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Bounded universal functions for sequences of holomorphic self-maps of the disk

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Abstract. We give several characterizations of those sequences of holomorphic self-maps $\{\phi_n\}_{n\geq1}$ of the unit disk for which there exists a function F in the unit ball $\mathcal{B}=\{f\in H^\infty:\|f\|_\infty\leq1\}$ of H^∞ such that the orbit $\{F\circ\phi_n:n\in\mathbb{N}\}$ is locally uniformly dense in $\mathcal{B}$. Such a function F is said to be a $\mathcal{B}$-universal function. One of our conditions is stated in terms of the hyperbolic derivatives of the functions ϕ_n. As a consequence we will see that if ϕ_n is the nth iterate of a map ϕ of $\mathbb{D}$ into $\mathbb{D}$, then $\{\phi_n\}_{n\geq1}$ admits a $\mathcal{B}$-universal function if and only if ϕ is a parabolic or hyperbolic automorphism of $\mathbb{D}$. We show that whenever there exists a $\mathcal{B}$-universal function, then this function can be chosen to be a Blaschke product. Further, if there is a $\mathcal{B}$-universal function, we show that there exist uniformly closed subspaces consisting entirely of universal functions.

1. Introduction

Let H^∞ denote the algebra of bounded analytic functions on the unit disk $\mathbb{D}$, and let $H(\mathbb{D})$ denote the space of functions analytic on $\mathbb{D}$ with the local uniform topology. In this paper we will consider sequences of holomorphic self-maps of $\mathbb{D}$, $\{\phi_n\}_{n\geq1}$, and functions that have unexpected behavior with respect to these maps, the so-called *universal functions*. Our setting will be the following: X will be $H(\mathbb{D})$ or a subset of H^∞. For a holomorphic self-map ϕ of $\mathbb{D}$ we define the composition operator C_ϕ on X by $C_\phi(f)=f\circ\phi$.

Definition 1.1. A function $f\in X$ is said to be X-*universal* for $\{\phi_n\}_{n\geq1}$ (or, equivalently, we say that $\{C_{\phi_n}\}_{n\geq1}$ admits an X-universal function f) if the set $\{f\circ\phi_n:n\in\mathbb{N}\}$ is locally uniformly dense in X. Also, we will call a function $f\in H^\infty$ *universal* with respect to $\{\phi_n\}_{n\geq1}$ if $\{f\circ\phi_n:n\in\mathbb{N}\}$ is locally uniformly dense in

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$\{g\in H^\infty:\|g\|_\infty\le\|f\|_\infty\}$. In other words, $f\in H^\infty$ is universal if $\{f\circ\phi_n:n\in\mathbb{N}\}$ is as big as it possibly can be.

Bounded universal functions for invertible composition operators on the space $H^\infty(\Omega)$, where Ω is a planar domain, were studied in [14], [17] and [19] and, for the case in which $\Omega\subseteq\mathbb{C}^n$, in [1], [2], [8], [11] and [13]. In this paper, we will restrict consideration to the case in which $\Omega=\mathbb{D}$ while varying our space X. The space X may be the closed unit ball, $\mathcal{B}$, of H^∞, it may denote the set $\mathcal{S}$ of functions in $\mathcal{B}$ that do not vanish on $\mathbb{D}$, or it may be the space $H(\mathbb{D})$. The aim of this paper is to study X-universality for noninvertible composition operators on H^∞.

If we let $\{p_n\}_{n\ge1}$ denote a sequence of finite Blaschke products that is dense in $\mathcal{B}$, then it is very easy to come up with a $\mathcal{B}$-universal function for the self-maps $\{p_n\}_{n\ge1}$: the identity function works, for example. Similarly the atomic inner function S is $\mathcal{S}$-universal for $\{p_n\}_{n\ge1}$. The more interesting case is one in which the p_n's do not fill up much of $\mathcal{B}$; for example, a sequence $\{p_n\}_{n\ge1}$ such that $p_n\to1$ uniformly on compact sets. Thus we were lead to the following question: under what conditions on the sequence of self-maps does there exist a $\mathcal{B}$-universal function? an $\mathcal{S}$-universal function? an $H(\mathbb{D})$-universal function? In this paper we will give a complete answer to these questions.

We will always consider a sequence $\{\phi_n\}_{n\ge1}$ of holomorphic self-maps with $\phi_n(0)\to1$. Our sequence need not be a sequence of iterates and it need not be a sequence of automorphisms. Indeed, the more interesting cases, for the purposes of the study in this paper, are those in which this is not the case. Our results can be summarized as follows. In Section 2, which was motivated by work in [13], [14], [15] and [17], we show (Theorem 2.1) that the existence of a $\mathcal{B}$-universal function, a $\mathcal{B}$-universal Blaschke product, and an $\mathcal{S}$-universal singular inner function are equivalent. Such functions exist precisely when

$$\limsup_{n\to\infty}\frac{|\phi_n'(0)|}{1-|\phi_n(0)|^2}=1.$$

In Section 3, which follows the line of thought in papers such as [1], [3], [5] and [19], we show (Theorem 3.1) that these conditions are equivalent to the existence of a uniformly closed, infinite-dimensional vector space generated by Blaschke products such that every function in the vector space is universal in the sense of Definition 1.1. In Section 4 we present several related results, including a discussion of the special cases in which the sequence of self-maps is a sequence of iterates of a particular function. The final section of the paper includes necessary and sufficient conditions for $H(\mathbb{D})$-universality as well as examples of sequences of self-maps that admit $H(\mathbb{D})$-universal functions but not $\mathcal{B}$-universal functions.

2. Universality in the ball of H^∞

2.1. The main result

In [17] Heins showed that there exists a sequence of automorphisms of $\mathbb{D}$ that admits a universal Blaschke product. The main result in [14] shows that for every sequence $\{z_n\}_{n\geq 1}$ in $\mathbb{D}$ with $|z_n|\to 1$ there exists a Blaschke product B such that $\{B((z+z_n)/(1+\bar{z}_n z)):n\in\mathbb{N}\}$ is locally uniformly dense in $\mathcal{B}$ and the proof given there indicates how the construction of the Blaschke product proceeds. In [15], the paper focuses on the construction of an $\mathcal{S}$-universal singular inner function for a sequence of automorphisms. The proof of Theorem 2.1 below is quite different. First, we do not assume that the self-maps are automorphisms; thus the proof requires the development of new techniques. Second, in our proof of the existence of an $\mathcal{S}$-universal singular inner function, rather than constructing the singular inner function (as in [15]) we use the existence of the $\mathcal{B}$-universal Blaschke product to obtain the singular inner function. This existence proof uses only elementary methods. Here is the main result of this section:

Theorem 2.1. *Let $\{\phi_n\}_{n\geq 1}$ be a sequence of holomorphic self-maps of the unit disk $\mathbb{D}$ such that $\phi_n(0)\to 1$ as $n\to\infty$. The following conditions are equivalent*:

(A) *the sequence $\{C_{\phi_n}\}_{n\geq 1}$ admits a $\mathcal{B}$-universal function*;

(B) *the sequence $\{C_{\phi_n}\}_{n\geq 1}$ admits a $\mathcal{B}$-universal Blaschke product*;

(C) *the sequence $\{C_{\phi_n}\}_{n\geq 1}$ admits an $\mathcal{S}$-universal singular inner function*;

(D) *the set $\mathcal{B}_m=\{u\circ\phi_n:u\in\mathcal{B}$ and $n\geq m\}$ is locally uniformly dense in $\mathcal{B}$ for every $m\geq 1$*;

(E) $\limsup_{n\to\infty}|\phi_n'(0)|/(1-|\phi_n(0)|^2)=1$.

Before we begin the proof of Theorem 2.1, we point out that this theorem also holds under the weaker assumption that $|\phi_n(0)|\to 1$ as $n\to\infty$. The necessary modifications of the proof below will be left to the reader.

The proof proceeds as follows: we show that (D) $\Rightarrow$ (B) $\Rightarrow$ (A) $\Rightarrow$ (E) $\Rightarrow$ (D). Then we complete the proof by showing that (B) $\Rightarrow$ (C) $\Rightarrow$ (E). The implication (D) $\Rightarrow$ (B) is really the key to the proof.

2.2. Proof of (D) $\Rightarrow$ (B)

Our proofs will use the following elementary fact several times:

Fact 2.2. If a sequence of holomorphic functions on a domain is bounded by 1 and converges to 1 at some point of the domain, then the sequence converges to 1 uniformly on compact subsets of the domain.

The proof of (D) $\Rightarrow$ (B) relies on the following two lemmas:

Lemma 2.3. *The set of finite Blaschke products B with $B(1)=1$ is dense in $\mathcal{B}$ for the topology of local uniform convergence.*

Proof. By Carathéodory's theorem (see [10, p. 6]) the set of finite Blaschke products is dense in $\mathcal{B}$. Thus, for $f\in\mathcal{B}$ and $K\subseteq\mathbb{D}$ compact, there exists a finite Blaschke product B such that $|f-B|<\varepsilon/2$ on K. Use [16, Lemma 2.10] to obtain an automorphism b of $\mathbb{D}$ with $b(1)=\overline{B(1)}$ and $|b-1|<\varepsilon/2$ on K. The function bB is a finite Blaschke product and $|bB-f|<\varepsilon$ on K, which proves our claim. $\square$

The second lemma establishes the existence of certain "anti-peak" functions. In what follows, for $f\in H(\mathbb{D})$ and $K\subseteq\mathbb{D}$ compact, we let $\|f\|_K=\sup\{|f(z)|:z\in K\}$.

