Ark. Mat., **47** (2009), 295–312
DOI: 10.1007/s11512-008-0075-z

Schatten–von Neumann properties for Fourier integral operators with non-smooth symbols, I

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Abstract. We consider Fourier integral operators with symbols in modulation spaces and non-smooth phase functions whose second orders of derivatives belong to certain types of modulation space. We prove continuity and Schatten–von Neumann properties of such operators when acting on L^2.

0. Introduction

In [5], A. Boulkhemair considers a certain class of Fourier integral operators where the corresponding symbols are defined without any explicit regularity assumptions and with only small regularity assumptions on the phase functions. The symbol class here is, in the present paper, denoted by $M^{\infty,1}$ and contains $S^0_{0,0}$, the set of smooth functions which are bounded together with all their derivatives. In time-frequency analysis, the set $M^{\infty,1}$ is known as a particular modulation space. (See e.g. [9], [10] and [13], or the definition below.) Boulkhemair then proves that such operators are uniquely extendible to continuous operators on L^2. In particular it follows that pseudo-differential operators with symbols in $M^{\infty,1}$ are L^2-continuous, which was proved by J. Sjöstrand in [21], where it seems that $M^{\infty,1}$ was used for the first time in this context.

More recent contributions to the theory of Fourier integral operators with non-smooth symbols are presented in [17], [18] and [19]. For example, in [18], Ruzhansky and Sugimoto investigate, among other things, L^2-estimates for Fourier integral operators with symbol classes which contain non-smooth functions, e.g. Besov spaces and local Sobolev–Kato spaces.

In this paper we consider Fourier integral operators where the symbol classes are given by $M^{p,q}$, where $p,q\in[1,\infty]$, and with phase functions satisfying similar conditions as in [5]. We discuss continuity of such operators when acting on modulation spaces, and prove Schatten–von Neumann properties when acting on L^2.

In order to be more specific we recall some definitions. Assume that $p,q\in[1,\infty]$. Then the *modulation space* $M^{p,q}(\mathbf{R}^n)$ is the set of all $f\in\mathcal{S}'(\mathbf{R}^n)$ such that

$$\|f\|_{M^{p,q}}\equiv\biggl(\int_{\mathbf{R}^n}\biggl(\int_{\mathbf{R}^n}|\mathcal{F}(f\chi(\,\cdot\,-x))(\xi)|^p\,dx\biggr)^{q/p}\,d\xi\biggr)^{1/q}<\infty \tag{0.1}$$

(with obvious modification when $p=\infty$ or $q=\infty$). Here $\mathcal{F}$ is the *Fourier transform* on $\mathcal{S}'(\mathbf{R}^n)$ which is given by

$$\mathcal{F}f(\xi)=\hat{f}(\xi)\equiv(2\pi)^{-n/2}\int_{\mathbf{R}^n}f(x)e^{-i\langle x,\xi\rangle}\,dx$$

when $f\in\mathcal{S}(\mathbf{R}^n)$, and $\chi\in\mathcal{S}(\mathbf{R}^n)\setminus\{0\}$ is called a *window function* which is kept fixed.

During the last twenty years, modulation spaces have been an active field of research (see e.g. [8], [9], [10], [11], [13], [16], [23] and [26]). They are rather similar to Besov spaces (see [2], [22] and [26] for sharp embeddings) and it has turned out that they are useful to have in background in time-frequency analysis and to some extent also in pseudo-differential calculus.

Next we discuss the definition of Fourier integral operators. For convenience we restrict ourselves to operators which belong to $\mathcal{L}(\mathcal{S}(\mathbf{R}^n),\mathcal{S}'(\mathbf{R}^n))$. Here we let $\mathcal{L}(V_1,V_2)$ denote the set of all linear and continuous operators from V_1 to V_2, when V_1 and V_2 are topological vector spaces. For any appropriate $a\in\mathcal{S}'(\mathbf{R}^{2n+m})$ (the *symbol*) and real-valued $\varphi\in C(\mathbf{R}^{2n+m})$ (the *phase function*), the Fourier integral operator $\operatorname{Op}_\varphi(a)$ is defined by the formula

$$\operatorname{Op}_\varphi(a)f(x)=(2\pi)^{-n}\iint_{\mathbf{R}^{m+n}}a(x,y,\zeta)f(y)e^{i\varphi(x,y,\zeta)}\,dy\,d\zeta, \tag{0.2}$$

when $f\in\mathcal{S}(\mathbf{R}^n)$. Here the integrals should be interpreted in distribution sense, if necessary. By letting $m=n$, and choosing symbols and phase functions in appropriate ways, it follows that the pseudo-differential operator

$$\operatorname{Op}(a)f(x)=(2\pi)^{-n}\iint_{\mathbf{R}^{2n}}a(x,y,\zeta)f(y)e^{i\langle x-y,\zeta\rangle}\,dy\,d\zeta$$

is a special case of Fourier integral operators. Furthermore, if $t\in\mathbf{R}$ is fixed, and a is an appropriate function or distribution on $\mathbf{R}^{2n}$ instead of $\mathbf{R}^{3n}$, then the definition of the latter pseudo-differential operators cover the definition of pseudo-differential operators of the form

$$a_t(x,D)f(x)=(2\pi)^{-n}\iint_{\mathbf{R}^{2n}}a((1-t)x+ty,\zeta)f(y)e^{i\langle x-y,\zeta\rangle}\,dy\,d\zeta. \tag{0.3}$$

On the other hand, in the framework of harmonic analysis it follows that the map $a\mapsto a_t(x,D)$ from $\mathcal{S}(\mathbf{R}^{2n})$ to $\mathcal{L}(\mathcal{S}(\mathbf{R}^n),\mathcal{S}'(\mathbf{R}^n))$ is uniquely extendible to a bijection from $\mathcal{S}'(\mathbf{R}^{2n})$ to $\mathcal{L}(\mathcal{S}(\mathbf{R}^n),\mathcal{S}'(\mathbf{R}^n))$.

In the literature it is usually assumed that a and φ in (0.2) are smooth functions. For example, if $a\in\mathcal{S}(\mathbf{R}^{2n+m})$ and $\varphi\in C^\infty(\mathbf{R}^{2n+m})$ satisfy $\varphi^{(\alpha)}\in S^0_{0,0}(\mathbf{R}^{2n+m})$ for all multi-indices α with $|\alpha|\ge N$ for some integer $N\ge 0$, then it is easily seen that $\mathrm{Op}_\varphi(a)$ is continuous on $\mathcal{S}(\mathbf{R}^n)$ and extends to a continuous map from $\mathcal{S}'(\mathbf{R}^n)$ to $\mathcal{S}(\mathbf{R}^n)$. In [1] it is proved that if $\varphi^{(\alpha)}\in S^0_{0,0}(\mathbf{R}^{2n+m})$ for all multi-indices α with $|\alpha|=2$ and satisfies

$$\left|\det\begin{pmatrix}\varphi''_{x,y} & \varphi''_{x,\zeta}\\ \varphi''_{y,\zeta} & \varphi''_{\zeta,\zeta}\end{pmatrix}\right|\ge d \tag{0.4}$$

for some $d>0$, then the definition of Op_φ extends uniquely to any $a\in S^0_{0,0}(\mathbf{R}^{2n+m})$, and then $\mathrm{Op}_\varphi(a)$ is continuous on $L^2(\mathbf{R}^n)$. Next assume that φ instead satisfies $\varphi^{(\alpha)}\in M^{\infty,1}(\mathbf{R}^{3n})$ for all multi-indices α with $|\alpha|=2$ and that (0.4) holds for some $d>0$. This implies that the condition on φ is relaxed since $S^0_{0,0}\subseteq M^{\infty,1}$. Then Boulkhemair improves the result in [1] by proving that the definition of Op_φ extends uniquely to any $a\in M^{\infty,1}(\mathbf{R}^{2n+m})$, and that $\mathrm{Op}_\varphi(a)$ is still continuous on $L^2(\mathbf{R}^n)$.

