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Deterministic and probabilistic discrepancies

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Abstract. In this paper, we compare some deterministic and probabilistic techniques in the study of upper bounds in problems related to certain mean square discrepancie with respect to balls in the d-dimensional unit torus, and show that the quality of these techniques depends in an intricate way on the dimension d under consideration.

1. Introduction

Let $\mathbb{T}^d=[-\frac{1}{2},\frac{1}{2}]^d$ denote the d-dimensional unit torus, where $d\geq 1$ is fixed. Suppose that $\mathcal{P}$ is a distribution of $N=M^d$ points in $\mathbb{T}^d$, where M is a positive integer. For every positive real number $r<\frac{1}{2}$, let B_r denote the ball centred at $0\in\mathbb{T}^d$ and with radius r, with characteristic function χ_{B_r}. We are interested in the discrepancy

$$D(N)=\left(\int_{\mathbb{T}^d}\left|N|B_r|-\sum_{p\in\mathcal{P}}\chi_{B_r-t}(p)\right|^2 dt\right)^{1/2} \tag{1}$$

of the finite set $\mathcal{P}$ with respect to the family of all translates B_r-t of the ball B_r in $\mathbb{T}^d$.

In particular, we are interested in the above problem when the points in $\mathcal{P}$ are obtained by modifications of the standard lattice. More precisely, for every positive integer M, the standard lattice is the set

$$L_M=\left\{\left(\frac{r_1}{M},...,\frac{r_d}{M}\right):r_1,...,r_d\in\{0,1,...,M-1\}\right\}. \tag{2}$$

We shall denote a typical point in L_M by p.

Let $d\mu$ denote a probability measure on $\mathbb{T}^d$. For every $p\in L_M$, let $d\mu_p$ denote the translation of $d\mu$ by $p\in L_M$, so that for any integrable function f in $\mathbb{T}^d$, we have

$$\int_{\mathbb{T}^d} f(t)\,d\mu_p=\int_{\mathbb{T}^d} f(t-p)\,d\mu.$$

We now average the discrepancy $D(N)$ in $L^2(\mathbb{T}^d, d\mu_p)$ for every $p\in L_M$, and consider

$$D^2_{d\mu}(N)=\int_{\mathbb{T}^d}\cdots\int_{\mathbb{T}^d}\int_{\mathbb{T}^d}\left|N|B_r|-\sum_{p\in L_M}\chi_{B_r-t}(u_p)\right|^2 dt\prod_{s\in L_M}d\mu_s, \tag{3}$$

where, for every point $p\in L_M$, the probabilistic variable u_p is associated with the probabilistic measure $d\mu_p$. Note that the cardinality of L_M is N.

In Section 2, we shall show that a simple orthogonality argument leads to the explicit formula

$$D^2_{d\mu}(N)=N\sum_{0\neq k\in\mathbb{Z}^d}|\widehat{\chi}_{B_r}(k)|^2(1-|\hat{\mu}(k)|^2)+N^2\sum_{0\neq h\in\mathbb{Z}^d}|\widehat{\chi}_{B_r}(Mh)|^2|\hat{\mu}(Mh)|^2, \tag{4}$$

in terms of the Fourier transforms of the characteristic function χ_{B_r} and the measure $d\mu$.

Let us first consider an extreme case where we take $d\mu=dt$, the Lebesgue measure on $\mathbb{T}^d$. Here one can hardly speak about modification of the standard lattice, as every point u_j is chosen totally at random in $\mathbb{T}^d$ and we end up considering a Monte Carlo estimate of the discrepancy $D(N)$. Since $\hat{\mu}(0)=1$ and $\hat{\mu}(k)=0$ for every non-zero $k\in\mathbb{Z}^d$, the identity (4) becomes

$$D^2_{dt}(N)=N\sum_{0\neq k\in\mathbb{Z}^d}|\widehat{\chi}_{B_r}(k)|^2=N(\|\chi_{B_r}\|^2_{L^2(\mathbb{T}^d,dt)}-|B_r|^2)=N(|B_r|-|B_r|^2).$$

Note that this is independent of the dimension d and that the ball B_r can be replaced by any measurable subset of $\mathbb{T}^d$ with diameter less than 1. We shall not consider this case further.

In general, we are governed by the following lower bound result of Beck [1] and Montgomery [13]. See also Brandolini, Colzani and Travaglini [5].

Theorem 1.1. *There exists a positive constant c_d, depending only on d, such that for every finite set $\mathcal{Q}$ of N points in $\mathbb{T}^d$, we have*

$$\int_0^{1/2}\int_{\mathbb{T}^d}\left|N|B_r|-\sum_{q\in\mathcal{Q}}\chi_{B_r-t}(q)\right|^2 dt\,dr\geq c_dN^{1-1/d}. \tag{5}$$

It is known that Theorem 1.1 is essentially best possible; see, for example, Beck and Chen [3], Chen [7] or Travaglini [16]. The purpose of this paper is to compare some of these approaches.

We shall consider the case when $d\mu=\delta_0$, the Dirac measure concentrated at the origin. In this case, the Fourier transform $\hat{\mu}$ is identically equal to 1, so that the identity (4) becomes

$$D_{\delta_0}^2(N)=N^2\sum_{0\neq h\in\mathbb{Z}^d}|\widehat{\chi}_{B_r}(Mh)|^2. \tag{6}$$

We shall also consider the case when $d\mu=d\lambda=\lambda(t)\,dt$, where

$$\lambda(t)=N\chi_{[-1/2M,1/2M]^d}(t)$$

denotes the characteristic function of the small cube $[-1/2M,1/2M]^d$, suitably normalized. It is well known that for every $k=(k_1,...,k_d)\in\mathbb{Z}^d$, the Fourier transform

$$\hat{\lambda}(k)=N\prod_{i=1}^d\frac{\sin(\pi k_i/M)}{\pi k_i},$$

with obvious modification when $k_i=0$ for some $i=1,...,d$. Since $\hat{\lambda}(Mh)=0$ for every non-zero $h\in\mathbb{Z}^d$, the identity (4) becomes

$$\begin{aligned} D_{d\lambda}^2(N)&=N\sum_{0\neq k\in\mathbb{Z}^d}|\widehat{\chi}_{B_r}(k)|^2(1-|\hat{\lambda}(k)|^2) \\ &=N(\|\chi_{B_r}\|^2_{L^2(\mathbb{T}^d,dt)}-|B_r|^2)-N(\|\chi_{B_r}*\lambda\|^2_{L^2(\mathbb{T}^d,dt)}-|B_r|^2) \\ &=N(\|\chi_{B_r}\|^2_{L^2(\mathbb{T}^d,dt)}-\|\chi_{B_r}*\lambda\|^2_{L^2(\mathbb{T}^d,dt)}) \\ &=N(|B_r|-\|\chi_{B_r}*\lambda\|^2_{L^2(\mathbb{T}^d,dt)}). \end{aligned} \tag{7}$$

We shall compare the deterministic discrepancy $D_{\delta_0}(N)$ and the probabilistic discrepancy $D_{d\lambda}(N)$.