Lemma 2.4. *Let $K\subseteq\mathbb{D}$ be a compact subset of $\mathbb{D}$ and $\varepsilon>0$. There exists a function $\gamma\in A(\mathbb{D})$ such that $\gamma(1)=0$, $\|\gamma-1\|_K<\varepsilon$ and $\|\gamma\|_\infty<1$.*

Proof. For a positive integer n, define the function γ_n by

$$\gamma_n(z)=\left(1-\frac{\varepsilon}{2}\right)\left[\frac{1-z}{2}\right]^{1/n} \quad \text{for } z\in\mathbb{D}.$$

Then if n is large enough, γ_n satisfies all the required conditions. $\square$

Under assumption (D) of Theorem 2.1 above, we prove the following proposition, which is the main ingredient in the proof of the theorem:

Proposition 2.5. *Let $\{\phi_n\}_{n\geq 1}$ be a sequence of holomorphic self-maps of $\mathbb{D}$ such that $\phi_n(0)\to 1$ and for any $m\geq 1$, the set $\mathcal{B}_m$ is dense in $\mathcal{B}$. Then for any $m_0\geq 1$, any $\varepsilon>0$, any function f in $\mathcal{B}$ and any compact $K\subseteq\mathbb{D}$, there exist a finite Blaschke product B and an integer $m\geq m_0$ such that*

(a) $B(1)=1$;

(b) $\|B-1\|_K<\varepsilon$;

(c) $\|B\circ\phi_m-f\|_K<\varepsilon$.

Proof. Let K, $\varepsilon>0$ and f be given as above. It is always possible to assume that $\|f\|_\infty<1-\eta<1$ for some $\eta>0$. By Fact 2.2, since $\phi_n(0)\to 1$ we know that $\{\phi_n\}_{n\geq 1}$ converges uniformly to 1 on K. Let γ be the anti-peak function given by Lemma 2.4. There exists $\alpha>0$ such that $|z-1|<\alpha$ implies $|\gamma(z)|<\min(\varepsilon,\eta)$. We fix a peak function ψ such that $\psi(1)=1$, $\|\psi\|_K<\varepsilon$, and for $|z-1|>\alpha$, $|\psi(z)|<1-\|\gamma\|_\infty$ (for instance, we can take ψ to be a sufficiently large power of $z\mapsto(1+z)/2$). For this peak function ψ, there exists a number β with $0<\beta<\alpha$ such that for $|z-1|<\beta$, we have $|\psi(z)-1|<\varepsilon$. Finally since $\{\phi_n\}_{n\geq 1}$ converges to 1 uniformly on K, we can choose an integer $m_1 \geq m_0$ such that for every $m \geq m_1$, $\phi_m(K)$ is contained in the disk $\{z \in \mathbb{D} : |z-1| < \beta\}$.

Now we apply our assumption to choose an integer $m \geq m_1$ and a function $u \in \mathcal{B}$ such that

$$\|u \circ \phi_m - f\|_K < \varepsilon.$$

Since f has norm less than $1-\eta$, modifying u if necessary we can assume that $\|u\|_\infty \leq 1-\eta$. Let

$$h = (1-\varepsilon)\gamma + u\psi.$$

We claim that the function h solves the problem, except that it is not a Blaschke product and $h(1)$ is not equal to 1. Once we have established this claim, we will modify h into a Blaschke product that will truly solve the problem.

If $z \in \phi_m(K)$, we have

$$|h(z) - u(z)| \leq (1-\varepsilon)|\gamma(z)| + |\psi(z) - 1| \leq \varepsilon(1-\varepsilon) + \varepsilon,$$

where the last inequality uses the fact that $\phi_m(K) \subseteq \{z \in \mathbb{D} : |z-1| < \beta\}$. Thus, $h \circ \phi_m$ is close to f on K.

If z is in K, then

$$|h(z) - 1| \leq \varepsilon + |\gamma(z) - 1| + \varepsilon \leq 3\varepsilon.$$

Now observe that $|h(z)| \leq 1$ for any $z \in \mathbb{D}$: Indeed if $|z-1| > \alpha$ this follows from the property of ψ, whereas for $|z-1| < \alpha$ this is a consequence of the values of $|\gamma(z)|$ and $\|u\|_\infty$.

To obtain the required Blaschke product, apply Lemma 2.3 to replace the function h by a Blaschke product B with $B(1)=1$ such that the required uniform approximations of (b) and (c) on K hold. □

We now show that the proof of (D) ⇒ (B) is a consequence of Proposition 2.5.

Proof of (D) ⇒ (B) *in Theorem* 2.1. Let $\{f_l\}_{l \geq 1}$ be a dense sequence of elements of $\mathcal{B}$, and $K_l = \overline{\mathbb{D}}(0, 1-2^{-l})$ be an exhaustive sequence of compact subsets of $\mathbb{D}$. We claim that there exist finite Blaschke products $\{B_l\}_{l \geq 1}$ and a sequence $\{n_l\}_{l \geq 1}$ of integers such that for every $l \geq 1$,

$$B_l(1) = 1, \tag{1}$$

$$\|B_l - 1\|_{K_l} < 2^{-l}, \tag{2}$$

and

$$\|C_{\phi_{n_l}}(B_1...B_k)-f_l\|_{K_l} < 2^{-(l+1)} \quad \text{for every } k \geq l. \tag{3}$$

Now we show how to obtain a $\mathcal{B}$-universal Blaschke product using properties (1), (2) and (3). Let

$$B = \prod_{l\geq 1} B_l.$$

This infinite product converges, by (2), and therefore B is a Blaschke product. Once the claim is established we can use (3) to conclude that $\|C_{\phi_{n_l}}(B)-f_l\|_{K_l}<2^{-l}$ for every $l\geq 1$. Thus we will be done once we have established the existence of the aforementioned Blaschke products.

The construction of the B_l's and the n_l's is done by induction on l using Proposition 2.5. The case $l=1$ is nothing more than Proposition 2.5. If the construction has been carried out until step $l-1$, we choose B_l and n_l large enough by Proposition 2.5 so that

(i) $B_l(1)=1$;
(ii) $\|B_l-1\|_K<2^{-l}$, where $K=K_l\cup\bigcup_{j\leq l-1}\phi_{n_j}(K_j)$;
(iii) $\|C_{\phi_{n_l}}(B_1...B_{l-1})-1\|_{K_l}<2^{-(l+1)}$;
(iv) $\|C_{\phi_{n_l}}(B_l)-f_l\|_{K_l}<2^{-(l+1)}$.

By (i) we know that $B_j(1)=1$ for every $j\leq l-1$, so the functions $B_j\circ\phi_k$ tend to 1 uniformly on compact sets as $k\to\infty$, and from this (iii) follows. From (iii) and (iv) it follows that

$$\|C_{\phi_{n_l}}(B_1...B_l)-f_l\|_{K_l} < 2^{-l}.$$

Now that the B_l has been constructed, for every $k\geq l$ we get

$$\begin{aligned}\|C_{\phi_{n_l}}(B_1...B_k)-f_l\|_{K_l} &\leq \|C_{\phi_{n_l}}(B_1...B_l)-f_l\|_{K_l} + \sum_{j=l+1}^{k} \|B_j\circ\phi_{n_l}\|_{K_l} \\ &\leq 2^{-l} + \sum_{j=l+1}^{\infty} 2^{-j} \leq 2^{-l+1}.\end{aligned}$$

This completes the proof of (D) $\Rightarrow$ (B). □

Now it is clear that (B) $\Rightarrow$ (A). We turn to the rest of the proof of Theorem 2.1.

2.3. Proof of (A) $\Rightarrow$ (E)

By assumption (A), there exists a function $f\in\mathcal{B}$ and an increasing sequence $\{n_k\}_{k\geq 1}$ such that $f\circ\phi_{n_k}\to z$ uniformly on compact sets as $n_k\to\infty$. In particular

$|(f\circ\phi_{n_k})'(0)|=|\phi'_{n_k}(0)|\,|f'(\phi_{n_k}(0))|\to 1$, so

$$\frac{|\phi'_{n_k}(0)|}{1-|\phi_{n_k}(0)|^2}(1-|\phi_{n_k}(0)|^2)|f'(\phi_{n_k}(0))|\to 1.$$

By the Schwarz–Pick lemma $(1-|\phi_{n_k}(0)|^2)|f'(\phi_{n_k}(0))|\le 1$ for every n_k and

$$\frac{|\phi'_{n_k}(0)|}{1-|\phi_{n_k}(0)|^2}\le 1$$

for all n_k. So we can conclude that

$$\frac{|\phi'_{n_k}(0)|}{1-|\phi_{n_k}(0)|^2}\to 1.$$

This proves (E).