In Section 2 we discuss Schatten–von Neumann properties for Fourier integral operators which are related to those which were considered by Boulkhemair. More precisely, we prove that if $p\in[1,\infty]$ and $a\in M^{p,1}(\mathbf{R}^{2n+m})$ then $\mathrm{Op}_\varphi(a)$ belongs to $\mathcal{I}_p$, the set of Schatten–von Neumann operators of order $p\in[1,\infty]$ on $L^2(\mathbf{R}^n)$. Recall that an operator T on $L^2(\mathbf{R}^n)$ is a *Schatten–von Neumann operator* of order p if it is linear and continuous on $L^2(\mathbf{R}^n)$, and satisfies

$$\|T\|_{\mathcal{I}_p}\equiv\sup\biggl(\sum_{j=1}^{\infty}|(Tf_j,g_j)|^p\biggr)^{1/p}<\infty.$$

Here the supremum should be taken over all orthonormal sequences $\{f_j\}_{j=1}^\infty$ and $\{g_j\}_{j=1}^\infty$ in $L^2(\mathbf{R}^n)$.

Furthermore, if $1\le q\le\min(p,p')$, $m=n$ and instead $a(x,y,\zeta)=b(x,\zeta)$ for some $b\in M^{p,q}(\mathbf{R}^{2n})$, and in addition

$$|\det(\varphi''_{y,\zeta})|\ge d \tag{0.5}$$

for some constant $d>0$, then we prove that $\mathrm{Op}_\varphi(a)\in\mathcal{I}_p$. Here and in what follows we let $p'\in[1,\infty]$ denote the conjugate exponent of $p\in[1,\infty]$, i.e. $1/p+1/p'=1$. When proving these results we first prove that they hold in the case $p=1$. The remaining cases are then consequences of Boulkhemair's result, interpolation and duality.

Finally we remark that a continuation of the present paper, which involves discussions of Fourier integral operators in the context of weighted modulation spaces, is under preparation by the authors.

1. Preliminaries

In this section we discuss the basic properties for modulation spaces. The proofs are in many cases omitted since they can be found in [6], [7], [8], [9], [10], [11], [12], [13], [24], [25] and [26].

We start by discussing notation. The duality between a topological vector space and its dual is denoted by $\langle\,\cdot\,,\cdot\,\rangle$. For admissible a and b in $\mathcal{S}'(\mathbf{R}^n)$, we set $(a,b)=\langle a,\overline{b}\rangle$, and it is obvious that $(\,\cdot\,,\cdot\,)$ on L^2 is the usual scalar product.

Next assume that $\mathcal{B}_1$ and $\mathcal{B}_2$ are topological spaces. Then $\mathcal{B}_1\hookrightarrow\mathcal{B}_2$ means that $\mathcal{B}_1$ is continuously embedded in $\mathcal{B}_2$. In the case that $\mathcal{B}_1$ and $\mathcal{B}_2$ are Banach spaces, $\mathcal{B}_1\hookrightarrow\mathcal{B}_2$ is equivalent to $\mathcal{B}_1\subseteq\mathcal{B}_2$ and $\|x\|_{\mathcal{B}_2}\le C\|x\|_{\mathcal{B}_1}$, for some constant $C>0$ which is independent of $x\in\mathcal{B}_1$.

Assume that $p,q\in[1,\infty]$, and that $\chi\in\mathcal{S}(\mathbf{R}^n)\setminus\{0\}$. Then recall that the (*classical*) *modulation space* $M^{p,q}(\mathbf{R}^n)$ is the set of all $f\in\mathcal{S}'(\mathbf{R}^n)$ such that (0.1) holds. We note that the definition of $M^{p,q}(\mathbf{R}^n)$ is independent of the choice of window χ, and that different choices of χ give rise to equivalent norms. (See Proposition 1.1 below.) For conveniency we also set $M^p=M^{p,p}$.

The following proposition is a consequence of well-known facts in [9] and [13]. Recall that we let p' denote the conjugate exponent of p, i.e. $1/p+1/p'=1$ should be fulfilled.

Proposition 1.1. *Assume that* $p,q,p_j,q_j\in[1,\infty]$ *for* $j=1,2$. *Then the following are true*:

(1) *If* $\chi\in M^1(\mathbf{R}^n)\setminus\{0\}$, *then* $f\in M^{p,q}(\mathbf{R}^n)$ *if and only if* (0.1) *holds, that is* $M^{p,q}(\mathbf{R}^n)$ *is independent of the choice of* χ. *Moreover,* $M^{p,q}$ *is a Banach space under the norm in* (0.1), *and different choices of* χ *give rise to equivalent norms*;

(2) *If* $p_1\le p_2$ *and* $q_1\le q_2$ *then*

$$\mathcal{S}(\mathbf{R}^n)\hookrightarrow M^{p_1,q_1}(\mathbf{R}^n)\hookrightarrow M^{p_2,q_2}(\mathbf{R}^n)\hookrightarrow\mathcal{S}'(\mathbf{R}^n);$$

(3) *The* L^2-*product* $(\,\cdot\,,\cdot\,)$ *on* $\mathcal{S}$ *extends to a continuous map from* $M^{p,q}(\mathbf{R}^n)\times M^{p',q'}(\mathbf{R}^n)$ *to* $\mathbf{C}$. *On the other hand, if* $\|a\|=\sup|(a,b)|$, *where the supremum is taken over all* $b\in M^{p',q'}(\mathbf{R}^n)$ *such that* $\|b\|_{M^{p',q'}}\le 1$, *then* $\|\cdot\|$ *and* $\|\cdot\|_{M^{p,q}}$ *are equivalent norms*;

(4) *If* $p,q<\infty$, *then* $\mathcal{S}(\mathbf{R}^n)$ *is dense in* $M^{p,q}(\mathbf{R}^n)$. *The dual space of* $M^{p,q}(\mathbf{R}^n)$ *can be identified with* $M^{p',q'}(\mathbf{R}^n)$ *through the form* $(\,\cdot\,,\cdot\,)$. *Moreover,* $\mathcal{S}(\mathbf{R}^n)$ *is weakly dense in* $M^\infty(\mathbf{R}^n)$.

Proposition 1.1(1) permits us to be rather vague concerning the choice of $\chi\in M^1\setminus\{0\}$ in (0.1). For example, if $C>0$ is a constant and Ω is a subset of $\mathcal{S}'$, then $\|a\|_{M^{p,q}}\le C$ for every $a\in\Omega$, means that the inequality holds for some choice of

$\chi\in M^1\setminus\{0\}$ and every $a\in\Omega$. Evidently, for any other choice of $\chi\in M^1\setminus\{0\}$, a similar inequality is true although C may have to be replaced by a larger constant, if necessary.