The deterministic and probabilistic discrepancies are related to systematic and stratified (or jittered) samplings in statistics in a rather natural way; see, for example, Bellhouse [4] or Kollig and Keller [10].

The Fourier transform $\widehat{\chi}_{B_r}$ of the characteristic function of the ball B_r is described in terms of the Bessel functions. For every $k\in\mathbb{Z}^d$, we have

$$\widehat{\chi}_{B_r}(k)=\begin{cases} |B_r|=\dfrac{\pi^{d/2}r^d}{\Gamma\big(1+\frac{d}{2}\big)}, & \text{if } k=0, \\ r^{d/2}|k|^{-d/2}J_{d/2}(2\pi r|k|), & \text{if } k\neq0, \end{cases} \tag{8}$$

where $J_{d/2}$ is the Bessel function of order $d/2$. The properties of the Bessel functions lead to quite different conclusions in our investigation, depending on the value of the dimension d. Accordingly, we need to split our discussion into two separate cases.

Our first result covers what may be termed the *usual case*.

Theorem 1.2. *Suppose that* $d\not\equiv 1 \pmod 4$.

(i) *For all sufficiently large* d, *the inequality* $D_{d\lambda}(M^d) < D_{\delta_0}(M^d)$ *holds for all sufficiently large values of* M.

(ii) *For* $d=2$ *and* $r=\frac{1}{4}$, *the inequality* $D_{\delta_0}(M^2) < D_{d\lambda}(M^2)$ *holds for all sufficiently large values of* M.

Our next result covers what may be termed the *exceptional case*.

Theorem 1.3. *Suppose that* $d\equiv 1 \pmod 4$.

(i) *For all sufficiently large* d, *the inequality* $D_{d\lambda}(M^d) < D_{\delta_0}(M^d)$ *holds for infinitely many values of* M.

(ii) *For every* d, *the inequality* $D_{\delta_0}(M^d) < D_{d\lambda}(M^d)$ *holds for infinitely many values of* M.

(iii) *For* $d=1$, *the inequality* $D_{\delta_0}(M) < D_{d\lambda}(M)$ *holds for every value of* M.

The peculiarity of the case $d\equiv 1 \pmod 4$ has arisen in earlier work. See, for example, Konyagin, Skriganov and Sobolev [11], where the peculiar distribution of lattice points with respect to balls in these dimensions is discussed. We shall see later in Section 6 that a closer analysis of the Bessel functions that arise in (8) reveals that simultaneous diophantine approximation plays a key role in the study of this special case.

Remark. For a general introduction to discrepancy theory, the reader is referred to the books by Beck and Chen [2], Drmota and Tichy [8] and Matoušek [12]. The reader is also referred to the book by Chazelle [6] where many applications to randomness and complexity are discussed.

Notation. Throughout this paper, we write $f=O_\nu(g)$ to indicate the existence of an implicit positive constant A_ν, depending at most on ν, such that $|f|\leq A_\nu g$. This implicit constant may change from one occurrence to the next. On the other hand, we write $|f|\leq C_\nu g$ to indicate that the explicit constant C_ν does not change from one occurrence to the next.

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2. The explicit formula

In this short section, we apply an orthogonality argument to deduce the explicit formula (4). Applying Parseval's identity to (3), we obtain

$$
\begin{aligned}
(9)\quad D^2_{d\mu}(N) &= \int_{\mathbb{T}^d} \cdots \int_{\mathbb{T}^d} \sum_{0\neq k\in\mathbb{Z}^d} |\widehat{\chi}_{B_r}(k)|^2 \Big| \sum_{p\in L_M} e^{2\pi i k\cdot u_p} \Big|^2 \prod_{s\in L_M} d\mu_s \\
&= \sum_{0\neq k\in\mathbb{Z}^d} |\widehat{\chi}_{B_r}(k)|^2 \int_{\mathbb{T}^d} \cdots \int_{\mathbb{T}^d} \sum_{p,q\in L_M} e^{2\pi i k\cdot u_p} e^{-2\pi i k\cdot u_q} \prod_{s\in L_M} d\mu_s \\
&= \sum_{0\neq k\in\mathbb{Z}^d} |\widehat{\chi}_{B_r}(k)|^2 \sum_{p,q\in L_M} \int_{\mathbb{T}^d} \cdots \int_{\mathbb{T}^d} e^{2\pi i k\cdot u_p} e^{-2\pi i k\cdot u_q} \prod_{s\in L_M} d\mu_s \\
&= \sum_{0\neq k\in\mathbb{Z}^d} |\widehat{\chi}_{B_r}(k)|^2 \Bigg(N + \sum_{\substack{p,q\in L_M \\ p\neq q}} \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} e^{2\pi i k\cdot u_p} e^{-2\pi i k\cdot u_q} \, d\mu_p \, d\mu_q \Bigg).
\end{aligned}
$$

For $p\neq q$, we clearly have

$$
\begin{aligned}
\int_{\mathbb{T}^d} \int_{\mathbb{T}^d} e^{2\pi i k\cdot u_p} e^{-2\pi i k\cdot u_q} \, d\mu_p \, d\mu_q &= \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} e^{2\pi i k\cdot (u_p-p)} e^{-2\pi i k\cdot (u_q-q)} \, d\mu \, d\mu \\
&= e^{-2\pi i k\cdot p} e^{2\pi i k\cdot q} \int_{\mathbb{T}^d} e^{2\pi i k\cdot u_p} \, d\mu \int_{\mathbb{T}^d} e^{-2\pi i k\cdot u_q} \, d\mu \\
&= |\hat{\mu}(k)|^2 e^{-2\pi i k\cdot p} e^{2\pi i k\cdot q},
\end{aligned}
$$

and so

$$
\begin{aligned}
(10)\quad \sum_{\substack{p,q\in L_M \\ p\neq q}} \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} e^{2\pi i k\cdot u_p} e^{-2\pi i k\cdot u_q} \, d\mu_p \, d\mu_q &= |\hat{\mu}(k)|^2 \sum_{\substack{p,q\in L_M \\ p\neq q}} e^{-2\pi i k\cdot p} e^{2\pi i k\cdot q} \\
&= |\hat{\mu}(k)|^2 \Bigg(\sum_{p,q\in L_M} e^{-2\pi i k\cdot p} e^{2\pi i k\cdot q} - N \Bigg) \\
&= |\hat{\mu}(k)|^2 \Bigg(\Big| \sum_{p\in L_M} e^{2\pi i k\cdot p} \Big|^2 - N \Bigg).
\end{aligned}
$$

The formula (4) follows immediately from combining (9), (10) and the orthogonality relationship

$$
\sum_{p\in L_M} e^{2\pi i k\cdot p} = \begin{cases} N, & \text{if } k\in M\mathbb{Z}^d, \\ 0, & \text{otherwise.} \end{cases}
$$

3. An upper bound for probabilistic discrepancy

In this section, we establish the following simple result which we need for both Theorems 1.2 and 1.3.