2.4. Proof of (E) $\Rightarrow$ (D)

Passing to a subsequence we can assume that $\{\phi_n(0)\}_{n\ge 1}$ is a thin sequence such that $\phi_n(0)\to 1$ and

$$\lim_{n\to\infty}\frac{|\phi'_n(0)|}{1-|\phi_n(0)|^2}=1.$$

Recall that a sequence of distinct points $\{z_n\}_{n\ge 1}$ in $\mathbb{D}$ is said to be *thin* if

$$\lim_{n\to\infty}\prod_{\substack{j=1\\ j\ne n}}^{\infty}\rho(z_j,z_n)=1,$$

where ρ denotes the pseudo-hyperbolic distance on $\mathbb{D}$.

Let B denote the interpolating Blaschke product corresponding to the sequence $\{\phi_n(0)\}_{n\ge 1}$. Note that the assumption that $\{\phi_n(0)\}_{n\ge 1}$ is thin means that

$$\lim_{n\to\infty}(1-|\phi_n(0)|^2)|B'(\phi_n(0))|=1.$$

Since $(B\circ\phi_n)'(0)=B'(\phi_n(0))\phi'_n(0)$, we obtain

$$(B\circ\phi_n)'(0)=B'(\phi_n(0))(1-|\phi_n(0)|^2)\,\frac{\phi'_n(0)}{1-|\phi_n(0)|^2}.$$

Now, using the fact that B is thin and our hypothesis, we get

$$|(B\circ\phi_n)'(0)|=|B'(\phi_n(0))|(1-|\phi_n(0)|^2)\,\frac{|\phi'_n(0)|}{1-|\phi_n(0)|^2}\to 1.$$

Since $\{B\circ\phi_n\}_{n\geq 1}$ is a bounded family of analytic functions a normal families argument implies that there is a subsequence $\{B\circ\phi_{n_j}\}_{j\geq 1}$ and an analytic function f of norm at most one such that $B\circ\phi_{n_j}\to f$. But $(B\circ\phi_{n_j})(0)=0$ for every n_j and $|(B\circ\phi_{n_j})'(0)|\to 1$, so $f(0)=0$ and $|f'(0)|=1$. By Schwarz's lemma, $f(z)=\lambda z$ for some $|\lambda|=1$. Writing $v=\bar{\lambda}B$ we see that $v\circ\phi_{n_j}\to z$ uniformly on compact sets. Thus, for every $h\in\mathcal{B}$, we see that $(h\circ v)\circ\phi_{n_j}$ converges locally uniformly to h. Hence (D) of Theorem 2.1 is satisfied, so we have proved the equivalence of the assertions (A), (B), (D), and (E) of Theorem 2.1.

We turn now to the case of universal singular inner functions.

2.5. Proof of (B) ⇒ (C)

Consider the atomic singular inner function

$$S(z)=\exp\left(\frac{z+1}{z-1}\right).$$

By our assumption we can choose a Blaschke product B that is universal with respect to $\{\phi_n\}_{n\geq 1}$. Now $S\circ B$ is a singular inner function. If $f\in\mathcal{S}$, let h be a solution of the equation $\log f=-(1+h)/(1-h)$ with $h\in\mathcal{B}$. Note that $f=S\circ h$. Since B is universal for $\{\phi_n\}_{n\geq 1}$ we can choose $\{\phi_{n_k}\}_{k\geq 1}$ such that $B\circ\phi_{n_k}\to h$. Then $(S\circ B)\circ\phi_{n_k}\to S\circ h=f$, and thus $S\circ B$ is $\mathcal{S}$-universal for $\{C_{\phi_n}\}_{n\geq 1}$.

2.6. Proof of (C) ⇒ (E)

Our study here will be aided by a result that is related to a conjecture of Krzyz. Writing a function f of $\mathcal{S}$ as

$$f(z)=\sum_{n=0}^{\infty}a_n z^n,$$

the Krzyz conjecture is concerned with estimating the size of a_n. He conjectured that $\max_{f\in\mathcal{S}}|a_n|=2/e$ for $n\geq 1$ and this occurs if and only if

$$f(z)=\lambda\exp\left(\frac{\mu z^n+1}{\mu z^n-1}\right),$$

where $|\mu|=|\lambda|=1$. Showing that $\max|a_1|=2/e$ is, however, not difficult (see, for example [21]). For the reader's convenience we present a proof here.

Lemma 2.6. $\sup\{|f'(0)|:f\in\mathcal{S}\}=2/e.$

Proof. Let $f\in\mathcal{S}$. Without loss of generality we can assume that $f(0)>0$ and that the function f is not constant. Choose the principal branch of the logarithm and let $g(z)=-\log f(z)$ with $g(0)>0$. Then g is a holomorphic function on $\mathbb{D}$ with positive real part. We have

$$g(z)=\frac{1}{2\pi}\int_0^{2\pi}\frac{re^{it}+z}{re^{it}-z}\operatorname{Re} g(re^{it})\,dt+i\operatorname{Im} g(0).$$

Hence

$$g'(0)=\frac{1}{2\pi}\int_0^{2\pi}\frac{2re^{it}}{(re^{it})^2}\operatorname{Re} g(re^{it})\,dt.$$

Consequently, for all $r\in]0,1[$,

$$|g'(0)|\leq\frac{2}{r}\int_0^{2\pi}\operatorname{Re} g(re^{it})\,\frac{dt}{2\pi}=\frac{2}{r}\operatorname{Re} g(0).$$

Thus $|g'(0)|\leq 2|g(0)|$, from which we conclude that

$$|f'(0)|=|f(0)|\,|g'(0)|\leq 2|f(0)|\,|g(0)|=-2f(0)\log f(0).$$

Since the maximum of the function $-x\log x$ on the interval $[0,1]$ is equal to $1/e$, we see that $|f'(0)|\leq 2/e$. □

For $a\in\mathbb{D}$, let τ_a denote the automorphism given by

$$\tau_a(z)=\frac{a-z}{1-\bar{a}z}.$$

Note that τ_a is its own inverse.

We turn to the proof of the implication (C) ⇒ (E). The idea is the same as in the proof of (A) ⇒ (E). By assumption, there exists $f\in\mathcal{S}$ such that

$$f\circ\phi_{n_k}\to S \quad \text{uniformly on compact sets.}$$

Thus

$$|(f\circ\phi_{n_k})'(0)|=|f'(\phi_{n_k}(0))\phi_{n_k}'(0)|\to\left|S(z)\left(\frac{-2}{(1-z)^2}\right)\right|_{z=0}=\frac{2}{e}.$$

This can be rewritten as

$$(1-|\phi_{n_k}(0)|^2)|f'(\phi_{n_k}(0))|\frac{|\phi_{n_k}'(0)|}{1-|\phi_{n_k}(0)|^2}\to\frac{2}{e}. \tag{4}$$

By Lemma 2.6, if $F\in\mathcal{S}$, then $|F'(0)|\le 2/e$. This implies that for $a\in\mathbb{D}$ we have

$$|(F\circ\tau_a)'(0)|=|F'(a)(1-|a|^2)|\le\frac{2}{e}.$$

Thus, letting $F=f$ and $a=\phi_{n_k}(0)$ we get that

$$|f'(\phi_{n_k}(0))|(1-|\phi_{n_k}(0)|^2)\le\frac{2}{e}$$

for every n_k. Since the quantity $|\phi'_{n_k}(0)|/(1-|\phi_{n_k}(0)|^2)$ in (4) above is less than or equal to 1 for all n_k, we must have

$$\frac{|\phi'_{n_k}(0)|}{1-|\phi_{n_k}(0)|^2}\to 1,$$

which shows that (E) holds.

2.7. Some consequences and remarks

Our first corollary is the following result:

Corollary 2.7. *Let $\{\phi_n\}_{n\ge 1}$ be a sequence of holomorphic self-maps of $\mathbb{D}$ with $\phi_n(0)\to 1$. Then*

$$\limsup_{n\to\infty}\frac{|\phi'_n(0)|}{1-|\phi_n(0)|^2}=1\quad\textit{if and only if}\quad\limsup_{n\to\infty}\frac{(1-|a|^2)|\phi'_n(a)|}{1-|\phi_n(a)|^2}=1$$

for every $a\in\mathbb{D}$.

Proof. One direction is clear, so suppose that

$$\limsup_{n\to\infty}\frac{|\phi'_n(0)|}{1-|\phi_n(0)|^2}=1.$$

By Theorem 2.1 there exists a universal function $u\in\mathcal{B}$ for $\{C_{\phi_n}\}_{n\ge 1}$. Let $a\in\mathbb{D}$ and consider $\{\phi_n\circ\tau_a\}_{n\ge 1}$. If $f\in\mathcal{B}$, then $f\circ\tau_a^{-1}\in\mathcal{B}$. Therefore, $u\circ\phi_{n_k}\to f\circ\tau_a^{-1}$ for some subsequence $\{\phi_{n_k}\}_{k\ge 1}$. Thus $u\circ(\phi_{n_k}\circ\tau_a)\to f$ and $\{\phi_n\circ\tau_a\}_{n\ge 1}$ has a $\mathcal{B}$-universal function too. Again applying Theorem 2.1 we get

$$\limsup_{n\to\infty}\frac{|(\phi_n\circ\tau_a)'(0)|}{1-|\phi_n(\tau_a(0))|^2}=\limsup_{n\to\infty}\frac{(1-|a|^2)|\phi'_n(a)|}{1-|\phi_n(a)|^2}=1,$$

completing the proof of the corollary. □

The next corollary tells us that the set of $\mathcal{B}$-universal functions is quite large.