It is also convenient to let $\mathcal{M}^{p,q}(\mathbf{R}^n)$ be the completion of $\mathcal{S}(\mathbf{R}^n)$ under the norm $\|\cdot\|_{M^{p,q}}$. Then $\mathcal{M}^{p,q}\subseteq M^{p,q}$ with equality if and only if $p<\infty$ and $q<\infty$. It follows that most of the properties which are valid for $M^{p,q}(\mathbf{R}^n)$ also hold for $\mathcal{M}^{p,q}(\mathbf{R}^n)$.

We also need to use multiplication properties of modulation spaces. The proof of the following proposition is omitted since the result can be found in [9], [10], [25] and [26].

Proposition 1.2. *Assume that $p,p_j,q_j\in[1,\infty]$ for $j=0,...,N$ satisfy*

$$\frac{1}{p_1}+...+\frac{1}{p_N}=\frac{1}{p_0}\quad \textit{and}\quad \frac{1}{q_1}+...+\frac{1}{q_N}=N-1+\frac{1}{q_0}.$$

Then $(f_1,...,f_N)\mapsto f_1...f_N$ from $\mathcal{S}(\mathbf{R}^n)\times...\times\mathcal{S}(\mathbf{R}^n)$ to $\mathcal{S}(\mathbf{R}^n)$ extends uniquely to a continuous map from $M^{p_1,q_1}(\mathbf{R}^n)\times...\times M^{p_N,q_N}(\mathbf{R}^n)$ to $M^{p_0,q_0}(\mathbf{R}^n)$, and

$$\|f_1...f_N\|_{M^{p_0,q_0}}\le C\|f_1\|_{M^{p_1,q_1}}...\|f_N\|_{M^{p_N,q_N}}$$

for some constant C which is independent of $f_j\in M^{p_j,q_j}(\mathbf{R}^n)$, $j=1,...,N$.

Furthermore, if $u_0=0$ when $p<\infty$, $f\in M^{p,1}(\mathbf{R}^n)$, and u and v are entire functions on $\mathbf{C}$ with expansions

$$u(z)=\sum_{k=0}^{\infty}u_kz^k\quad \textit{and}\quad v(z)=\sum_{k=0}^{\infty}|u_k|z^k,$$

then $u(f)\in M^{p,1}(\mathbf{R}^n)$, and

$$\|u(f)\|_{M^{p,1}}\le Cv(C\|f\|_{M^{p,1}}),$$

for some constant C which is independent of $f\in M^{p,1}(\mathbf{R}^n)$.

Remark 1.3. Assume that $p,q,q_1,q_2\in[1,\infty]$. Then the following properties for modulation spaces hold:

(1) If $q_1\le\min(p,p')$ and $q_2\ge\max(p,p')$, then $M^{p,q_1}\subseteq L^p\subseteq M^{p,q_2}$. In particular, $M^2=L^2$;

(2) $M^{p,q}(\mathbf{R}^n)\hookrightarrow C(\mathbf{R}^n)$ if and only if $q=1$;

(3) $M^{1,\infty}$ is a convolution algebra which contains all measures on $\mathbf{R}^n$ with bounded mass;

(4) $M^{p,q}\cap\mathcal{E}'=\mathcal{F}L^q\cap\mathcal{E}'$. Furthermore, if B is a ball with radius r, then

$$C_{r,n}^{-1}\|\hat{f}\|_{L^q}\le\|f\|_{M^{p,q}}\le C_{r,n}\|\hat{f}\|_{L^q},\quad f\in\mathcal{E}'(B),$$

for some constant $C_{r,n}$ which only depends on r and n;

(5) M^p is invariant under Fourier transformation. A similar fact holds for partial Fourier transforms.

(See e.g. [6], [7], [9], [10], [11], [12], [13] and [26].)

For future references we note that the constant $C_{r,n}$ is independent of the center of the ball B in (4) in Remark 1.3.

In our investigations we need the following characterization of modulation spaces.

Proposition 1.4. *Let $\{x_\alpha\}_{\alpha\in I}$ be a lattice in $\mathbf{R}^n$, $B_\alpha=x_\alpha+B$ where $B\subseteq\mathbf{R}^n$ is an open ball, and assume that $f_\alpha\in\mathcal{E}'(B_\alpha)$ for every $\alpha\in I$. Also assume that $p,q\in[1,\infty]$. Then the following are true*:

(1) *if*

$$f=\sum_{\alpha\in I} f_\alpha \quad \textit{and} \quad F(\xi)\equiv\Big(\sum_{\alpha\in I}|\hat f_\alpha(\xi)|^p\Big)^{1/p}\in L^q(\mathbf{R}^n), \tag{1.1}$$

then $f\in M^{p,q}$, and $f\mapsto\|F\|_{L^q}$ defines a norm on $M^{p,q}$ which is equivalent to $\|\cdot\|_{M^{p,q}}$ in (0.1);

(2) *if in addition $\bigcup_{\alpha\in I}B_\alpha=\mathbf{R}^n$, $\chi\in C_0^\infty(B)$ satisfies $\sum_{\alpha\in I}\chi(\,\cdot-x_\alpha)=1$, $f\in M^{p,q}(\mathbf{R}^n)$, and $f_\alpha=f\chi(\,\cdot-x_\alpha)$, then $f_\alpha\in\mathcal{E}'(B_\alpha)$ and* (1.1) *is fulfilled.*

Proof. (1) Assume that $\chi\in C_0^\infty(\mathbf{R}^n)\setminus\{0\}$ is fixed. Since there is a bound of overlapping supports of f_α, we obtain

$$|\mathcal{F}(f\chi(\,\cdot-x))(\xi)|\le\sum_{\alpha\in I}|\mathcal{F}(f_\alpha\chi(\,\cdot-x))(\xi)|\le C\Big(\sum_{\alpha\in I}|\mathcal{F}(f_\alpha\chi(\,\cdot-x))(\xi)|^p\Big)^{1/p}$$

for some constant C. From the support properties of χ, it follows that for some balls B' and $B'_\alpha=x_\alpha+B'$ we get

$$\begin{aligned}\Big(\int_{\mathbf{R}^n}|\mathcal{F}(f\chi(\,\cdot-x))(\xi)|^p\,dx\Big)^{1/p}&\le C_1\Big(\sum_{\alpha\in I}\int_{B'_\alpha}|\mathcal{F}(f_\alpha\chi(\,\cdot-x))(\xi)|^p\,dx\Big)^{1/p}\\&\le C_2\Big(\sum_{\alpha\in I}\int_{B'_\alpha}(|\hat f_\alpha|*|\widehat\chi|)(\xi)^p\,dx\Big)^{1/p}\\&\le C_3\Big(\sum_{\alpha\in I}(|\hat f_\alpha|*|\widehat\chi|)(\xi)^p\Big)^{1/p}\\&\le C_3(F*|\widehat\chi|)(\xi)\end{aligned}$$

for some constants C_1, C_2 and C_3. Here we have used Minkowski's inequality in the last inequality. By applying the L^q-norm and using Young's inequality we get

$$\|f\|_{M^{p,q}} \le C''\big\|F*|\widehat{\chi}|\big\|_{L^q} \le C''\|F\|_{L^q}\|\widehat{\chi}\|_{L^1}.$$

Hence, since we have assumed that $F\in L^q$, it follows that $\|f\|_{M^{p,q}}$ is finite. This proves (1).