Lemma 3.1. *For every sufficiently large positive integer M, we have*

$$D^2_{d\lambda}(M^d)\leq\frac{\pi^{d/2}d^{3/2}r^{d-1}M^{d-1}}{2\Gamma(1+d/2)}.$$

Proof. Note that from (7), we have

$$D^2_{d\lambda}(M^d)=M^d(|B_r|-\|\chi_{B_r}*\lambda\|^2_{L^2(\mathbb{T}^d,dt)}),$$

and it is easy to see that

$$(\chi_{B_r}*\lambda)(t)=M^d\left|B_r\cap\left(\left[-\frac{1}{2M},\frac{1}{2M}\right]^d+t\right)\right|\geq 0. \tag{11}$$

Note in particular that $(\chi_{B_r}*\lambda)(t)=1$ if $|t|<r-\sqrt{d}/2M$. Suppose now that the integer M is sufficiently large. Ignoring the non-negative contribution to the term $\|\chi_{B_r}*\lambda\|^2_{L^2(\mathbb{T}^d,dt)}$ from those values of $t\in\mathbb{T}^d$ satisfying $|t|\geq r-\sqrt{d}/2M$, we see that

$$\begin{aligned}D^2_{d\lambda}(M^d)&\leq M^d|B_r|-M^d\int_{|t|<r-\sqrt{d}/2M}dt=M^d(|B_r|-|B_{r-\sqrt{d}/2M}|)\\&=\frac{M^d\pi^{d/2}}{\Gamma(1+d/2)}\left(r^d-\left(r-\frac{\sqrt{d}}{2M}\right)^d\right)\leq\frac{M^d\pi^{d/2}}{\Gamma(1+d/2)}dr^{d-1}\left(r-\left(r-\frac{\sqrt{d}}{2M}\right)\right)\end{aligned}$$

from which the result follows easily. □

4. The usual case

For all sufficiently large values of d, the inequality

$$\frac{\pi^{d/2}d^{3/2}}{2\Gamma(1+d/2)}<\frac{\pi^{-2}2^{-d}d}{1000}$$

clearly holds. Part (i) of Theorem 1.2 follows at once from Lemma 3.1 and the result below.

Lemma 4.1. *Suppose that $d\not\equiv 1 \pmod 4$. Then for every sufficiently large positive integer M, we have*

$$D^2_{\delta_0}(M^d)\geq\frac{\pi^{-2}2^{-d}dr^{d-1}M^{d-1}}{1000}.$$

Proof. Combining (6) and (8), we have

$$D^2_{\delta_0}(M^d)=M^d\sum_{0\neq h\in\mathbb{Z}^d} r^d|h|^{-d}J^2_{d/2}(2\pi rM|h|). \tag{12}$$

Recall that the Bessel functions have the asymptotic expansion

$$J_\nu(x)=\sqrt{\frac{2}{\pi x}}\cos\left(x-\frac{\nu\pi}{2}-\frac{\pi}{4}\right)+O_\nu\left(\frac{1}{x^{3/2}}\right)\quad\text{for } \nu>-\frac{1}{2}. \tag{13}$$

See, for example, Stein and Weiss [15, Chapter IV, Lemma 3.11]. It follows that for $h\neq 0$ and sufficiently large M, we have

$$J_{d/2}(2\pi rM|h|)=\frac{1}{\pi r^{1/2}M^{1/2}|h|^{1/2}}\left(\cos\left(2\pi rM|h|-\frac{(d+1)\pi}{4}\right)+O_d\left(\frac{1}{rM|h|}\right)\right). \tag{14}$$

For every real number $\alpha\in\mathbb{R}$, let $\|\alpha\|=\min_{n\in\mathbb{Z}}|\alpha-n|$ denote the distance of α to the nearest integer. We have two cases.

Case 1. Suppose that the integer M is sufficiently large and satisfies

$$\left\|2rM-\frac{d+1}{4}-\frac{1}{2}\right\|\geq\frac{1}{10},\quad\text{so that }\left|\cos\left(2\pi rM-\frac{(d+1)\pi}{4}\right)\right|\geq\frac{1}{5}.$$

Then it follows from (12) and (14) that

$$\begin{aligned}D^2_{\delta_0}(M^d)&\geq M^d\sum_{\substack{h\in\mathbb{Z}^d\\|h|=1}} r^d|h|^{-d}J^2_{d/2}(2\pi rM|h|)\\&=2dM^dr^dJ^2_{d/2}(2\pi rM)>\frac{\pi^{-2}dr^{d-1}M^{d-1}}{1000},\end{aligned}$$

clearly stronger than the required conclusion.

Case 2. Suppose that the integer M is sufficiently large and satisfies

$$\left\|2rM-\frac{d+1}{4}-\frac{1}{2}\right\|<\frac{1}{10}. \tag{15}$$

Since $d\not\equiv 1 \pmod 4$, it can be shown that

$$\left\|4rM-\frac{d+1}{4}-\frac{1}{2}\right\|>\frac{1}{20}, \tag{16}$$

so that

$$\left|\cos\left(4\pi rM-\frac{(d+1)\pi}{4}\right)\right|>\frac{1}{10}.$$

Then it follows from (12) and (14) that

$$\begin{aligned}D_{\delta_0}^2(M^d)&\geq M^d\sum_{\substack{h\in\mathbb{Z}^d\\|h|=2}}r^d|h|^{-d}J_{d/2}^2(2\pi rM|h|)\\&=2dM^dr^d2^{-d}J_{d/2}^2(4\pi rM)>\frac{\pi^{-2}2^{-d}dr^{d-1}M^{d-1}}{1000},\end{aligned}$$

again giving the required conclusion. □

We complete this section by making some comments. The derivation of (16) from (15) is a consequence of the simple observation that for any fixed rational number $b=\frac{1}{4},\frac{1}{2},\frac{3}{4}$, corresponding respectively to $d\equiv 2,3,4 \pmod 4$,

$$\|x-b\|<\frac{1}{10}\quad\Longrightarrow\quad\|2x-b\|>\frac{1}{20}. \tag{17}$$

However, the implication (17) does not remain valid if $b=0$, corresponding to $d\equiv 1 \pmod 4$. This gives rise to the exceptional case.