Corollary 2.8. *Let $\{\phi_n\}_{n\geq 1}$ be a sequence of holomorphic self-maps of $\mathbb{D}$ with $\phi_n(0)\to 1$. Suppose that $\{C_{\phi_n}\}_{n\geq 1}$ admits a $\mathcal{B}$-universal function. Then the set of $\mathcal{B}$-universal Blaschke products for $\{C_{\phi_n}\}_{n\geq 1}$ is dense in $\mathcal{B}$.*

Proof. By Theorem 2.1, $\{C_{\phi_n}\}_{n\geq 1}$ admits a universal Blaschke product B. We can write $B=\prod_{j=1}^{\infty} L_j$, where the L_j's are automorphisms of $\mathbb{D}$. Let $\{b_n\}_{n\geq 1}$ be a sequence of finite Blaschke products that are dense in $\mathcal{B}$ and satisfy $b_n(1)=1$ (see Lemma 2.3). Consider the tails $B_n=\prod_{j=n}^{\infty} L_j$. Then $\{b_n B_n : n\in\mathbb{N}\}$ is dense in $\mathcal{B}$ and each member of this set is $\mathcal{B}$-universal for $\{C_{\phi_n}\}_{n\geq 1}$. $\square$

Remark 2.9. In general, the product of two $\mathcal{B}$-universal functions is not $\mathcal{B}$-universal. For example, if B is a universal Blaschke product, then B^2 is not universal. However, the composition of two $\mathcal{B}$-universal functions is a $\mathcal{B}$-universal function: Suppose that B_1 and B_2 are $\mathcal{B}$-universal for $\{C_{\phi_n}\}_{n\geq 1}$. Then $B_1\circ B_2$ is $\mathcal{B}$-universal, too. To see this, note that since B_2 is universal, for every m there exists a subsequence $\{\phi_{n_j(m)}\}_{j\geq 1}$ such that $B_2\circ\phi_{n_j(m)}\to\phi_m$. So $B_1\circ\phi_m = \lim_{j\to\infty} B_1\circ(B_2\circ\phi_{n_j(m)})$ and therefore the orbit of B_1 under $\{C_{\phi_n}\}_{n\geq 1}$ is contained in the closure of the orbit of $B_1\circ B_2$. Since the former is dense in $\mathcal{B}$, the latter must be too. Thus $B_1\circ B_2$ is $\mathcal{B}$-universal.

3. Infinite-dimensional spaces of universal functions

3.1. The main result

In this section we follow the line of thought of papers such as [1], [3], [5] and [19] and study how large the subspaces of universal functions can be. We say that a uniformly closed subspace V of $H(\mathbb{D})$ is topologically generated by the set E if the linear span of E is uniformly dense in V.

Theorem 3.1. *Let $\{\phi_n\}_{n\geq 1}$ be a sequence of holomorphic self-maps of $\mathbb{D}$. The following are equivalent*:

(A) *the sequence $\{C_{\phi_n}\}_{n\geq 1}$ admits a $\mathcal{B}$-universal function*;

(B) *the set $\mathcal{B}_q^p=\{(u_1\circ\phi_n, ..., u_p\circ\phi_n) : n\geq q,$ and $u_j\in\mathcal{B}, 1\leq j\leq p\}$ is dense in $\mathcal{B}^p$ for any integers $p, q\geq 1$*;

(C) *there exists a uniformly closed infinite-dimensional vector subspace V of H^∞, topologically generated by Blaschke products and linearly isometric to ℓ^1, with the property that every function in V is universal with respect to $\{C_{\phi_n}\}_{n\geq 1}$.*

The proof of this theorem requires Lemmas 2.3 and 2.4 of the previous section as well as Proposition 3.2 below, which is the "multidimensional version" of Proposition 2.5. The proof is exactly the same, so we omit it.

Proposition 3.2. *Let $\{\phi_n\}_{n\geq 1}$ be a sequence of holomorphic self-maps of $\mathbb{D}$ such that $\phi_n(0)\to 1$ and for any $p,q\geq 1$, the set $\mathcal{B}^p_q=\{(u_1\circ\phi_n,...,u_p\circ\phi_n):n\geq q$ and $u_j\in\mathcal{B}, 1\leq j\leq p\}$ is dense in $\mathcal{B}^p$. Then, for any $p\geq 1$, any $m_0\geq 1$, any $\varepsilon>0$, any functions $(f_1,...,f_p)$ in $\mathcal{B}^p$ and any $K\subseteq\mathbb{D}$ compact, there exist finite Blaschke products $B_1,...,B_p$ and an integer $m\geq m_0$ such that*

$$\|B_j\circ\phi_m-f_j\|_K<\varepsilon,\ \|B_j-1\|_K<\varepsilon \text{ and } B_j(1)=1 \quad \text{for all } j\in\{1,...,p\}.$$

The proof of Theorem 3.1 is very similar to the one that appears in [3]. A special case of this theorem was proved using maximal ideal space techniques in [19]. Before we prove the theorem, we present a lemma that will be useful in its proof:

Lemma 3.3. *Let $\{\phi_n\}_{n\geq 1}$ be a sequence of holomorphic self-maps of $\mathbb{D}$. Then the uniform limit of universal functions in H^∞ is a universal function with respect to $\{C_{\phi_n}\}_{n\geq 1}$.*

Proof. Let $\{f_n\}_{n\geq 1}$ be a sequence of universal functions in H^∞ that converge uniformly to f on $\mathbb{D}$ and let K be a compact subset of $\mathbb{D}$. Let $g\in H^\infty$ with $\|g\|_\infty\leq\|f\|_\infty$. Since $\|(\|f_n\|_\infty/\|f\|_\infty)g\|_\infty\leq\|f_n\|_\infty$ and f_n is universal, there exists a subsequence $\{\phi_{j_k(n)}\}_{k\geq 1}$ of $\{\phi_j\}_{j\geq 1}$ (depending on n), such that $f_n\circ\phi_{j_k(n)}\to(\|f_n\|_\infty/\|f\|_\infty)g$ locally uniformly in $\mathbb{D}$. So

$$\begin{aligned}\|f\circ\phi_{j_k(n)}-g\|_K &\leq \|f\circ\phi_{j_k(n)}-f_n\circ\phi_{j_k(n)}\|_\infty\\ &\quad+\left\|f_n\circ\phi_{j_k(n)}-g\frac{\|f_n\|_\infty}{\|f\|_\infty}\right\|_K+\left\|\frac{\|f_n\|_\infty}{\|f\|_\infty}g-g\right\|_\infty,\end{aligned}$$

from which the result follows. □

Here is the proof of Theorem 3.1.

3.2. Proof of (A) ⇒ (B)

For $k=1,...,p$, let $f_k\in\mathcal{B}$. By assumption (A), there exists a universal function $u\in\mathcal{B}$ and a sequence $\{n_j\}_{j\geq 1}$ of integers such that $u\circ\phi_{n_j}\to z$. Then $f_k\circ u\circ\phi_{n_j}\to f_k$ uniformly on compact sets as $j\to\infty$. Therefore

$$\{(u_1\circ\phi_n,...,u_p\circ\phi_n):n\geq q \text{ and } u_j\in\mathcal{B}, 1\leq j\leq p\}$$

is dense in $\mathcal{B}^p$ for every $q\in\mathbb{N}$. This yields (B).

Since it is clear that (C) ⇒ (A), our proof will be complete if we show that (B) ⇒ (C). Our proof follows closely the proof of Theorem 1 in [3], with minor modifications so as to adapt the proof in [3] to the situation we discuss here.

3.3. Proof of (B) $\Rightarrow$ (C)

To begin the proof, let $\{\alpha_n\}_{n\geq 1}$ be a countable dense sequence of points on the unit circle with $\alpha_1=1$ and let $\{h_n\}_{n\geq 1}$ be a sequence in $\mathcal{B}$ that is locally uniformly dense. There is no loss of generality in assuming that h_n is not identically equal to 1. Write $\mathbb{D}$ as $\mathbb{D}=\bigcup_{n=1}^{\infty} K_n$, where K_n is compact and $K_n\subseteq K_{n+1}^{\circ}$ for every n.