The assertion (2) follows immediately from the general theory of modulation spaces. (See e.g. [13] and [14].) The proof is complete. □

Next we discuss (complex) interpolation properties for modulation spaces. Such properties were carefully investigated in [9] for classical modulation spaces, and thereafter extended in several directions in [11], where interpolation properties for coorbit spaces were established. As a consequence of [11] we have the following proposition.

Proposition 1.5. *Assume that* $0<\theta<1$ *and that* $p,q,p_1,p_2,q_1,q_2\in[1,\infty]$ *satisfy*

$$\frac{1}{p}=\frac{1-\theta}{p_1}+\frac{\theta}{p_2} \quad \text{and} \quad \frac{1}{q}=\frac{1-\theta}{q_1}+\frac{\theta}{q_2}.$$

Then

$$(\mathcal{M}^{p_1,q_1}(\mathbf{R}^n),\mathcal{M}^{p_2,q_2}(\mathbf{R}^n))_{[\theta]}=\mathcal{M}^{p,q}(\mathbf{R}^n).$$

Next we recall some facts from Chapter 18 in [15] concerning pseudo-differential operators. Assume that $a\in\mathcal{S}(\mathbf{R}^{2n})$, and that $t\in\mathbf{R}$ is fixed. Then the pseudo-differential operator $a_t(x,D)$ in (0.3) is a linear and continuous operator on $\mathcal{S}(\mathbf{R}^n)$, as remarked in the introduction. For general $a\in\mathcal{S}'(\mathbf{R}^{2n})$, the pseudo-differential operator $a_t(x,D)$ is defined as the continuous operator from $\mathcal{S}(\mathbf{R}^n)$ to $\mathcal{S}'(\mathbf{R}^n)$ with distribution kernel

$$K_{t,a}(x,y)=(2\pi)^{-n/2}(\mathcal{F}_2^{-1}a)((1-t)x+ty,y-x). \tag{1.2}$$

Here $\mathcal{F}_2F$ is the partial Fourier transform of $F(x,y)\in\mathcal{S}'(\mathbf{R}^{2n})$ with respect to y. This definition makes sense, as the mappings $\mathcal{F}_2$ and $F(x,y)\mapsto F((1-t)x+ty,y-x)$ are homeomorphisms on $\mathcal{S}'(\mathbf{R}^{2n})$. We also note that this definition of $a_t(x,D)$ agrees with the operator in (0.3) when $a\in\mathcal{S}(\mathbf{R}^{2m})$.

Furthermore, for any fixed $t\in\mathbf{R}$, the map $a\mapsto a_t(x,D)$ is bijective from $\mathcal{S}'(\mathbf{R}^{2n})$ to $\mathcal{L}(\mathcal{S}(\mathbf{R}^n),\mathcal{S}'(\mathbf{R}^n))$ (see [15]).

In particular, if $a\in\mathcal{S}'(\mathbf{R}^{2m})$ and $s,t\in\mathbf{R}$, then there is a unique $b\in\mathcal{S}'(\mathbf{R}^{2m})$ such that $a_s(x,D)=b_t(x,D)$. By straightforward applications of Fourier's inversion formula, it follows that

$$a_s(x,D)=b_t(x,D) \quad \Longleftrightarrow \quad b(x,\zeta)=e^{i(t-s)\langle D_x,D_\zeta\rangle}a(x,\zeta) \tag{1.3}$$

(cf. Section 18.5 in [15].)

We end this section by recalling some facts on Schatten–von Neumann operators from the introduction, and pseudo-differential operators.

The set $\mathcal{I}_p$ is a Banach space which increases with $p\in[1,\infty]$, and if $p<\infty$, then $\mathcal{I}_p$ is contained in the set of compact operators on L^2. Furthermore, $\mathcal{I}_1$, $\mathcal{I}_2$ and $\mathcal{I}_\infty$ agree with the set of trace-class operators, Hilbert–Schmidt operators and continuous operators on L^2, respectively, with the same norms.

Next we discuss complex interpolation properties of Schatten–von Neumann classes. Let $p,p_1,p_2\in[1,\infty]$ and let $0\le\theta\le1$. Then

$$\mathcal{I}_p=(\mathcal{I}_{p_1},\mathcal{I}_{p_2})_{[\theta]}, \quad \text{when } \frac{1}{p}=\frac{1-\theta}{p_1}+\frac{\theta}{p_2}. \tag{1.4}$$

We refer to [20] for a brief discussion of Schatten–von Neumann operators.

We also recall some facts on Schatten–von Neumann properties in the calculus of pseudo-differential operators. For any $t\in\mathbf{R}$ and $p\in[1,\infty]$, let $s_{t,p}(\mathbf{R}^{2n})$ be the set of all $a\in\mathcal{S}'(\mathbf{R}^{2n})$ such that $a_t(x,D)\in\mathcal{I}_p$. Also set $\|a\|_{s_{t,p}}\equiv\|a_t(x,D)\|_{\mathcal{I}_p}$. By using the fact that $a\mapsto a_t(x,D)$ is a bijective map from $\mathcal{S}'(\mathbf{R}^{2n})$ to $\mathcal{L}(\mathcal{S}(\mathbf{R}^n),\mathcal{S}'(\mathbf{R}^n))$, it follows that the map $a\mapsto a_t(x,D)$ restricts to an isometric bijection from $s_{t,p}(\mathbf{R}^{2n})$ to $\mathcal{I}_p$.

The proof of the following proposition is omitted since it can be found in [26], and to some extent in [14].

Proposition 1.6. *Assume that $p,q_1,q_2\in[1,\infty]$ are such that $q_1\le\min(p,p')$ and $q_2\ge\max(p,p')$. Then the following are true*:

(1) $M^{p,q_1}(\mathbf{R}^{2n})\subseteq s_{t,p}(\mathbf{R}^{2n})\subseteq M^{p,q_2}(\mathbf{R}^{2n})$;

(2) *the operator kernel of $a_t(x,D)$ belongs to $M^p(\mathbf{R}^{2n})$ if and only if the symbol $a\in M^p(\mathbf{R}^{2n})$.*

2. Schatten–von Neumann properties of Fourier integral operators

In this section we discuss Schatten–von Neumann operators for Fourier integral operators with symbols in $M^{p,q}$. In the first part we assume that the symbol $a(x,y,\zeta)$ belongs to $M^{p,1}(\mathbf{R}^{2n+m})$ while in the second part we consider a more tricky case when the symbol is constant with respect to the y-variable, and belongs to $M^{p,q}(\mathbf{R}^{2n})$ with respect to the remaining variables x and ζ.

Certain parts of these investigations depend on the following lemmas.