5. The two-dimensional case

Calculation shows that in the 2-dimensional case, the probabilistic discrepancy $D_{d\lambda}(M^2)$ and deterministic discrepancy $D_{\delta_0}(M^2)$ are very close in value. We therefore need rather precise estimates. We shall only consider the case $r=\frac{1}{4}$; for simplicity, we write B to denote the disc $B_{1/4}$. Then it follows from (7) that

$$D_{d\lambda}^2(M^2)=M^2(|B_r|-\|\chi_{B_r}*\lambda\|_{L^2(\mathbb{T}^2,dt)}^2)=M^2\left(\frac{\pi}{16}-\int_{\mathbb{T}^2}\left(\int_B\lambda(t-v)\,dv\right)^2dt\right), \tag{18}$$

where

$$\int_B\lambda(t-v)\,dv=M^2\left|B\cap\left(\left[-\frac{1}{2M},\frac{1}{2M}\right]^2+t\right)\right|. \tag{19}$$

Clearly we always have

$$0\leq\int_B\lambda(t-v)\,dv\leq 1. \tag{20}$$

Let $t=\rho\Theta=\rho(\cos\theta,\sin\theta)$. We shall make use of symmetry and assume for simplicity that $0<\theta\leq\pi/4$.

Our first step is to show that if the point t is far enough from the boundary of the disc B, then the square

$$\left[-\frac{1}{2M},\frac{1}{2M}\right]^2+t$$

lies either completely inside B or completely outside B. Indeed, simple geometric considerations give the precise information that

$$\int_B \lambda(t-v)\,dv=\begin{cases}1, & \text{if } \rho\le\frac{1}{4}-\frac{\cos\theta+\sin\theta}{M},\\ 0, & \text{if } \rho\ge\frac{1}{4}+\frac{\cos\theta+\sin\theta}{M}.\end{cases}\tag{21}$$

Next, for those points t that are close to the boundary of B, we then need to obtain a good approximation of the quantity (19). We shall see in a moment that, up to $O(M^{-1})$, we have the identity

$$\int_B \lambda(t-v)\,dv=\begin{cases}1, & \text{if } \rho\le\frac{1}{4}-\frac{\cos\theta+\sin\theta}{2M},\\ 0, & \text{if } \rho\ge\frac{1}{4}+\frac{\cos\theta+\sin\theta}{2M}.\end{cases}\tag{22}$$

We need to study the set

$$B\cap\left(\left[-\frac{1}{2M},\frac{1}{2M}\right]^2+t\right)$$

when the translated square intersects the boundary of the disc B. We consider the situation as in Figure 1. Here the line segment PS is tangent to the boundary of B. Let $\mathcal{B}_\Theta$ denote the half plane containing the origin and having PS as part of its

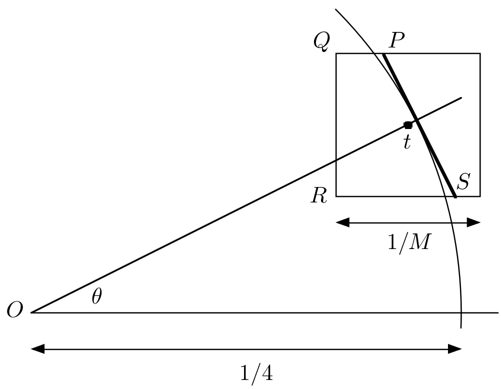

Figure 1.

boundary. Then

$$\int_B \lambda(t-v)\,dv = M^2\left|\mathcal{B}_\Theta\cap\left(\left[-\frac{1}{2M},\frac{1}{2M}\right]^2+t\right)\right|+O(M^{-1}),$$

where the set

$$\mathcal{B}_\Theta\cap\left(\left[-\frac{1}{2M},\frac{1}{2M}\right]^2+t\right) \tag{23}$$

is represented in Figure 1 by the quadrilateral $PQRS$, although as t varies, this may become a triangle or pentagon. To compute the area of the intersection (23), we consider Figure 2.

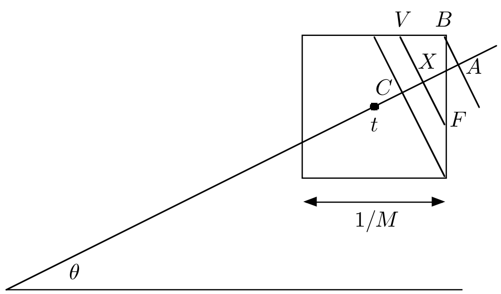

Figure 2.

Note that

$$|tC| = \frac{\cos\theta-\sin\theta}{2M}, \quad |tA| = \frac{\cos\theta+\sin\theta}{2M} \quad \text{and} \quad |VBF| = \frac{|AX|^2}{\sin 2\theta}.$$

It follows that with an error of at most $O(M^{-1})$, we have (22) and

$$\int_B \lambda(t-v)\,dv = \begin{cases} 1-\dfrac{\left(M\left(\rho-\frac{1}{4}\right)+\frac{1}{2}(\cos\theta+\sin\theta)\right)^2}{\sin 2\theta}, \\ \qquad\qquad \text{if } \dfrac{1}{4}-\dfrac{\cos\theta+\sin\theta}{2M}\le\rho<\dfrac{1}{4}-\dfrac{\cos\theta-\sin\theta}{2M}, \\ \dfrac{1}{2}-\dfrac{M}{\cos\theta}\left(\rho-\dfrac{1}{4}\right), \qquad \text{if } \dfrac{1}{4}-\dfrac{\cos\theta-\sin\theta}{2M}\le\rho<\dfrac{1}{4}+\dfrac{\cos\theta-\sin\theta}{2M}, \\ \dfrac{\left(M\left(\frac{1}{4}-\rho\right)+\frac{1}{2}(\cos\theta+\sin\theta)\right)^2}{\sin 2\theta}, \\ \qquad\qquad \text{if } \dfrac{1}{4}+\dfrac{\cos\theta-\sin\theta}{2M}\le\rho<\dfrac{1}{4}+\dfrac{\cos\theta+\sin\theta}{2M}. \end{cases} \tag{24}$$

Squaring the expression (19) and integrating with respect to t, we obtain

$$\int_{\mathbb{T}^2}\left(\int_B \lambda(t-v)\,dv\right)^2 dt = I_1+I_2+I_3+I_4+I_5+E. \tag{25}$$