Consider the infinite matrix A defined by

$$\begin{pmatrix} \alpha_1 h_1 & \alpha_1 h_2 & \alpha_1 h_2 & \alpha_2 h_2 & \alpha_2 h_2 & \alpha_1 h_3 & \alpha_1 h_3 & \alpha_1 h_3 & \dots \\ 1 & \alpha_1 h_2 & \alpha_2 h_2 & \alpha_1 h_2 & \alpha_2 h_2 & \alpha_1 h_3 & \alpha_1 h_3 & \alpha_1 h_3 & \dots \\ 1 & 1 & 1 & 1 & 1 & \alpha_1 h_3 & \alpha_2 h_3 & \alpha_3 h_3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where the second block (that is, the 2nd through 5th column) contains all possible combinations of (α_1,α_2) multiplied by h_2 (there are four possibilities, $(\alpha_k h_2,\alpha_j h_2)$ where $k=1,2$ and $j=1,2$, listed as the first two entries in columns 2, 3, 4, and 5), the third block (6th through 32nd column) contains all possible combinations of $(\alpha_1,\alpha_2,\alpha_3)$ multiplied by the function h_3, and so on. Then each element of the matrix A is of the form $\alpha_{s(k,j)}h_{t(j)}$, where $s(k,j)$ is determined by both the column and row of the entry and $t(j)$ is determined solely by the column. Let $M(j)$ denote the number of terms in column j that are different from 1, that is that are written in the form $\alpha_{s(k,j)}h_{t(j)}$. By induction on j, we will choose finite Blaschke products $B_{k,j}$ and a sequence of integers $\{m_j\}_{j\geq 1}$ such that

$$\|B_{k,j}\circ\phi_{m_j}-\alpha_{s(k,j)}h_{t(j)}\|_{K_j}<\frac{1}{2^j} \quad \text{for all } k\leq M(j); \tag{5}$$

$$B_{k,j}=1 \quad \text{for } k>M(j); \tag{6}$$

$$|B_{k,l}\circ\phi_{m_j}(0)-1|<\frac{1}{2^j} \quad \text{for all } l<j \text{ and all } k; \tag{7}$$

$$|B_{k,j}\circ\phi_{m_l}(0)-1|<\frac{1}{2^j} \quad \text{for all } l<j \text{ and all } k; \tag{8}$$

$$\|B_{k,j}-1\|_{K_j}<\frac{1}{2^j} \quad \text{for all } k\leq M(j); \tag{9}$$

$$B_{k,j}(1)=1 \quad \text{for all } k. \tag{10}$$

We note that the first step of the induction follows from Proposition 3.2.

Now we assume that the construction has been carried out until step $j-1$ and show how to complete step j. Using the continuity at the point 1 of the (finite) set of functions $(B_{k,l})_{k\leq M(l),\, l<j}$, we choose an integer m_j such that, for every $l<j$, for every $k\geq 1$ and for every $m\geq m_j$, $|B_{k,l}\circ\phi_m(0)-1|<1/2^j$. Now let $K=K_j\cup\bigcup_{l<j}\{\phi_{m_l}(0)\}$. The functions $B_{k,j}$ are automatically given by Proposition 3.2.

We define a function B_k on $\mathbb{D}$ by

$$B_k = \prod_{j=1}^{\infty} B_{k,j}$$

and establish the following claim:

Claim 3.4. The following assertions hold:

$$\text{For each } k, \text{ the function } B_k \text{ is a Blaschke product;} \tag{11}$$

$$\prod_{j\neq l} B_{k,j} \circ \phi_{m_l}(0) \to 1 \text{ as } l\to\infty; \tag{12}$$

$$|B_k \circ \phi_{m_l} - B_{k,l} \circ \phi_{m_l}| \to 0 \text{ uniformly on compact sets as } l\to\infty. \tag{13}$$

Proof of Claim 3.4. By (9), the infinite product

$$\prod_{j=1}^{\infty} B_{k,j} = \prod_{j:M(j)<k} B_{k,j} \prod_{j:M(j)\geq k} B_{k,j}$$

converges uniformly to B_k on compact subsets of $\mathbb{D}$, which proves the first part of the claim.

We turn to showing that $\prod_{j\neq l} B_{k,j} \circ \phi_{m_l}(0) \to 1$ uniformly on compact sets. Recall that for complex numbers z_j, $j=1, ..., m$, with $|z_j| \leq 1$, we have

$$\left|1 - \prod_{j=1}^{m} z_j\right| \leq \sum_{j=1}^{m} |1 - z_j|.$$

We will use this inequality to establish the rest of the claim as follows:

$$\begin{aligned}
\left|1 - \prod_{j\neq l} B_{k,j} \circ \phi_{m_l}(0)\right| &\leq \sum_{j\neq l} |1 - B_{k,j} \circ \phi_{m_l}(0)| \\
&\leq \sum_{j=l+1}^{\infty} |1 - B_{k,j} \circ \phi_{m_l}(0)| + \sum_{j=1}^{l-1} |1 - B_{k,j} \circ \phi_{m_l}(0)| \\
&\leq \sum_{j=l+1}^{\infty} \frac{1}{2^j} + \sum_{j=1}^{l-1} \frac{1}{2^l} \\
&= l \frac{1}{2^l},
\end{aligned}$$

where the first sum was estimated using (8) and the second one using (7). This completes the proof that $\prod_{j\neq l} B_{k,j} \circ \phi_{m_l}(0)$ converges to 1.

To prove the third assertion of the claim, we note that

$$|B_k\circ\phi_{m_l}-B_{k,l}\circ\phi_{m_l}|=\left|\left(1-\prod_{j\neq l}B_{k,j}\circ\phi_{m_l}\right)B_{k,l}\circ\phi_{m_l}\right|.$$

Now we have proved that $\prod_{j\neq l}B_{k,j}\circ\phi_{m_l}(0)\to 1$, $|\prod_{j\neq l}B_{k,j}\circ\phi_{m_l}|\leq 1$, and Fact 2.2, so we can apply these results to conclude that $\prod_{j\neq l}B_{k,j}\circ\phi_{m_l}$ converges to 1 uniformly on compact sets. This, in turn, implies that

$$|B_k\circ\phi_{m_l}-B_{k,l}\circ\phi_{m_l}|\to 0\quad\text{uniformly on compact sets,}$$

completing the proof of the claim. □

Now let the vector space V be the uniform closure of the vector space generated by the functions B_j, $j=1,2,\dots$. We must show that if $f\in V$, then f is universal for $\{g\in H^\infty:\|g\|_\infty\leq\|f\|_\infty\}$. Since this is clear for $f=0$, we can assume that f is not identically equal to 0, and if we divide f by its norm, we see that we can reduce our study to functions of norm 1. By Lemma 3.3 the uniform limit of universal functions is universal, so it is enough to consider functions that are in the span of the B_j's. Given these considerations, we now choose $f\in V$ with $\|f\|_\infty=1$ and $f=\sum_{j=1}^n\lambda_jB_j$. Let $h\in H^\infty(\mathbb{D})$ with $\|h\|_\infty\leq 1$. Now let $\mu_j=\bar{\lambda}_j/|\lambda_j|$ (where $\mu_j=1$ if $\lambda_j=0$). Let K_r be one of our compact sets. Choose a column, k, in the matrix such that the function $h_{t(k)}$ appearing in that column satisfies $\|h-h_{t(k)}\|_{K_r}<\varepsilon/(n\sum_{j=1}^n|\lambda_j|)$ and the column has the n permuted values $\alpha_{s(l,k)}$, for $l=1,\dots,n$ in the correct order so that

$$\max_{l=1,\dots,n}|\alpha_{s(l,k)}-\mu_l|<\frac{\varepsilon}{n\sum_{j=1}^n|\lambda_j|}.$$

Then using Claim 3.4 and property (5) on K_r for sufficiently large k we get

$$f\circ\phi_{m_k}=\sum_{l=1}^n\lambda_l(B_l\circ\phi_{m_k})\approx\sum_{l=1}^n\lambda_lB_{l,k}\circ\phi_{m_k}\approx\sum_{l=1}^n\lambda_l(\alpha_{s(l,k)}h_{t(k)})\quad\text{on } K_r.$$

So

$$\begin{aligned}\left\|f\circ\phi_{m_k}-\sum_{j=1}^n|\lambda_j|h\right\|_{K_r}&\approx\left\|\sum_{j=1}^n\lambda_j\alpha_{s(j,k)}h_{t(k)}-\sum_{j=1}^n|\lambda_j|h\right\|_{K_r}\\&=\left\|\sum_{j=1}^n\lambda_j\mu_jh_{t(k)}+\sum_{j=1}^n\lambda_j(\alpha_{s(j,k)}-\mu_j)h_{t(k)}-\sum_{j=1}^n|\lambda_j|h\right\|_{K_r}\\&\leq\left\|\sum_{j=1}^n|\lambda_j|(h_{t(k)}-h)\right\|_{K_r}+\varepsilon\\&<2\varepsilon.\end{aligned}$$

Now we can choose any function in $\mathcal{B}$ in place of h, for instance the constant function 1. For any $\delta>0$, we find that there exists an integer m'_k such that

$$\left| f\circ\phi_{m'_k}(0) - \sum_{j=1}^{n} |\lambda_j| \right| < \delta.$$

This yields $\sum_{j=1}^{n} |\lambda_j| \le \|f\|_\infty + \delta$, and since δ is arbitrary, we get $\sum_{j=1}^{n} |\lambda_j| = 1 = \|f\|_\infty$. Thus, $\|f\circ\phi_{m_k} - h\|_{K_r} < 2\varepsilon$, and f is universal. By homogeneity, our vector space is infinite-dimensional, isometric to ℓ^1, and each function $f\in V$ can be written as $f=\sum_{j=1}^{\infty} a_j B_j$ where the sequence $\{a_j\}_{j\ge1}$ belongs to ℓ^1. This finishes the proof of Theorem 3.1.