Lemma 2.1. *Assume that* $f \in M^{\infty,1}(\mathbf{R}^n)$ *and that* $\chi \in C_0^\infty(B)$*, where* B *is the unit ball with center at origin. Then the following are true*:

(1) *if* $t \in [0,1]$*, then* $f(t\,\cdot\,) \in M^{\infty,1}(\mathbf{R}^n)$*, and for some constant* C*, independent of* $f \in M^{\infty,1}$ *and* $t \in [0,1]$,

$$\|f(t\,\cdot\,)\|_{M^{\infty,1}} \le C\|f\|_{M^{\infty,1}};$$

(2) *if*

$$g_{x_0}(x) = \chi(x-x_0)\int_0^1 (1-t) f(t(x-x_0)+x_0)\,dt$$

for some $x_0 \in \mathbf{R}^n$*, then* $g \in M^{\infty,1}$*, and for some constant* C *which is independent of* x_0,

$$\|g_{x_0}\|_{M^{\infty,1}} \le C\|f\|_{M^{\infty,1}}.$$

Proof. The assertion (1) is an immediate consequence of Proposition 3.2 in [5]. (See also [22] for more general dilation properties.) In order to prove (2) we note that the $M^{\infty,1}$-norm of $\chi(\,\cdot\,-x_0)$ is independent of x_0. Hence Proposition 1.2 and the first part of the proposition give

$$\begin{aligned}\|g_{x_0}\|_{M^{\infty,1}} &\le C_1\|\chi(\,\cdot\,-x_0)\|_{M^{\infty,1}}\int_0^1 (1-t)\|f(t\,\cdot\,+(1-t)x_0)\|_{M^{\infty,1}}\,dt\\ &= C_1\|\chi\|_{M^{\infty,1}}\int_0^1 (1-t)\|f(t\,\cdot\,)\|_{M^{\infty,1}}\,dt \le C_2\|f\|_{M^{\infty,1}}\end{aligned}$$

for some constants C_1 and C_2. The proof is complete. □

Lemma 2.2. *Assume that* $B \subseteq \mathbf{R}^n$ *is a ball,* $\varphi \in C^2(\mathbf{R}^n)$ *is real-valued and satisfies* $\varphi^{(\alpha)} \in M^{\infty,1}$ *for all multi-indices* α *with* $|\alpha|=2$*, and that* $f \in M^{1,q}(\mathbf{R}^n) \cap \mathcal{E}'(B)$*. Then* $fe^{i\varphi} \in M^{1,q}(\mathbf{R}^n)$*, and for some constant* C*, which only depends on* n *and the radius of the ball,*

$$\|fe^{i\varphi}\|_{M^{1,q}} \le C\|f\|_{M^{1,q}}\exp(C\|\varphi''\|_{M^{\infty,1}}).$$

Proof. We may assume that B is the unit ball which is centered at origin. By Taylor expansion it follows that $\varphi = \psi_1+\psi_2$, where

$$\psi_1(x) = \varphi(0)+\langle\varphi'(0),x\rangle \quad\text{and}\quad \psi_2(x) = \int_0^1 (1-t)\langle\varphi''(tx);x,x\rangle\,dt.$$

Since multiplications by modulations do not affect the modulation space norms we obtain

$$\|fe^{i\psi_1}\|_{M^{1,q}} = \|f\|_{M^{1,q}}.$$

Furthermore, if $\chi\in C_0^\infty(\mathbf{R}^n)$ satisfies that $\chi(x)=1$ on B, then it follows from Lemma 2.1 that $\|\chi\psi_2\|_{M^{\infty,1}}\le C\|\varphi''\|_{M^{\infty,1}}$. Hence by Proposition 1.2 it follows that $\|e^{i\chi\psi_2}\|_{M^{\infty,1}}\le C\exp(C\|\varphi''\|_{M^{\infty,1}})$ for some constant C. This gives

$$\begin{aligned}\|fe^{i\varphi}\|_{M^{1,q}} = \|(fe^{i\psi_1})e^{i\chi\psi_2}\|_{M^{1,q}} &\le C\|fe^{i\psi_1}\|_{M^{1,q}}\|e^{i\chi\psi_2}\|_{M^{\infty,1}}\\ &\le C_1\|f\|_{M^{1,q}}\exp(C_1\|\varphi''\|_{M^{\infty,1}})\end{aligned}$$

for some constants C and C_1. This proves the assertion. □

We may now prove the following.

Proposition 2.3. *Assume that* $a\in M^1(\mathbf{R}^{2n+m})$, *and that* $\varphi\in C(\mathbf{R}^{2n+m})$ *is real-valued and satisfies* $\varphi^{(\alpha)}\in M^{\infty,1}$ *for all multi-indices* α *with* $|\alpha|=2$. *Then the distribution kernel of the operator* $\operatorname{Op}_\varphi(a)$ *in* (0.2) *belongs to* $M^1(\mathbf{R}^{2n})$.

In particular, $\operatorname{Op}_\varphi(a)\in\mathcal{I}_1$.

For the proof as well as later on, it is convenient to use the notation $X, Y, Z, ...$ for triples of the form $(x,y,\zeta)\in\mathbf{R}^{2n+m}$.

Proof. Let $\{X_\alpha\}_{\alpha\in I}$ be a lattice in $\mathbf{R}^{2n+m}$, $\chi\in C_0^\infty(\mathbf{R}^{2n+m})$ be such that $\sum_{\alpha\in I}\chi(\,\cdot\,-X_\alpha)=1$, and let $a_\alpha=a\cdot\chi(\,\cdot\,-X_\alpha)$. From Lemma 2.1 it follows that

$$\|ae^{i\varphi}\|_{M^1}\le\sum_{\alpha\in I}\|a_\alpha e^{i\varphi}\|_{M^1}\le C\biggl(\sum_{\alpha\in I}\|a_\alpha\|_{M^1}\biggr)\exp(C_1\|\varphi''\|_{M^{\infty,1}}).$$

Furthermore, by Remark 1.3(4) and Proposition 1.4 we have

$$\sum_{\alpha\in I}\|a_\alpha\|_{M^1}\le C\sum_{\alpha\in I}\|\hat a_\alpha\|_{L^1}\le C'\|a\|_{M^1}$$

for some constants C and C'. Summing up, we have proved that

$$\|ae^{i\varphi}\|_{M^1}\le C\|a\|_{M^1}\exp(C\|\varphi''\|_{M^{\infty,1}}) \tag{2.1}$$

for some constant C, which in particular shows that $ae^{i\varphi}\in M^1(\mathbf{R}^{2n+m})$. Hence $ae^{i\varphi}\in M^1$.

Next we recall that the map

$$f(x_1,x_2)\longmapsto\int_{\mathbf{R}^{n_2}}f(x_1,x_2)\,dx_2$$

is continuous from $M^1(\mathbf{R}^{n_1+n_2})$ to $M^1(\mathbf{R}^{n_1})$ when $x_j\in\mathbf{R}^{n_j}$. (See e.g. [13] and [25].) Hence if

$$K(x,y)\equiv(2\pi)^{-n}\int_{\mathbf{R}^m}a(x,y,\zeta)e^{i\varphi(x,y,\zeta)}\,d\zeta$$

is the kernel of $\operatorname{Op}_\varphi(a)$, it follows that $K\in M^1(\mathbf{R}^{2n})$.

The last part of the assertion is now a consequence of Proposition 1.6, and the result follows. $\square$

Theorem 2.4. *Assume that* $p\in[1,\infty]$ *and that* $a\in M^{p,1}(\mathbf{R}^{2n+m})$. *Also assume that* $\varphi\in C^2(\mathbf{R}^{2n+m})$ *is real-valued and satisfies* (0.4) *for some* $d>0$, *and that* $\varphi^{(\alpha)}\in M^{\infty,1}$ *for all multi-indices* α *with* $|\alpha|=2$. *Then* $\operatorname{Op}_\varphi(a)\in\mathcal{I}_p$.