In (25), we write

$$\begin{aligned} I_1 &= 8\int_0^{\pi/4}\int_0^{1/4-(\cos\theta+\sin\theta)/2M}\left(\int_B \lambda(\rho\Theta-v)\,dv\right)^2 \rho\,d\rho\,d\theta \\ &= 8\int_0^{\pi/4}\int_0^{1/4-(\cos\theta+\sin\theta)/2M}\rho\,d\rho\,d\theta = \frac{\pi}{16}-\frac{1}{M}+O\left(\frac{1}{M^2}\right), \end{aligned} \tag{26}$$

in view of (22). We also write

$$I_5 = 8\int_0^{\pi/4}\int_{1/4+(\cos\theta+\sin\theta)/2M}^{\infty}\left(\int_B \lambda(\rho\Theta-v)\,dv\right)^2 \rho\,d\rho\,d\theta = O\left(\frac{1}{M^2}\right), \tag{27}$$

in view of (21) and (22). In view of (24), we can write

$$\begin{aligned} I_2 &= 8\int_0^{\pi/4}\int_{1/4-(\cos\theta+\sin\theta)/2M}^{1/4-(\cos\theta-\sin\theta)/2M}\left(1-\frac{\left(M\left(\rho-\frac{1}{4}\right)+\frac{1}{2}(\cos\theta+\sin\theta)\right)^2}{\sin 2\theta}\right)^2 \rho\,d\rho\,d\theta, \\ I_3 &= 8\int_0^{\pi/4}\int_{1/4-(\cos\theta-\sin\theta)/2M}^{1/4+(\cos\theta-\sin\theta)/2M}\left(\frac{1}{2}-\frac{M}{\cos\theta}\left(\rho-\frac{1}{4}\right)\right)^2 \rho\,d\rho\,d\theta, \\ I_4 &= 8\int_0^{\pi/4}\int_{1/4+(\cos\theta-\sin\theta)/2M}^{1/4+(\cos\theta+\sin\theta)/2M}\left(\frac{\left(M\left(\frac{1}{4}-\rho\right)+\frac{1}{2}(\cos\theta+\sin\theta)\right)^2}{\sin 2\theta}\right)^2 \rho\,d\rho\,d\theta. \end{aligned}$$

Then in view of (24), we have

$$E \ll \int_0^{\pi/4}\int_{1/4-(\cos\theta+\sin\theta)/2M}^{1/4+(\cos\theta+\sin\theta)/2M}\frac{1}{M}\rho\,d\rho\,d\theta = O\left(\frac{1}{M^2}\right). \tag{28}$$

Using the substitution $s=M\left(\rho-\frac{1}{4}\right)+\frac{1}{2}(\cos\theta+\sin\theta)$, we have

$$I_2 = \frac{2}{M}\int_0^{\pi/4}\int_0^{\sin\theta}\left(1-\frac{s^2}{\sin 2\theta}\right)^2 ds\,d\theta + O\left(\frac{1}{M^2}\right).$$

Using the substitution $s=M\left(\rho-\frac{1}{4}\right)-\frac{1}{2}(\cos\theta+\sin\theta)$ and symmetry, we have

$$I_4 = \frac{2}{M}\int_0^{\pi/4}\int_{-\sin\theta}^{0}\left(\frac{s^2}{\sin 2\theta}\right)^2 ds\,d\theta + O\left(\frac{1}{M^2}\right).$$

Combining these two, a straightforward calculation gives

$$I_2+I_4=\frac{1}{M}\left(\frac{8}{5}-\frac{11\sqrt{2}}{30}-\frac{2}{3}\log\bigl(1+\sqrt{2}\bigr)\right)+O\left(\frac{1}{M^2}\right). \tag{29}$$

Using the substitution $s=M\bigl(\rho-\frac{1}{4}\bigr)$ and symmetry, a straightforward calculation gives

$$\begin{aligned} I_3&=\frac{2}{M}\int_0^{\pi/4}\int_{-(\cos\theta-\sin\theta)/2}^{(\cos\theta-\sin\theta)/2}\left(\frac{1}{2}-\frac{s}{\cos\theta}\right)^2 ds\,d\theta+O\left(\frac{1}{M^2}\right)\\ &=\frac{1}{M}\left(\frac{\sqrt{2}}{3}-\frac{2}{3}+\frac{1}{2}\log\bigl(1+\sqrt{2}\bigr)\right)+O\left(\frac{1}{M^2}\right). \end{aligned} \tag{30}$$

Combining (25)–(30), we conclude that

$$\int_{\mathbb{T}^2}\left(\int_B\lambda(t-v)\,dv\right)^2 dt=\frac{\pi}{16}-\frac{1}{M}\left(\frac{1}{15}+\frac{\sqrt{2}}{30}+\frac{1}{6}\log\bigl(1+\sqrt{2}\bigr)\right)+O\left(\frac{1}{M^2}\right),$$

and so it follows from (18) that

$$D_{d\lambda}^2(M^2)=\left(\frac{1}{15}+\frac{\sqrt{2}}{30}+\frac{1}{6}\log\bigl(1+\sqrt{2}\bigr)\right)M+O(1). \tag{31}$$

We compare this with $D_{\delta_0}(M^2)$. Combining (6), (8) and (13), we have

$$\begin{aligned} D_{\delta_0}(M^2)&=\frac{M^2}{16}\sum_{0\neq h\in\mathbb{Z}^2}\frac{1}{|h|^2}J_1^2\left(\frac{\pi M|h|}{2}\right)\\ &=\frac{M}{4\pi^2}\sum_{0\neq h\in\mathbb{Z}^2}\frac{1}{|h|^3}\cos^2\left(\frac{\pi M|h|}{2}-\frac{3\pi}{4}\right)+O(1)=S_1+S_2+O(1), \end{aligned} \tag{32}$$

where

$$S_1=\frac{M}{\pi^2}\sum_{m=1}^{\infty}\frac{1}{m^3}\cos^2\left(\frac{\pi Mm}{2}-\frac{3\pi}{4}\right)=\frac{M}{2\pi^2}\sum_{m=1}^{\infty}\frac{1}{m^3}, \tag{33}$$

and

$$\begin{aligned} S_2&=\frac{M}{\pi^2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{(m^2+n^2)^{3/2}}\cos^2\left(\frac{\pi M(m^2+n^2)^{1/2}}{2}-\frac{3\pi}{4}\right)\\ &\le\frac{M}{\pi^2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{(m^2+n^2)^{3/2}}. \end{aligned} \tag{34}$$

Combining (32)–(34), we have

$$D_{\delta_0}(M^2) \leq \left(\frac{1}{2\pi^2} \sum_{m=1}^{\infty} \frac{1}{m^3} + \frac{1}{\pi^2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{(m^2+n^2)^{3/2}} \right) M + O(1). \tag{35}$$

Part (ii) of Theorem 1.2 now follows from (31) and (35), on noting that

$$\frac{1}{15} + \frac{\sqrt{2}}{30} + \frac{1}{6} \log\left(1+\sqrt{2}\right) > 0.26,$$

and that

$$\frac{1}{2\pi^2} \sum_{m=1}^{\infty} \frac{1}{m^3} + \frac{1}{\pi^2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{(m^2+n^2)^{3/2}} < 0.26.$$

6. The exceptional case

For all sufficiently large values of d, the inequality

$$\frac{\pi^{d/2} d^{3/2}}{2\Gamma(1+d/2)} < \frac{\pi^{-2} d}{10}$$

clearly holds. Part (i) of Theorem 1.3 follows at once from Lemma 3.1 and the following result.