If we choose the Blaschke generators more carefully, we obtain the following proposition:

Proposition 3.5. *Let $\{\phi_n\}_{n\ge1}$ be a sequence of holomorphic self-maps of $\mathbb{D}$ having the property that $\lim_{n\to\infty}\phi_n(0)=1$. Suppose that $\{C_{\phi_n}\}_{n\ge1}$ admits a $\mathcal{B}$-universal function f. Then there exists a subspace V of $H(\mathbb{D})$ that is dense in $H(\mathbb{D})$ in the local uniform topology, closed in the sup-norm topology, and such that every element of V is bounded and universal for $\{C_{\phi_n}\}_{n\ge1}$.*

Proof. We begin by applying Theorem 3.1 to obtain a uniformly closed vector space V generated by $\mathcal{B}$-universal Blaschke products B_l. Then we modify the B_l's: Let $\{p_l\}_{l\ge1}$ denote the sequence of monomials

$$\{p_l\}_{l\ge1} = (1,1,z,1,z,z^2,1,z,z^2,z^3,1,z,z^2,z^3,z^4,1,z,...)$$

and let $C_l=p_lB_l$. Since $p_l(1)=1$ for every l, each of these new functions is $\mathcal{B}$-universal for $\{C_{\phi_n}\}_{n\ge1}$. As in the proof of Theorem 3.1 above, every function in the uniformly closed vector space W generated by the C_l is universal. We claim that $C_l\to1$ locally uniformly. From this claim it will follow that the closure of W in the local uniform topology contains the polynomials and therefore is dense in $H(\mathbb{D})$. Thus, once we establish our claim, Proposition 3.5 will be proved.

So let $K\subseteq\mathbb{D}$ be a compact subset of $\mathbb{D}$. Choose an index l. Returning to the notation of the previous theorem (recall that $M(j)$ denotes the number of terms in column j that are different from 1), let j_l denote that smallest index for which $M(j_l)\ge l$ for the first time. Choose l_0 so that $K\subseteq K_{l_0}$ and $2^{-j_l}<\varepsilon/2$ whenever $j_l\ge M(j_l)\ge l\ge l_0$. On K, by (6) and (9), we have

$$|B_l-1| \le \sum_{j=1}^{j_l-1} |B_{lj}-1| + \sum_{j=j_l}^{\infty} |B_{lj}-1| \le 0 + \sum_{j=j_l}^{\infty} 2^{-j} < \varepsilon.$$

This proves the claim and completes the proof of the proposition. □

Remark 3.6. Suppose that E is a subspace of $H(\mathbb{D})$ that is closed in the topology of uniform convergence on compact sets. If E is infinite-dimensional, then we claim that E contains a function that is unbounded on $\mathbb{D}$. Indeed, if this were not the case, then E would be an infinite-dimensional closed subspace of $H^2(\mathbb{D})$ consisting of bounded functions, which is a contradiction (see, for example, [20, p. 117, Theorem 5.2]).

4. Applications, examples and remarks

4.1. Sequences of iterates

Here is our first application of Theorem 2.1. We let ϕ be a self-map of $\mathbb{D}$ and $\phi_{[n]}$ denote the nth iterate of ϕ.

Corollary 4.1. *Let ϕ be a holomorphic self-map of $\mathbb{D}$. Then the sequence of iterates $\{C_{\phi_{[n]}}\}_{n\geq 1}$ of C_ϕ admits a $\mathcal{B}$-universal function if and only if ϕ is a parabolic or hyperbolic automorphism.*

Proof. If ϕ is an automorphism, then $|\phi'(0)|/(1-|\phi(0)|^2)=1$. Since the orbits of parabolic and hyperbolic automorphisms in $\mathbb{D}$ cluster at a single boundary point (the so-called Denjoy–Wolff fixed point, see [22, p. 78]), the result follows from Theorem 2.1.

Now suppose that there is a $\mathcal{B}$-universal function u for $\{\phi_{[n]}\}_{n\geq 1}$. Since $\phi_{[n]}$ is the nth iterate of ϕ, we see that $u\circ\phi_{[n]}=(u\circ\phi_{[n-1]})\circ\phi$, and so

$$\{u\circ\phi_{[n]} : u\in\mathcal{B} \text{ and } n\in\mathbb{N}\}=\{u\circ\phi : u\in\mathcal{B}\}.$$

By assumption $\{u\circ\phi_{[n]}:u\in\mathcal{B}$ and $n\in\mathbb{N}\}$ is dense in $\mathcal{B}$, so $\{u\circ\phi:u\in\mathcal{B}\}$ is dense, too. Hence there exists $u_n\in\mathcal{B}$ such that $u_n\circ\phi\to z$ uniformly on compact sets. Thus

$$(u_n\circ\phi)'(0)=(1-|\phi(0)|^2)\, u_n'(\phi(0))\frac{\phi'(0)}{1-|\phi(0)|^2}\to 1.$$

Since $(1-|\phi(0)|^2)|u_n'(\phi(0))|\leq 1$, we must have $|\phi'(0)|/(1-|\phi(0)|^2)=1$. By the Schwarz lemma, ϕ must be an automorphism. Of course, ϕ cannot have a fixed point in $\mathbb{D}$, so ϕ is a parabolic or hyperbolic automorphism (see [22]). $\square$

4.2. A corollary on hyperbolic distance

The following lemma is well known (see, for example, [10, p. 405] or [9, p. 943]).

Lemma 4.2. *Let f be a holomorphic self-map of $\mathbb{D}$ such that $f(0)=0$ and $|f'(0)|\geq\delta$. Then for every $\eta<\delta$ we have $D(0,\eta(\delta-\eta)/(1-\eta\delta))\subseteq f(\eta\mathbb{D})$.*

For $z\in\mathbb{D}$ we let

$$L_a(z)=\frac{|a|}{a}\frac{a-z}{1-\bar{a}z}$$

and $D_\rho(a,\varepsilon)=\{z\in\mathbb{D}:\rho(z,a)=|L_a(z)|<\varepsilon\}$, the pseudohyperbolic disk of center a and radius ε. Let the pseudohyperbolic diameter of a set S be denoted by $r_\rho(S)$.

Corollary 4.3. *Let $\{\phi_n\}_{n\geq 1}$ be a sequence of holomorphic self-maps of $\mathbb{D}$ having the property that $\phi_n(0)\to 1$. If $\{C_{\phi_n}\}_{n\geq 1}$ admits a $\mathcal{B}$-universal function, then*

$$\limsup_{k\to\infty} r_\rho(\phi_k(\eta_k\mathbb{D}))=1$$

for some sequence $\{\eta_k\}_{k\geq 1}$, where $0<\eta_k<1$ with $\eta_k\to 1$.

Proof. Suppose that $\{C_{\phi_n}\}_{n\geq 1}$ admits a $\mathcal{B}$-universal function. By Theorem 2.1 we know that there exists a subsequence, denoted $\{\phi_{n_k}\}_{k\geq 1}$, such that

$$\lim_{n_k\to\infty}\frac{|\phi'_{n_k}(0)|}{1-|\phi_{n_k}(0)|^2}=1.$$

Let $u_k=\tau_{\phi_{n_k}(0)}\circ\phi_{n_k}$. A straightforward computation shows that

$$u'_k(0)=-\frac{\phi'_{n_k}(0)}{1-|\phi_{n_k}(0)|^2},$$

so $\delta_k:=|u'_k(0)|\to 1$. Choose $\eta_k=1-\sqrt{1-\delta_k^2}$. Then $\eta_k<\delta_k$ and

$$r_k:=\eta_k\left(\frac{\delta_k-\eta_k}{1-\delta_k\eta_k}\right)\to 1,\quad \text{as } k\to\infty.$$

By Lemma 4.2, $D_\rho(0,r_k)\subseteq u_k(\eta_k\mathbb{D})$. Hence

$$D_\rho(\phi_{n_k}(0),r_k)\subseteq\phi_{n_k}(\eta_k\mathbb{D}).$$

An alternative proof of Corollary 4.3 goes as follows: By Cauchy's integral formula

$$u'_k(0)=\int_0^{2\pi}\frac{u_k(\eta_k e^{i\theta})e^{-i\theta}}{\eta_k}\,\frac{d\theta}{2\pi}.$$

It is easy to see that this implies the existence of a $\theta_0\in]0,2\pi[$ such that

$$\frac{|u_k(\eta_k e^{i\theta_0})|}{\eta_k}\geq|u'_k(0)|.$$

Let $z_k:=\eta_k e^{i\theta_0}$. Recalling that $u_k(0)=0$ we see that $|u_k(z_k)|\geq\eta_k|u'_k(0)|$. Hence

$$\rho(\phi_{n_k}(z_k),\phi_{n_k}(0))=\rho(u_k(z_k),u_k(0))=|u_k(z_k)|\to 1.\quad\square$$

4.3. Somewhere dense orbits

We now present a result along the same lines as Bourdon and Feldman [7] for abstract hypercyclicity phenomena. Our result is much weaker than Bourdon and Feldman's, as it is simplified by our assumption and previous work.