Proof. In view of Theorem 3.1 in [5] and Proposition 2.3, the result is true when $p\in\{1,\infty\}$. For general p, the result now follows by interpolation, using Theorem 4.1.2 in [3], Proposition 1.5 and (1.4). $\square$

Next we discuss Fourier integral operators in (0.2) when a is a distribution of $2n$ variables. This situation is not covered in Theorem 2.4 when $p<\infty$, due to the fact that the distribution $(x,y,\zeta)\mapsto a(x,0,\zeta)$ does not belong to $M^{p,1}(\mathbf{R}^{3n})$ when $a(x,y,\zeta)\in M^{p,1}(\mathbf{R}^{3n})$.

Theorem 2.5. *Assume that* $p\in[1,\infty]$, $t_1,t_2\in\mathbf{R}$, $d>0$, *and that* $\varphi\in C(\mathbf{R}^{3n})$ *is real-valued and satisfies* $\varphi^{(\alpha)}\in M^{\infty,1}$ *for all multi-indices* α *with* $|\alpha|=2$ *and*

$$|\det(t_2\varphi''_{x,\zeta}(x,y,\zeta)-t_1\varphi''_{y,\zeta}(x,y,\zeta))|\geq d. \tag{2.2}$$

Then the map

$$a\longmapsto K_{a,\varphi}(x,y)\equiv\int_{\mathbf{R}^n}a(t_1x+t_2y,\zeta)e^{i\varphi(x,y,\zeta)}\,d\zeta$$

from $\mathcal{S}(\mathbf{R}^{2n})$ *to* $\mathcal{S}'(\mathbf{R}^{2n})$ *extends uniquely to a continuous map on* $M^p(\mathbf{R}^{2n})$.

For the proof we need the following lemma.

Lemma 2.6. *Assume that* $f\in M^{\infty,1}(\mathbf{R}^n)$, $\chi\in C_0^\infty(\mathbf{R}^n)$ *and* $x\in\mathbf{R}^n$, *and let*

$$h_{x,j,k}(y)=\chi(y)\int_0^1(1-t)f(x+ty)y_jy_k\,dt.$$

Then there is a constant C *and a function* $g\in L^1(\mathbf{R}^n)$ *such that* $\|g\|_{L^1}\leq C\|f\|_{M^{\infty,1}}$ *and* $|\mathcal{F}(h_{x,j,k})(\xi)|\leq g(\xi)$.

Proof. Let $\psi(y)=\psi_{j,k}(y)=\chi(y)y_jy_k$. By a change of variables we obtain

$$\begin{aligned}\mathcal{F}(h_{x,j,k})(\xi)&=\int_0^1(1-t)\int_{\mathbf{R}^n}f(x+ty)\psi(y)e^{-i\langle y,\xi\rangle}\,dy\,dt\\&=\int_0^1t^{-n}(1-t)\mathcal{F}\Big(f\psi\Big(\frac{\cdot-x}{t}\Big)\Big)\Big(\frac{\xi}{t}\Big)e^{i\langle x,\xi\rangle/t}\,dt.\end{aligned} \tag{2.3}$$

Hence, if

$$g(\xi)\equiv\int_0^1 t^{-n}(1-t)\sup_{x\in\mathbf{R}^n}\Big|\mathcal{F}\Big(f\psi\Big(\frac{\cdot-x}{t}\Big)\Big)\Big(\frac{\xi}{t}\Big)\Big|\,dt, \tag{2.4}$$

then it follows that $|\mathcal{F}(h_{x,j,k})(\xi)|\le g(\xi)$. We have to prove that $\|g\|_{L^1}\le C\|f\|_{M^{\infty,1}}$ for some constant C.

Assume that $r>0$ is chosen so that the support of χ is contained in the closed ball B_r with radius r and center at the origin, and let $\psi_1(x)=e^{-|x|^2}$ and $\psi_2\in C_0^\infty(\mathbf{R}^n)$ be such that $\psi_2(x)=e^{|x|^2}$ when $x\in B_r$. Also assume that $0\le t\le 1$. By straightforward computations we get

$$\Big|\mathcal{F}\Big(f\psi\Big(\frac{\cdot-x}{t}\Big)\Big)(\xi)\Big|\le(2\pi)^{-n/2}\Big(|\mathcal{F}(f\psi_1(\,\cdot-x))|*\Big|\mathcal{F}\Big(\psi\Big(\frac{\cdot-x}{t}\Big)\psi_2(\,\cdot-x)\Big)\Big|\Big)(\xi),$$

where the convolution should be taken with respect to the ξ-variable only. Since

$$\Big|\mathcal{F}\Big(\psi\Big(\frac{\cdot-x}{t}\Big)\psi_2(\,\cdot-x)\Big)\Big|=\Big|\mathcal{F}\Big(\psi\Big(\frac{\cdot}{t}\Big)\psi_2\Big)\Big|$$

we therefore get

$$\Big|\mathcal{F}\Big(f\psi\Big(\frac{\cdot-x}{t}\Big)\Big)(\xi)\Big|\le(2\pi)^{-n/2}(|\mathcal{F}(f\psi_1(\,\cdot-x))|*F_t)(\xi), \tag{2.5}$$

where

$$F_t(\xi)\equiv\Big|\mathcal{F}\Big(\psi\Big(\frac{\cdot}{t}\Big)\psi_2\Big)(\xi)\Big|.$$

We need to estimate F_t. Let Ω_0 be the closed unit ball in $\mathbf{R}^n$ and let Ω_j be the set of all $\xi\in\mathbf{R}^n$ outside the unit ball such that $|\xi_j|\ge|\xi|/2n$. Then $\bigcup_{j=0}^\infty\Omega_j=\mathbf{R}^n$, and since $F_t(\xi)\le(2\pi)^{-n/2}\|\psi(\,\cdot/t)\psi_2\|_{L^1}$ it follows that

$$F_t(\xi)\le Ct^n,\quad \xi\in\Omega_0. \tag{2.6}$$

Furthermore, if $N\ge0$ is an integer and $\xi\in\Omega_j$, then by integration by parts, and the fact that $0\le t\le1$, it follows that

$$\begin{aligned}|\xi|^N|F_t(\xi)|&\le C_1|\xi_j^N F_t(\xi)|\\&=C_2t^n\Big|\int_{\mathbf{R}^n}\psi(y)\psi_2(ty)\Big(\frac{D_{y_j}}{t}\Big)^N(e^{-it\langle y,\xi\rangle})\,dy\Big|\\&=C_2t^{n-N}\Big|\int_{\mathbf{R}^n}\big(D_{y_j}^N(\psi(y)\psi_2(ty))\big)e^{-it\langle y,\xi\rangle}\,dy\Big|\\&\le C_2t^{n-N}\int_{\mathbf{R}^n}|D_{y_j}^N(\psi(y)\psi_2(ty))|\,dy\\&\le C_3t^{n-N}\end{aligned}$$

for some constant C_1, C_2 and C_3 which are independent of j. Hence, by taking geometric means of the latter estimates, it follows that for any real number $s\ge0$, there is a constant C_s such that

$$|\xi|^s|F_t(\xi)|\le C_st^{n-s},\quad |\xi|\ge 1.$$

A combination of this estimate, (2.6) and the fact that $0\le t\le 1$ now gives

$$|F_t(\xi)|\le C_st^{n-s}\langle\xi\rangle^{-s},\quad \xi\in\mathbf{R}^n, \tag{2.7}$$

for some constant C_s which is independent of ξ. Here $\langle\xi\rangle=(1+|\xi|^2)^{1/2}$.