Lemma 6.1. *Suppose that* $d \equiv 1 \pmod 4$. *Then for infinitely many positive integers* M, *we have*

$$D_{\delta_0}^2(M^d) \geq \frac{\pi^{-2} d r^{d-1} M^{d-1}}{10}. \tag{36}$$

Proof. Clearly it follows from (12) that

$$D_{\delta_0}^2(M^d) \geq M^d \sum_{\substack{h \in \mathbb{Z}^d \\ |h|=1}} r^d |h|^{-d} J_{d/2}^2(2\pi r M |h|) = 2d M^d r^d J_{d/2}^2(2\pi r M).$$

Using the asymptotic expansion (13) for the Bessel function, we obtain that

$$D_{\delta_0}^2(M^d) \geq 2d\pi^{-2} r^{d-1} M^{d-1} \cos^2\left(2\pi r M - \frac{\pi}{2}\right) + O(M^{d-2}). \tag{37}$$

Since $0 < 2r < 1$, the inequality $\|2rM\| \geq \frac{1}{4}$ is satisfied by infinitely many positive integers M. For these integers M, we clearly have

$$\cos^2\left(2\pi r M - \frac{\pi}{2}\right) \geq \frac{1}{2}.$$

The inequality (36) now follows from (37) if M is sufficiently large. □

We next turn our attention to part (ii) of Theorem 1.3.

Lemma 6.2. *There exists a positive constant $c_{d,r}$, depending at most on the dimension d and the radius r, such that for every positive integer M, we have*

$$D_{d\lambda}^2(M^d) \geq c_{d,r} M^{d-1}. \tag{38}$$

Proof. We remark that we cannot deduce this result from Theorem 1.1, since the left-hand side of (5) involves an average over the radius r of the balls B_r. Instead, we start from the simple observation that

$$\|\chi_{B_r} * \lambda\|_{L^1(\mathbb{T}^d,dt)} \leq \|\lambda\|_{L^1(\mathbb{T}^d,dt)} \|\chi_{B_r}\|_{L^1(\mathbb{T}^d,dt)} = |B_r|.$$

Then it follows from (7) that

$$\begin{aligned} D_{d\lambda}^2(M^d) &= M^d(|B_r| - \|\chi_{B_r} * \lambda\|_{L^2(\mathbb{T}^d,dt)}^2) \\ &= M^d(|B_r| - \|\chi_{B_r} * \lambda\|_{L^1(\mathbb{T}^d,dt)}) \\ &\quad + M^d(\|\chi_{B_r} * \lambda\|_{L^1(\mathbb{T}^d,dt)} - \|\chi_{B_r} * \lambda\|_{L^2(\mathbb{T}^d,dt)}^2) \\ &\geq M^d(\|\chi_{B_r} * \lambda\|_{L^1(\mathbb{T}^d,dt)} - \|\chi_{B_r} * \lambda\|_{L^2(\mathbb{T}^d,dt)}^2) \\ &= M^d \int_{\mathbb{T}^d} ((\chi_{B_r} * \lambda)(t) - (\chi_{B_r} * \lambda)^2(t))\, dt \\ &= M^d \int_{\mathbb{T}^d} (\chi_{B_r} * \lambda)(t)(1 - (\chi_{B_r} * \lambda)(t))\, dt. \end{aligned} \tag{39}$$

Note that $0 \leq (\chi_{B_r} * \lambda)(t) \leq 1$ for every $t \in \mathbb{T}^d$, so that always

$$(\chi_{B_r} * \lambda)(t)(1 - (\chi_{B_r} * \lambda)(t)) \geq 0. \tag{40}$$

Recall next the relationship (11). We now use an elaboration of an idea first used in the two-dimensional case in Section 5. Write $t = \rho\sigma$ in polar coordinates, where $\rho \geq 0$ and $\sigma \in \Sigma_{d-1}$. For every positive real number $r < \frac{1}{2}$ and every $\sigma \in \Sigma_{d-1}$, let $\mathcal{B}_{r,\sigma}$ denote the half space containing the origin and such that its boundary is perpendicular to σ and tangent to the surface of the ball B_r. Then it follows from (11) that

$$(\chi_{B_r} * \lambda)(t) = M^d \left| \mathcal{B}_{r,\sigma} \cap \left(\left[-\frac{1}{2M}, \frac{1}{2M} \right]^d + t \right) \right| + O_{d,r}\left(\frac{1}{M} \right), \tag{41}$$

where the implicit constant in the error term can be chosen independent of t. On the other hand, in view of (11) and the symmetry of the cube

$$\left[-\frac{1}{2M}, \frac{1}{2M} \right]^d$$

about its centre, we have $(\chi_{B_r}*\lambda)(r\sigma)=\frac{1}{2}+O_{d,r}(M^{-1})$. Noting that the function $\rho\mapsto(\chi_{B_r}*\lambda)(\rho\sigma)$ is non-increasing, we conclude that

$$(\chi_{B_r}*\lambda)(\rho\sigma)\leq\frac{1}{2}+O_{d,r}(M^{-1})\quad\text{whenever } r\leq\rho\leq r+\frac{1}{4M}.$$

Observe that in view of (11), the expression (40) is equal to zero unless the cube

$$\left[-\frac{1}{2M},\frac{1}{2M}\right]^d+t$$

intersects the boundary of the ball B_r; in other words, unless ρ is very close in value to r. Combining (39), (40) and (41), we obtain

$$\begin{aligned}D^2_{d\lambda}(M^d)&\geq M^d\int_{r\leq|t|\leq r+1/4M}(\chi_{B_r}*\lambda)(t)(1-(\chi_{B_r}*\lambda)(t))\,dt\\&\geq\frac{1}{3}M^d\int_{r\leq|t|\leq r+1/4M}(\chi_{B_r}*\lambda)(t)\,dt\\&\geq\frac{1}{3}M^{2d}\int_{r\leq|t|\leq r+1/4M}\left|\mathcal{B}_{r,\sigma}\cap\left(\left[-\frac{1}{2M},\frac{1}{2M}\right]^d+t\right)\right|dt+O_{d,r}(M^{d-2}).\end{aligned}$$

The inequality (38) now follows on noting that there is a positive constant a_d, depending only on the dimension d, such that

$$\left|\mathcal{B}_{r,\sigma}\cap\left(\left[-\frac{1}{2M},\frac{1}{2M}\right]^d+t\right)\right|\geq\frac{a_d}{M^d}$$

for every $\sigma\in\Sigma_{d-1}$ and whenever $r\leq\rho\leq r+1/4M$, provided that the positive integer M is sufficiently large. □