Corollary 4.4. *Let ϕ be a holomorphic self-map of $\mathbb{D}$ and $\phi_{[n]}$ the nth iterate. For $B_j \in \mathcal{B}$, let $E_j = \{B_j \circ \phi_{[n]} : n \in \mathbb{N}\}$. Suppose that for some $N \in \mathbb{N} \cup \{\infty\}$ we have $\bigcup_{j=1}^{N} \overline{E}_j = \mathcal{B}$. Then there exists j_0 such that B_{j_0} is $\mathcal{B}$-universal.*

Proof. First we note that our hypothesis implies that ϕ has no fixed point inside $\mathbb{D}$, for otherwise each $f \in \overline{E}_j$ would have the same value $B_j(p)$ at (the unique) fixed point p of ϕ. But the set $\bigcup \{B_j(p) : j=1, ..., N\}$ is at most countable, while the set $\{f(p) : f \in \mathcal{B}\}$ is uncountable. Hence, by the grand iteration theorem (see [22, p. 78]), the iterates $\phi_{[n]}$ converge locally uniformly to a boundary (fixed) point ω. We can assume that $\omega=1$. Since the identity function z is in the closure of at least one of the sets $\{B_j \circ \phi_{[n]} : n \in \mathbb{N}\}$ we see that condition (D) of Theorem 2.1 is satisfied. Since $\phi_{[n]}(0) \to 1$, we can apply the aforementioned theorem to conclude that $\{C_{\phi_{[n]}}\}_{n \geq 1}$ admits a $\mathcal{B}$-universal function. On the other hand, since $\mathcal{B}$ is a Baire set, there exists j_0 such that the set $\overline{\{B_{j_0} \circ \phi_{[n]} : n \in \mathbb{N}\}}$ contains an open set U. Moreover, by Corollary 2.8 the set of $\mathcal{B}$-universal functions for $\{\phi_{[n]}\}_{n \geq 1}$ is dense in $\mathcal{B}$, so U contains a universal function f. Thus, f belongs to the closed orbit of B_{j_0}. In particular, the orbit of f itself is contained in the closed orbit of B_{j_0}. Since the former is dense in $\mathcal{B}$, the latter must be as well. Thus B_{j_0} is $\mathcal{B}$-universal. $\square$

4.4. Left composition operators

Let $\mathcal{B}^0$ denote the set of functions in $\mathcal{B}$ such that $|f(z)| < 1$ for every $z \in \mathbb{D}$ and let $\mathcal{S}^0 = \mathcal{B}^0 \cap \mathcal{S}$. In the introduction we mentioned that the identity function is a $\mathcal{B}$-universal function for $\{C_{p_n}\}_{n \geq 1}$, where $\{p_n\}_{n \geq 1}$ is a sequence of finite Blaschke products dense in $\mathcal{B}$. Are there any other $\mathcal{B}$-universal functions of this trivial type? In other words, what kind of functions $U \in \mathcal{B}^0$ are there such that the map $f \mapsto U \circ f$ is a surjection of $\mathcal{B}^0$ onto itself? We may ask (an appropriate version of) the same question about $V \in \mathcal{S}^0$; when is $f \mapsto V \circ f$ a surjection of $\mathcal{B}^0$ onto $\mathcal{S}^0$? The answer is given by the following proposition.

The proof of the proposition uses the notion of prime and semiprime inner functions (see [23] and [12]). An inner function u is said to be *prime* if whenever $u = f \circ g$ for two holomorphic self-maps f and g of $\mathbb{D}$, then f or g is an automorphism of $\mathbb{D}$. An inner function u is said to be *semiprime* if whenever $u = f \circ g$ for two holomorphic self-maps f and g of $\mathbb{D}$, then f or g is a finite Blaschke product.

In [12, Proposition 1.2], it is shown that the only (normalized) nonprime decompositions of the atomic inner function are those for which $f=z^n$ and $g=S^{1/n}$. We are now ready for the proof of the proposition.

Proposition 4.5. (a) *Let $U\in\mathcal{B}$. Then the map $f\mapsto U\circ f$ is a surjection of $\mathcal{B}^0$ onto itself if and only if U is an automorphism of $\mathbb{D}$.*

(b) *Let $V\in\mathcal{S}$. Then the map $f\mapsto V\circ f$ is a surjection of $\mathcal{B}^0$ onto $\mathcal{S}^0$ if and only if $V=S\circ\tau$ for some automorphism τ of $\mathbb{D}$.*

Proof. (a) Suppose that $f\mapsto U\circ f$ is a surjection of $\mathcal{B}^0$ onto $\mathcal{B}^0$. Then there exists $f\in\mathcal{B}^0$ such that $U\circ f=z$. By [23, p. 847], U and f are inner functions. But every finite Blaschke product of prime degree (in this case z) is prime, so U or f must be an automorphism (see [23, p. 847] or [12, p. 254]). In either case U must be an automorphism. The converse is trivial.

(b) Suppose that $f\mapsto V\circ f$ is a surjection of $\mathcal{B}^0$ onto $\mathcal{S}^0$. Then there exists $f\in\mathcal{B}^0$ such that $V\circ f=S$. As above, V and f are inner functions. But S is semiprime (see [12, p. 255]); since $V\in\mathcal{S}$, we conclude that f is an automorphism and hence $V=S\circ\tau$. Again, the converse is trivial. $\square$

The atomic singular inner function has the following curious property: $S^\alpha=S\circ\tau$, where $\tau=(z-(1-\alpha)/(1+\alpha))/(1-z(1-\alpha)/(1+\alpha))$. Therefore, for every $\alpha>0$, the function $V=S^\alpha$ gives rise to a surjection from B^0 to S^0.

4.5. Some examples

Here are some examples of the sort of behavior we can expect. Our first example is a sequence of self-maps that are not automorphisms, but admit universal Blaschke products. The idea of looking at conformal maps that take a set slightly larger than $\mathbb{D}$ onto $\mathbb{D}$ can be modified to produce many examples of this type.

Example 4.6. Let U_n be domains of $\mathbb{C}$ satisfying $\mathbb{D}\subseteq U_n\subseteq\{z\in\mathbb{C}:|z|<1+\varepsilon_n\}$ where $\{\varepsilon_n\}_{n\geq 1}$ is a sequence of positive numbers tending to 0. Let $\{\phi_n\}_{n\geq 1}$ be a sequence of conformal maps of U_n onto $\mathbb{D}$ with $\phi_n(0)\to 1$. Then $\{C_{\phi_n}\}_{n\geq 1}$ admits a universal Blaschke product.

To see this, let B be a finite Blaschke product. For n sufficiently large, B will have no poles in U_n and we can define the function u_n by

$$u_n=\frac{B\circ\phi_n^{-1}}{\max\{|B(z)|:z\in U_n\}}.$$

Then $u_n \in \mathcal{B}$ and $u_n \circ \phi_n \to B$ on compact sets, since $\max\{|B(z)|: z \in U_n\}$ tends to 1 by the assumption on U_n. Hence $\{u \circ \phi_n : u \in \mathcal{B} \text{ and } n \in \mathbb{N}\}$ is dense in $\mathcal{B}$. The existence of the universal Blaschke product now follows from Theorem 2.1.

The next example describes the sequences $\{f_{\mu_n}\}_{n \geq 1}$ of normalized singular inner and outer functions that admit universal functions.

Example 4.7. Let $f \in \mathcal{S}$ be a normalized function; that is $f(0) > 0$. Then

$$f(z) = f_\mu(z) := \exp\biggl(-\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z}\, d\mu(t)\biggr),$$

where μ is a Borel measure on the unit circle $\mathbb{T}$ given by $\mu = -\log|f(e^{it})|\, dt/2\pi + \mu_s$ for some positive Borel measure μ_s singular with respect to the Lebesgue measure on $\mathbb{T}$. Let $\hat{\mu}(j) = \int_0^{2\pi} e^{-ijt}\, d\mu(t)$ denote the jth Fourier coefficient of μ. Then if the sequence of functions $\{f_{\mu_n}\}_{n \geq 1}$ is such that $f_{\mu_n}(0) \to 1$, it admits a $\mathcal{B}$-universal function if and only if

$$\limsup_{n\to\infty} \frac{|\hat{\mu}_n(1)|}{\sinh \hat{\mu}_n(0)} = 1. \tag{14}$$

Indeed, since $f'_\mu(z) = f_\mu(z) \int_0^{2\pi} -2e^{it}/(e^{it}-z)^2\, d\mu(t)$, we see that $f_\mu(0) = e^{-\hat{\mu}(0)}$ and $f'_\mu(0) = -2f_\mu(0)\hat{\mu}(1)$. Hence

$$\frac{|f'_{\mu_n}(0)|}{1-|f_{\mu_n}(0)|^2} = \frac{2e^{-\hat{\mu}_n(0)}|\hat{\mu}_n(1)|}{1-e^{-2\hat{\mu}_n(0)}} = \frac{|\hat{\mu}_n(1)|}{\sinh \hat{\mu}_n(0)}.$$

The result follows from Theorem 2.1.