By letting $s=n+\frac{1}{2}$ and combining (2.4), (2.5) and (2.7) it follows that

$$g(\xi)\le C\int_0^1 t^{-n}(1-t)t^{-1/2}\Big(\Big(\sup_{x\in\mathbf{R}^n}|\mathcal{F}(f\psi_1(\,\cdot\,-x))|\Big)*\langle\,\cdot\,\rangle^{-n-1/2}\Big)\Big(\frac{\xi}{t}\Big)\,dt.$$

Hence, by applying the L^1-norm on the latter inequality, and changing the variables of integration we get

$$\begin{aligned}\|g\|_{L^1}&\le C\int_0^1 t^{-n}(1-t)t^{-1/2}\iint_{\mathbf{R}^{2n}}\sup_{x\in\mathbf{R}^n}\Big|\mathcal{F}(f\psi_1(\,\cdot\,-x))\Big(\frac{\xi}{t}-\eta\Big)\Big|\langle\eta\rangle^{-n-1/2}\,d\xi\,d\eta\,dt\\ &=C'\int_{\mathbf{R}^n}\sup_{x\in\mathbf{R}^n}|\mathcal{F}(f\psi_1(\,\cdot\,-x))(\xi)|\,d\xi=C'\|f\|_{M^{\infty,1}},\end{aligned}$$

where

$$C'=C\|\langle\,\cdot\,\rangle^{-n-1/2}\|_{L^1}\int_0^1(1-t)t^{-1/2}\,dt<\infty.$$

This proves the assertion. □

Proof of Theorem 2.5. By letting

$$x_1=t_1x+t_2y,\quad y_1=s_1x+s_2y\quad\text{and}\quad \xi=\zeta$$

as new coordinates, where s_1 and s_2 are such that $s_1t_2\ne s_2t_1$, and observing that the space $M^{p,q}(\mathbf{R}^{2n})$ is invariant under pullbacks with automorphisms on $\mathbf{R}^{2n}$, it follows that we may assume that $t_1=1$ and $t_2=0$, and that the condition (2.2) is reduced to

$$|\det(\varphi''_{y,\xi}(x,y,\xi))|\ge d. \tag{2.2$'$}$$

First we assume that $p=1$ and that $a\in M^1\cap\mathcal{E}'$, and we let $\chi\in C_0^\infty(\mathbf{R}^n)$ and $\psi\in C_0^\infty(\mathbf{R}^{3n})$ be such that $\psi(x,y,\xi)=1$ when $a(x,\xi)\chi(y)\ne 0$. We also let $X_0=(0,y,0)$, $X=(x,z,\zeta)$ and

$$\begin{aligned}I_a(y,\xi,\eta)&=\mathcal{F}(K_{a,\varphi}(1\otimes\chi)(\,\cdot\,-(0,y))(\xi,\eta)\\ &=\int_{\mathbf{R}^{3n}}a(x,\zeta)e^{i\varphi(X)}\chi(z-y)e^{-i(\langle x,\xi\rangle+\langle z,\eta\rangle)}\,dX,\end{aligned}$$

and note that the L^1-norm of I_a is equivalent to the M^1-norm of $K_{a,\varphi}$ in view of Remark 1.3, since a has compact support. By a change of variables it follows that

$$I_a(y,\xi,\eta)=e^{i\langle y,\eta\rangle}\int_{\mathbf{R}^{3n}} a(x,\zeta)e^{i\varphi(X+X_0)}\chi(z)e^{-i(\langle x,\xi\rangle+\langle z,\eta\rangle)}\,dX.$$

In a similar way as in the proof of Lemma 2.2 we now set

$$\psi_{1,X_0}(X)=\varphi(X_0)+\langle\varphi'(X_0),X\rangle,$$
$$\psi_{2,X_0}(X)=\psi(X)\int_0^1(1-t)\langle\varphi''(X_0+tX)X,X\rangle\,dt.$$

Then an application of Taylor's formula on φ gives

$$\begin{aligned}|I_a(y,\xi,\eta)|&=\left|\int_{\mathbf{R}^{3n}} a(x,\zeta)\chi(z)e^{i\psi_{1,X_0}(X)}e^{-i(\langle x,\xi\rangle+\langle z,\eta\rangle)}e^{i\psi_{2,X_0}(X)}\,dX\right|\\&=(2\pi)^{-3n/2}|(\mathcal{F}(a\otimes\chi)*\mathcal{F}(e^{i\psi_{2,X_0}}))(\xi-\varphi'_x(X_0),-\varphi'_\zeta(X_0),\eta-\varphi'_y(X_0))|.\end{aligned}$$

As remarked above, we are interested in applying the L^1-norm on I_a. A problem here with the right-hand side in the latter equality is that $\mathcal{F}(e^{i\psi_{2,X_0}})$ depends on X_0. For this reason we set

$$\Phi_{k,X_0}\equiv\mathcal{F}(\psi_{2,X_0})*\dots*\mathcal{F}(\psi_{2,X_0}),$$

where $k\ge1$ is the number of factors in the convolution. An application of Lemma 2.5 then shows that there exists a function G such that $|\mathcal{F}(\psi_{2,X_0})|\le G$ and $\|G\|_{L^1}\le C\|\varphi''\|_{M^{\infty,1}}$ for some constant $C>0$. Hence if $\Psi_k\equiv G*\dots*G$ with k factors of G in the convolution, then it follows that $|\Phi_{k,X_0}|\le\Psi_k$ and that $\|\Psi_k\|_{L^1}\le C^k\|\varphi''\|^k_{M^{\infty,1}}$.

This implies that

$$|I_a(y,\xi,\eta)|\le\sum_{k=0}^\infty\frac{1}{k!}J_{a,k}(y,\xi,\eta), \tag{2.8}$$

where

$$\begin{aligned}J_{a,0}(y,\xi,\eta)&=|\mathcal{F}(a\otimes\chi)(\xi-\varphi'_x(X_0),-\varphi'_\zeta(X_0),\eta-\varphi'_y(X_0))|,\\J_{a,k}(y,\xi,\eta)&=(|\mathcal{F}(a\otimes\chi)|*|\Phi_{k,X_0}|)(\xi-\varphi'_x(X_0),-\varphi'_\zeta(X_0),\eta-\varphi'_y(X_0))\\&\le(|\mathcal{F}(a\otimes\chi)|*|\Psi_k|)(\xi-\varphi'_x(X_0),-\varphi'_\zeta(X_0),\eta-\varphi'_y(X_0)),\qquad k\ge1.\end{aligned}$$