Let the constant $c_{d,r}$ be given by Lemma 6.2. By the convergence of the series

$$\sum_{0\neq h\in\mathbb{Z}^d}r^{d-1}|h|^{-d-1},$$

there exists a positive constant $R_{d,r}$, depending at most on the dimension d and the radius r, such that

$$\sum_{\substack{h\in\mathbb{Z}^d\\|h|\geq\sqrt{R_{d,r}}}}r^{d-1}|h|^{-d-1}\leq\frac{1}{2}c_{d,r}.\tag{42}$$

Then it follows from (14) that

$$M^d \sum_{\substack{h\in\mathbb{Z}^d\\ |h|\geq\sqrt{R_{d,r}}}} r^d|h|^{-d}J^2_{d/2}(2\pi rM|h|) \leq M^{d-1} \sum_{\substack{h\in\mathbb{Z}^d\\ |h|\geq\sqrt{R_{d,r}}}} r^{d-1}|h|^{-d-1} \leq \frac{1}{2}c_{d,r}M^{d-1}.$$

In view of (12) and Lemma 6.2, we see that to establish part (ii) of Theorem 1.3, it remains to establish the following result.

Lemma 6.3. *Suppose that* $d\equiv 1 \pmod 4$. *Suppose further that the constants* $c_{d,r}$ *and* $R_{d,r}$ *are given by Lemma* 6.2 *and* (42). *Then for infinitely many positive integers* M, *we have*

$$\sum_{\substack{0\neq h\in\mathbb{Z}^d\\ |h|<\sqrt{R_{d,r}}}} r^d|h|^{-d}J^2_{d/2}(2\pi rM|h|) < \frac{c_{d,r}}{2M}. \tag{43}$$

Proof. Since $d\equiv 1 \pmod 4$, it follows from (13) that

$$J_{d/2}(2\pi rM|h|) = \frac{1}{\pi r^{1/2}M^{1/2}|h|^{1/2}} \sin 2\pi rM|h| + O_d\left(\frac{1}{r^{3/2}M^{3/2}|h|^{3/2}}\right),$$

so that

$$\sum_{\substack{0\neq h\in\mathbb{Z}^d\\ |h|<\sqrt{R_{d,r}}}} r^d|h|^{-d}J^2_{d/2}(2\pi rM|h|)$$

$$= \frac{1}{\pi^2} \sum_{\substack{0\neq h\in\mathbb{Z}^d\\ |h|<\sqrt{R_{d,r}}}} r^{d-1}|h|^{-d-1}M^{-1}\sin^2 2\pi rM|h| + O_{d,r}\left(\frac{1}{M^2}\right).$$

To complete the proof of the lemma, it suffices to show that there are infinitely many positive integers M such that

$$\sum_{\substack{0\neq h\in\mathbb{Z}^d\\ |h|<\sqrt{R_{d,r}}}} r^{d-1}|h|^{-d-1}\sin^2 2\pi rM|h| < \frac{\pi^2}{4}c_{d,r}. \tag{44}$$

To do this, note first that there exists a positive constant $C_{d,r}$, depending at most on the dimension d and the radius r, such that

$$\sum_{\substack{0\neq h\in\mathbb{Z}^d\\ |h|<\sqrt{R_{d,r}}}} r^{d-1}|h|^{-d-1}\sin^2 2\pi rM|h| \leq C_{d,r} \max_{1\leq j\leq R_{d,r}} \left\|2rM\sqrt{j}\right\|^2.$$

For every choice of r, the numbers $2r$ and $2r\sqrt{2}$ cannot both be rational. It follows from Dirichlet's simultaneous approximation theorem (see, for example, Hardy and Wright [9, Chapter XI]) that

$$\left\|2rM\sqrt{j}\right\| < M^{-1/R_{d,r}} \quad \text{for every } j=1,\dots,R_{d,r}.$$

The inequality (44) follows immediately if M is sufficiently large. □

We remark that Lemma 6.3 can be replaced by using a result of Parnovski and Sobolev [14]. Theorem 3.1 there leads to the inequality

$$D^2_{\delta_0}(M^d) \le \ell_{d,\varepsilon,r} M^{d-1} \log^{(-1+\varepsilon)/d}(M)$$

being satisfied by infinitely many positive integers M.

7. The one-dimensional case

In this penultimate section, we study the one-dimensional case and establish part (iii) of Theorem 1.3.

First of all, it follows from (7) that

$$D^2_{d\lambda}(M) = M\left(2r - \int_{\mathbb{T}} (\chi_{[-r,r]} * M\chi_{[-1/2M,1/2M]})^2(t)\,dt\right).$$

Note from (11) that

$$\begin{aligned}(\chi_{[-r,r]} * M\chi_{[-1/2M,1/2M]})(t) &= M\left|[-r,r] \cap \left(\left[-\frac{1}{2M}, \frac{1}{2M}\right] + t\right)\right| \\ &= \begin{cases} M\left(t+r+\dfrac{1}{2M}\right), & \text{if } -r-\dfrac{1}{2M} \le t \le -r+\dfrac{1}{2M}, \\ 1, & \text{if } -r+\dfrac{1}{2M} \le t \le r-\dfrac{1}{2M}, \\ -M\left(t-r-\dfrac{1}{2M}\right), & \text{if } \quad r-\dfrac{1}{2M} \le t \le r+\dfrac{1}{2M}. \end{cases}\end{aligned}$$

A simple calculation now gives

$$D^2_{d\lambda}(M) = \tfrac{1}{3}.$$

On the other hand, it follows from (12) that

$$D^2_{\delta_0}(M) = M \sum_{0 \ne h \in \mathbb{Z}} r|h|^{-1} J^2_{1/2}(2\pi rM|h|) = \frac{2}{\pi^2} \sum_{j=1}^{\infty} \frac{1}{j^2} \sin^2 2\pi rMj < \frac{2}{\pi^2} \sum_{j=1}^{\infty} \frac{1}{j^2} = \frac{1}{3}.$$

This completes the proof of part (iii) of Theorem 1.3.

8. Some asymptotics

Lemma 3.1 is a particular case of the following result which is asymptotic in nature and which, although not necessary for establishing our main results, may have independent interest. We thank the referee for having suggested this extra line of investigation.

