\end{aligned}
$$

The induction hypothesis allows us to conclude that $H_{k+1}(i)=0$, $i=1,...,k$, and the lemma is thus proved. □

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