Hence, by applying the L^1-norm on the latter estimates, and using the fact that a has compact support, we get

$$\begin{aligned}\|J_{a,0}\|_{L^1}&=\iiint_{\mathbf{R}^{3n}}|\mathcal{F}(a\otimes\chi)(\xi-\varphi'_x(X_0),-\varphi'_\zeta(X_0),\eta-\varphi'_y(X_0))|\,dy\,d\xi\,d\eta\\&=\iiint_{\mathbf{R}^{3n}}|\mathcal{F}(a\otimes\chi)(\xi,-\varphi'_\zeta(X_0),\eta)|\,dy\,d\xi\,d\eta\end{aligned}$$

$$
\begin{aligned}
&\le \frac{1}{d}\iiint_{\mathbf{R}^{3n}} |\mathcal{F}(a\otimes\chi)(\xi,x,\eta)|\,dx\,d\xi\,d\eta\\
&= \frac{1}{d}\|\hat{a}\|_{L^1}\\
&\le \frac{C}{d}\|a\|_{M^{\infty,1}},
\end{aligned}
$$

and

$$
\begin{aligned}
\|J_{a,k}\|_{L^1} &= \iiint_{\mathbf{R}^{3n}} (|\mathcal{F}(a\otimes\chi)|*|\Psi_k|)(\xi-\varphi'_x(X_0),-\varphi'_\zeta(X_0),\eta-\varphi'_y(X_0))\,dy\,d\xi\,d\eta\\
&= \|J_{a,0}\|_{L^1}\|\Psi_k\|_{L^1} \le \frac{C}{d}\|a\|_{M^{\infty,1}}(C\|\varphi''\|_{M^{\infty,1}})^k.
\end{aligned}
$$

In the first inequality we have used (2.2)$'$ and taking $x=-\varphi'_\zeta(X_0)$ as a new variable of integration in the y-direction. By combining these estimates we get

$$
\begin{aligned}
\|K_{a,\varphi}\|_{M^{1,1}} \le \|I_a\|_{L^1} \le \sum_{k=0}^{\infty}\frac{1}{k!}\|J_{a,k}\|_{L^1} &\le \frac{C}{d}\|a\|_{M^{\infty,1}}\sum_{k=0}^{\infty}\frac{1}{k!}(C\|\varphi''\|_{M^{\infty,1}})^k\\
&= \frac{C}{d}\|a\|_{M^1}\exp(C\|\varphi''\|_{M^{\infty,1}}).
\end{aligned}
$$

This proves the assertion in this case.

For general $a\in M^1$, the asserted continuity now follows by applying Proposition 1.4 in a way similar to the proof of Proposition 2.3. We leave the details to the reader.

Next we consider the case when $p=\infty$. Assume that $a,b\in M^1(\mathbf{R}^{2n})$, and let $\widetilde{\varphi}(x,y,\xi)=-\varphi(x,\xi,y)$. Then (2.2)$'$ also holds when φ is replaced by $\widetilde{\varphi}$. Hence, the first part of the proof shows that $K_{b,\widetilde{\varphi}}\in M^1$. Furthermore, by straightforward computations we have

$$
(K_{a,\varphi},b)=(a,K_{b,\widetilde{\varphi}}). \tag{2.9}
$$

In view of Proposition 1.1(3), it follows that the right-hand side in (2.9) makes sense if, more generally, a is an arbitrary element in $M^\infty(\mathbf{R}^{2n})$, and then

$$
|(a,K_{b,\widetilde{\varphi}})| \le \frac{C}{d}\|a\|_{M^\infty}\|b\|_{M^1}\exp(C\|\varphi''\|_{M^\infty,1})
$$

for some constant C which is independent of d, $a\in M^\infty$ and $b\in M^1$.

Hence, by letting $K_{a,\varphi}$ be defined as (2.9) when $a\in M^\infty$, it follows that $a\mapsto K_{a,\varphi}$ on M^1 extends to a continuous map on M^∞. Furthermore, since $\mathcal{S}$ is dense in M^∞ with respect to the weak* topology, it follows that this extension is unique. We have therefore proved the theorem for $p\in\{1,\infty\}$.

For general $p\in[1,\infty]$, the result now follows by interpolation, using Theorem 4.1.2 in [3] and Proposition 1.5. $\square$

Assume that $a\in M^\infty(\mathbf{R}^{2n})$, $t_1,t_2\in\mathbf{R}$, and that $\varphi\in C(\mathbf{R}^{3n})$ is real-valued and satisfies $\varphi^{(\alpha)}\in M^{\infty,1}$ for all multi-indices α with $|\alpha|=2$ and (2.2) for some $d>0$. Then we let the Fourier integral operator $\operatorname{op}_\varphi(a)=\operatorname{op}_{\varphi,t_1,t_2}(a)$ be the continuous operator from $\mathcal{S}(\mathbf{R}^n)$ to $\mathcal{S}'(\mathbf{R}^n)$ with kernel $K_{a,\varphi}$ in Theorem 2.5. The following result is an immediate consequence of Theorem 2.5 and Theorem 4.3 in [26].

Theorem 2.7. *Assume that $p\in[1,\infty]$, $a\in M^p(\mathbf{R}^{2n})$, $t_1,t_2\in\mathbf{R}$, and that $\varphi\in C(\mathbf{R}^{3n})$ is real-valued and satisfies $\varphi^{(\alpha)}\in M^{\infty,1}$ for all multi-indices α with $|\alpha|=2$. Also assume that* (2.2) *is fulfilled for some $d>0$. Then the definition of $\operatorname{op}_{\varphi,t_1,t_2}(a)$ from $\mathcal{S}(\mathbf{R}^n)$ to $\mathcal{S}'(\mathbf{R}^n)$ extends uniquely to a continuous map from $M^{p'}(\mathbf{R}^n)$ to $M^p(\mathbf{R}^n)$.*

By combining Theorems 2.4, 2.5 and interpolation, we obtain the following result.

Theorem 2.8. *Assume that $p,q\in[1,\infty]$ are such that $q\le\min(p,p')$, that $a\in M^{p,q}(\mathbf{R}^{2n})$, that $t_1,t_2\in\mathbf{R}$, and that $\varphi\in C(\mathbf{R}^{3n})$ is real-valued and satisfies $\varphi^{(\alpha)}\in M^{\infty,1}$ for all multi-indices α with $|\alpha|=2$. Also assume that* (0.4) *and* (2.2) *are fulfilled for some $d>0$. Then the definition of $\operatorname{op}_{\varphi,t_1,t_2}(a)$ from $\mathcal{S}(\mathbf{R}^n)$ to $\mathcal{S}'(\mathbf{R}^n)$ extends uniquely to a Schatten–von Neumann operator of order p on $L^2(\mathbf{R}^n)$.*

Proof. We may assume that $q=\min(p,p')$. First assume that $p\le 2$, and let $b\in\mathcal{S}'(\mathbf{R}^{2n})$ be chosen such that $b(x,D)=\operatorname{op}_\varphi(a)$. Then the operator kernel of b belongs to M^p, and since M^p is invariant under partial Fourier transformations in view of Remark 1.3(5), the result is a consequence of Proposition 1.6(1).

If instead $p=\infty$, then it follows from Theorem 2.4 that $\operatorname{op}_\varphi(a)$ is continuous on L^2, which proves the result in this case as well. The result now follows for general $p\in[2,\infty]$ by interpolation, using Theorem 4.1.2 in [3], Proposition 1.5 and (1.4). The proof is complete. $\square$

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Received June 15, 2007
published online June 6, 2008