Proposition 8.1. *For every dimension d and radius r, there exists a positive constant $C^*_{d,r}$, depending at most on d and r, such that*

$$\frac{D^2_{d\lambda}(M^d)}{M^{d-1}} \to C^*_{d,r}, \quad \text{as } M\to\infty.$$

Proof. Our starting point is the identity

$$D^2_{d\lambda}(M^d) = M^d(|B_r| - \|\chi_{B_r} * \lambda\|^2_{L^2(\mathbb{T}^d,dt)}), \tag{45}$$

at the beginning of the proof of Lemma 3.1. We recall that the function $\lambda(t)$ is supported in the small cube

$$\left[-\frac{1}{2M},\frac{1}{2M}\right]^d,$$

so that

$$\chi_{B_r}(t) = (\chi_{B_r} * \lambda)(t) \quad \text{whenever } t \notin B_{r+\sqrt{d}/M} \setminus B_{r-\sqrt{d}/M}. \tag{46}$$

Let

$$M_0 = \max\left\{\left[\frac{2\sqrt{d}}{r}\right], \left[\frac{2\sqrt{d}}{1-2r}\right]\right\},$$

where $[z]$ denotes the integer part of z. Then simple calculation shows that for every integer $M > M_0$, we have

$$r - \frac{\sqrt{d}}{M} > \frac{r}{2} \quad \text{and} \quad r + \frac{\sqrt{d}}{M} < \min\left\{2r, \frac{1}{2}\right\}.$$

For convenience, we shall write the integration in spherical coordinates in the form

$$\int_{\mathbb{R}^d} dt = \int_{\Sigma_{d-1}} \int_0^\infty \rho^{d-1}\omega(\sigma)\, d\rho\, d\sigma.$$

Since $|\rho-r|\leq\sqrt{d}/M$ when $t=\rho\sigma\in B_{r+\sqrt{d}/M}\setminus B_{r-\sqrt{d}/M}$, it follows from (45) and (46) that

$$\begin{aligned}
&D^2_{d\lambda}(M^d)\\
&\quad=M^d\int_{B_{r+\sqrt{d}/M}\setminus B_{r-\sqrt{d}/M}}\left(\chi_{B_r}(t)-M^{2d}\left|B_r\cap\left(\left[-\frac{1}{2M},\frac{1}{2M}\right]^d+t\right)\right|^2\right)dt\\
&\quad=\frac{\pi^{d/2}d^{3/2}r^{d-1}}{\Gamma(1+d/2)}M^{d-1}+O_{d,r}(M^{d-2})\\
&\qquad-M^d\int_{\Sigma_{d-1}}\omega(\sigma)\int_{r-\sqrt{d}/M}^{r+\sqrt{d}/M}M^{2d}\left|\mathcal{B}_{r,\sigma}\cap\left(\left[-\frac{1}{2M},\frac{1}{2M}\right]^d+\rho\sigma\right)\right|^2 r^{d-1}\,d\rho\,d\sigma,
\end{aligned}$$

where $\mathcal{B}_{r,\sigma}$ is the half space containing the origin 0 and defined analogously to the half plane $\mathcal{B}_\Theta$ in Section 5. In order to study the last term above, we consider the situation as in Figure 3, where the inner cube

$$\left[-\frac{1}{2M},\frac{1}{2M}\right]^d+t$$

is cut by the boundary of the half space $\mathcal{B}_{r,\sigma}$.

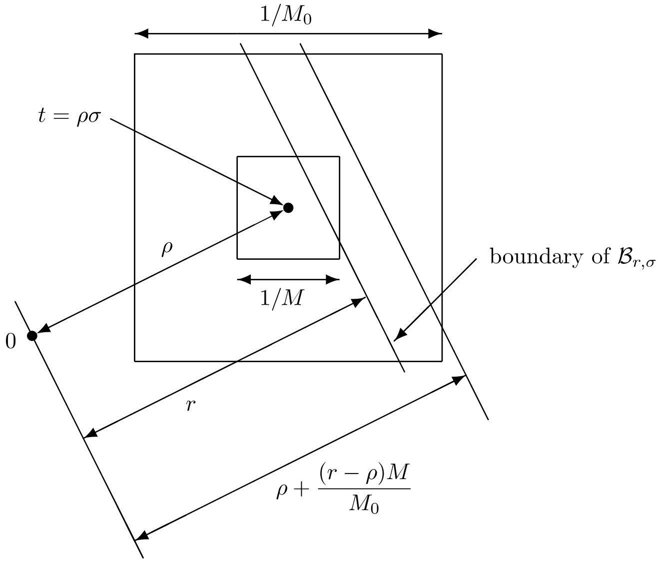

Figure 3.

The inner cube with side length $1/M$ has been dilated around its centre t to obtain a larger cube with side length $1/M_0$, and the boundary of the half space $\mathcal{B}_{r,\sigma}$ has been shifted accordingly. The ratio of the volume of the corresponding pieces of the two cubes with respect to the boundary of $\mathcal{B}_{r,\sigma}$ and its translate is M^d/M_0^d. We now need to translate the image of the boundary of $\mathcal{B}_{r,\sigma}$ back to its original position, and a simple calculation shows that

$$M^d\left|\mathcal{B}_{r,\sigma}\cap\left(\left[-\frac{1}{2M},\frac{1}{2M}\right]^d+\rho\sigma\right)\right| \\ = M_0^d\left|\mathcal{B}_{r,\sigma}\cap\left(\left[-\frac{1}{2M_0},\frac{1}{2M_0}\right]^d+\left(r+\frac{(\rho-r)M}{M_0}\right)\sigma\right)\right|.$$

Putting $s=r+(\rho-r)M/M_0$, we see that

$$D^2_{d\lambda}(M^d)=\frac{\pi^{d/2}d^{3/2}r^{d-1}}{\Gamma(1+d/2)}M^{d-1}+O_{d,r}(M^{d-2})-C^{**}_{r,d}M^{d-1},$$

where

$$C^{**}_{r,d}=r^{d-1}M_0^{2d+1}\int_{\Sigma_{d-1}}\omega(\sigma)\int_{r-\sqrt{d}/M_0}^{r+\sqrt{d}/M_0}\left|\mathcal{B}_{r,\sigma}\cap\left(\left[-\frac{1}{2M_0},\frac{1}{2M_0}\right]^d+s\sigma\right)\right|^2 ds\,d\sigma.$$

Finally we observe that

$$C^{*}_{r,d}=\frac{\pi^{d/2}d^{3/2}r^{d-1}}{\Gamma(1+d/2)}-C^{**}_{r,d}\neq 0$$

by Lemma 3.1, and this completes the proof. □

We complete our discussion by making a comment about $D^2_{\delta_0}(M^d)$. When $d\equiv 1 \pmod 4$, the oscillation of $D^2_{\delta_0}(M^d)$ is a basic ingredient for the main results of this paper. One may ask whether $D^2_{\delta_0}(M^d)$ shows an asymptotic behavior in the case $d\not\equiv 1 \pmod 4$. As far as we know, this is an open (and to us an interesting) question.

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