As an example of a sequence of such functions that admits a $\mathcal{B}$-universal function we mention the sequence of roots of the atomic inner function, $\phi_n = S^{1/n}$.

Now we turn to an example of a sequence that does not admit $\mathcal{B}$-universal functions.

Example 4.8. Let $\phi_n(z) = \alpha_n z + (1-\alpha_n)$, where $0 < \alpha_n < 1$ and $\alpha_n \to 0$. Then

$$\frac{\phi'_n(0)}{1-|\phi_n(0)|^2} \to \frac{1}{2}.$$

By Theorem 2.1, this family of self-maps does not admit $\mathcal{B}$-universal functions. This tells us that the geometric condition in Corollary 4.3 is not sufficient for guaranteeing the existence of $\mathcal{B}$-universal functions.

5. Universality on $H(\mathbb{D})$

Now we shall derive a necessary and sufficient condition for a sequence of self-maps of $\mathbb{D}$ to have the associated composition operators C_{ϕ_n} admit universal functions in the Fréchet space $H(\mathbb{D})$. The case in which ϕ_n is an automorphism is well known (see, for example, [4], [5], [6] and [18]). The situation here is easier than in the case of $\mathcal{B}$-universal functions, because we do not need to be concerned about the norm of the functions under construction. We state separately the following straightforward lemma:

Lemma 5.1. *Let $\{f_n\}_{n\geq 1}$ be a sequence of holomorphic functions on a disk $D(0,R)$. Suppose that the sequence $\{f_n\}_{n\geq 1}$ converges uniformly to an injective function f on $D(0,R)$. Let $0<r<R$. Then f_n is injective on $K:=\overline{D}(0,r)$ for sufficiently large n.*

Proof. Let z_0 be a point of $\overline{D}(0,r)$ and $r<r'<R$. For every n and every $z\in\partial D(0,r')$, we have $|f(z)-f(z_0)-(f_n(z)-f_n(z_0))|\leq 2\,\|f-f_n\|_{\overline{D}(0,r')}$ which tends to zero as n goes to infinity. Since f is injective on $\overline{D}(0,r')$,

$$\delta := \inf_{\substack{|z|=r'\\ |\xi|\leq r}} |f(z)-f(\xi)|$$

is positive. So, for n large enough, say $n\geq n_0$, where n_0 is independent of the point z_0, $|f(z)-f(z_0)-(f_n(z)-f_n(z_0))|<\delta\leq|f(z)-f(z_0)|$ for every $z\in\partial D(0,r')$. Hence $f-f(z_0)$ and $f_n-f_n(z_0)$ have the same number of zeroes in $D(0,r')$ by Rouché's theorem, and f_n is injective on $\overline{D}(0,r)$ whenever $n\geq n_0$. $\square$

Theorem 5.2. *Let $\{\phi_n\}_{n\geq 1}$ be a sequence of holomorphic self-maps of $\mathbb{D}$ such that $\phi_n(0)\to 1$. Then the following assertions are equivalent*:

(a) *the sequence of composition operators $\{C_{\phi_n}\}_{n\geq 1}$ admits an $H(\mathbb{D})$-universal function*;

(b) *there exists a subsequence $\{n_j\}_{j\geq 1}$ such that for all compact sets $K\subseteq\mathbb{D}$ there exists an integer j_0 such that ϕ_{n_j} is injective on K for all $j\geq j_0$.*

Proof. We first prove that (b) $\Rightarrow$ (a). Choose a sequence of compacts sets D_n (which can be chosen to be disks centered at 0, for example, D_n may be chosen to be the closure of $D(0,1-2^{-n})$) such that $\mathbb{D}=\bigcup_{n=1}^{\infty} D_n$ and $D_n\subseteq D_{n+1}^{\circ}$. We note that $\phi_n\to 1$ uniformly on every compact $K\subseteq\mathbb{D}$. Let $\{p_n\}_{n\geq 1}$ be a dense sequence in $H(\mathbb{D})$. We claim that there exist functions $f_n\in H(\mathbb{D})$ and a subsequence $\{k_n\}_{n\geq 1}$ of $\{n_j\}_{j\geq 1}$ so that

(A) $D_n\cap\phi_{k_j}(D_j)=\varnothing$ for every $j\leq n$ and $n\in\mathbb{N}$;

(B) $\phi_{k_n}(D_n)\cap\phi_{k_m}(D_m)=\varnothing$ for $n\neq m$;

(C) ϕ_{k_n} is injective on a neighborhood of D_n;
(D) $f_n(1)=0$;
(E) $|f_j|<2^{-j-n}$ on $\phi_{k_n}(D_n)$ for all $j<n$;
(F) $|f_n|<2^{-n}$ on $D_n\cup\bigcup_{j=1}^{n-1}\phi_{k_j}(D_j)$;
(G) $\|f_n-p_n\circ\phi_{k_n}^{-1}\|_{\phi_{k_n}(D_n)}<2^{-n}$.

The proof will be done by induction on n. Since the case $n=1$ works in the same manner as the general case, we indicate only how the general case works. So assume that $f_1,...,f_{n-1}$ and $k_1,...,k_{n-1}$ are given and satisfy (A)–(G). By our assumption and the fact that $\phi_n\to 1$ we can choose k_n so big that $D_n\cap\phi_{k_j}(D_j)=\varnothing$ for $j\le n$ as well as $\phi_{k_n}(D_n)\cap\phi_{k_m}(D_m)=\varnothing$ for $m<n$ and such that ϕ_{k_n} is injective on a neighborhood of D_n. Then $\phi_{k_n}^{-1}$ is well defined on $\phi_{k_n}(D_n)$. Since the compact set $\{1\}\cup D_n\cup\bigcup_{j=1}^{n}\phi_{k_j}(D_j)$ has a connected complement, by Runge's approximation theorem there is a polynomial q_n such that $|q_n(1)|\le 4^{-n}$ and (E)–(G) are satisfied with q_n in place of f_n and 2^{-s} replaced by 4^{-s}. By considering $f_n=q_n-q_n(1)$ we get assertions (D)–(G).

Next we claim that $f=\sum_{n=1}^{\infty}f_n$ is an $H(\mathbb{D})$-universal function for $\{C_{\phi_n}\}_{n\ge1}$. To check this, we note first that, due to (F), the series converges locally uniformly on $\mathbb{D}$. Now fix n. On D_n, use (E)–(G) to conclude that

$$\begin{aligned}|f\circ\phi_{k_n}-p_n| &\le \sum_{j<n}|f_j\circ\phi_{k_n}|+\sum_{j>n}|f_j\circ\phi_{k_n}|+|f_n\circ\phi_{k_n}-p_n|\\ &\le \sum_{j<n}2^{-j-n}+\sum_{j>n}2^{-j}+2^{-n}\to 0, \qquad \text{as } n\to\infty.\end{aligned}$$

This proves that f is an $H(\mathbb{D})$-universal function for $\{C_{\phi_n}\}_{n\ge1}$.

To prove that (a) $\Rightarrow$ (b), let f be an $H(\mathbb{D})$-universal function for $\{C_{\phi_n}\}_{n\ge1}$. Then there is a sequence n_j for which $f\circ\phi_{n_j}\to z$. Using Lemma 5.1, we conclude that for every compact set $K\subseteq\mathbb{D}$ and for j sufficiently large, the functions $f\circ\phi_{n_j}$ must be injective on K. Consequently, the same must hold for ϕ_{n_j}. □

Applying Theorem 5.2 to the sequence defined by $\phi_n=\alpha_n z+1-\alpha_n$, $\alpha_n\to 0$, which did not admit a $\mathcal{B}$-universal function, we see that this sequence does admit $H(\mathbb{D})$-universal functions.

Now consider a nonautomorphic, but injective, holomorphic, self-map ϕ of $\mathbb{D}$ whose Denjoy–Wolff point is located on the unit circle. In contrast to Corollary 4.1, there exists, by Theorem 5.2, an $H(\mathbb{D})$-universal function for the sequence of iterates of ϕ.

Corollary 5.3. *Let ϕ be a holomorphic self-map of $\mathbb{D}$. Then the sequence, $\{C_{\phi_{[n]}}\}_{n\ge1}$, of iterates of C_ϕ admits an $H(\mathbb{D})$-universal function if and only if ϕ is injective with Denjoy–Wolff fixed point located on the boundary.*

Proof. An elliptic automorphism or a self-map with fixed point in $\mathbb{D}$ cannot yield $H(\mathbb{D})$-universal functions. Thus, it follows from iteration theory (see [22]), that $\{\phi_{[n]}\}_{n\geq 1}$ converges locally uniformly to a boundary fixed point (the Denjoy–Wolff point) whenever $\{C_{\phi_{[n]}}\}_{n\geq 1}$ admits an $H(\mathbb{D})$-universal function. It is easy to see that ϕ must be injective: if not, we would have $\phi_{[n]}(z)=\phi_{[n]}(w)$ for each n and $z\neq w$ and each function in the (local uniform) closure of $\{H\circ\phi_{[n]}:n\in\mathbb{N}\}$ would have the same value at z and at w. The converse is a consequence of Theorem 5.2. $\square$

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