Ark. Mat., **48** (2010), 1–15
DOI: 10.1007/s11512-009-0100-x

A residue criterion for strong holomorphicity

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Abstract. We give a local criterion in terms of a residue current for strong holomorphicity of a meromorphic function on an arbitrary pure-dimensional analytic variety. This generalizes a result by A. Tsikh for the case of a reduced complete intersection.

1. Introduction

Let Z be an analytic variety in a neighborhood of the closed unit ball in $\mathbb{C}^n$, and let $\mathcal{I}_Z$ be the sheaf of holomorphic functions that vanish on Z. Then $\mathcal{O}_Z=\mathcal{O}/\mathcal{I}_Z$ is the sheaf of (strongly) holomorphic functions on Z. A meromorphic function on Z is a section of the sheaf $\mathcal{M}_Z$, where $\mathcal{M}_{Z,x}$ is the ring of quotients g/h, where $g,h\in\mathcal{O}_{Z,x}$ and h is a nonzerodivisor. Thus locally a meromorphic function ϕ is (represented by) g/h where g and h are holomorphic in the ambient space, h is generically nonvanishing on Z, and g'/h' is another representation of ϕ if and only if $gh'=g'h$ on Z.

If Z is given by a complete intersection, i.e., $Z=\{x; F_1(x)=...=F_p(x)=0\}$ and $\operatorname{codim} Z=p$, we have a well-defined $\bar\partial$-closed $(0,p)$-current

$$\mu^F=\bar\partial\frac{1}{F_p}\wedge...\wedge\bar\partial\frac{1}{F_1},$$

the Coleff–Herrera product [8], with support on Z. The following criterion was proved by Tsikh [17]; see also [12]:

Assume that the Jacobian $dF_1\wedge...\wedge dF_p$ is nonvanishing on $Z_{\rm reg}$. A meromorphic function ϕ on Z is (strongly) holomorphic on Z if and only if the current $\phi\mu^F$ is $\bar\partial$-closed.

The assumption on the Jacobian implies (and is in fact equivalent to) that the annihilator of μ^F is precisely $\mathcal{I}_Z$. The product $\phi\mu^F$ can be defined as the principal

The author was partially supported by the Swedish Research Council.

value

$$\lim_{\varepsilon\to 0}\chi\left(\frac{|h|}{\varepsilon}\right)(g/h)\mu^F, \tag{1.1}$$

where g/h is a (local) representation of ϕ and χ is (a possibly smooth approximant of) the characteristic function for the interval $[1,\infty)$, see Section 3. For further reference let us sketch a proof of Tsikh's theorem: If ϕ is strongly holomorphic, then it is represented by a function Φ that is holomorphic in the ambient space, and since μ^F is $\bar\partial$-closed it follows that $\phi\mu^F$ is. Conversely, assume that $\phi=g/h$, where g and h are holomorphic in the ambient space (and necessarily) h is generically nonvanishing on $Z_{\rm reg}$. Then formally at least, the assumption implies that

$$g\bar\partial\frac{1}{h}\wedge\bar\partial\frac{1}{F_p}\wedge\ldots\wedge\bar\partial\frac{1}{F_1}=0,$$

and since also $h, F_1, \ldots, F_p$ form a complete intersection it follows from the duality theorem, [10] and [14], that g is in the ideal generated by $h, F_1, \ldots, F_p$, i.e., $g=\alpha h+\alpha_1F_1+\ldots+\alpha_pF_p$. Thus ϕ is represented by $\alpha\in\mathcal{O}$ and so $\phi\in\mathcal{O}_Z$.

Remark 1.1. One should remark here that it is *not* possible to use the Lelong current $[Z]$; in fact, the meromorphic functions ϕ such that $\phi[Z]$ are $\bar\partial$-closed, form the wider class ω_Z^0 introduced in [6] and studied further in [12].

In this paper we generalize Tsikh's result in two ways. We consider an arbitrary variety Z of pure codimension p, and we consider also the nonreduced case, i.e., instead of $\mathcal{I}_Z$ we have an arbitrary pure-dimensional coherent ideal sheaf $\mathcal{I}$ with zero variety Z. To formulate our results we first have to discuss an appropriate generalization from [4] of the Coleff–Herrera product above.

In a neighborhood X of the closed unit ball there is a free resolution

$$0\longrightarrow\mathcal{O}(E_N)\xrightarrow{f_N}\ldots\xrightarrow{f_3}\mathcal{O}(E_2)\xrightarrow{f_2}\mathcal{O}(E_1)\xrightarrow{f_1}\mathcal{O}(E_0) \tag{1.2}$$

of the sheaf $\mathcal{O}/\mathcal{I}$. Here $\mathcal{O}(E_k)$ is the free sheaf associated to the trivial vector bundle E_k over X, and $E_0\simeq\mathbb{C}$ so that $\mathcal{O}(E_0)\simeq\mathcal{O}$. In [4] we defined, given Hermitian metrics on E_k, a residue current $R=R_p+R_{p+1}+\ldots$ with support on Z, where R_k is a $(0,k)$-current that takes values in $E_k\simeq{\rm Hom}(E_0,E_k)$, such that a holomorphic function ϕ is in $\mathcal{I}$ if and only if $\phi R=0$. For simplicity we think that we have some fixed global frames for E_k and choose the trivial metrics that they induce. In this way we can talk about *the* residue current associated with (1.2).

If $\mathcal{I}$ is Cohen–Macaulay, i.e., each stalk $\mathcal{I}_x$ is a Cohen–Macaulay ideal in $\mathcal{O}_x$ we can choose (1.2) such that $N=p$, and then $R=R_p$ is $\bar\partial$-closed. In general,

$f_{k+1}R_{k+1}-\bar\partial R_k=0$ for each k which can be written simply as $\nabla R=0$ if $\nabla=f-\bar\partial$ and $f=\bigoplus_k f_k$.

The assumption that $\mathcal{I}$ has pure dimension p means that in each local ring $\mathcal{O}_x$ all the associated primes have codimension p. As in the reduced case we have $\mathcal{O}_Z=\mathcal{O}/\mathcal{I}$. The sheaf of meromorphic functions is defined in precisely the same way as in the reduced case. Thus, if Φ and Φ' are meromorphic in the ambient space then they define the same meromorphic ϕ on Z if and only if $\Phi-\Phi'$ belongs to $\mathcal{I}$ generically on Z. In Section 3 we give a reasonable definition of ϕR for $\phi\in\mathcal{M}_Z$. Here is our basic result.

Theorem 1.2. *Suppose that $Z\sim\mathcal{I}$ has pure codimension p and let R be the residue current associated to a resolution of $\mathcal{O}/\mathcal{I}$. Then a meromorphic function ϕ on Z is* (*strongly*) *holomorphic if and only if*

$$\nabla(\phi R)=0. \tag{1.3}$$

If $\mathcal{I}$ is Cohen–Macaulay and $N=p$ in (1.2), then $R=R_p$ and so (1.3) means that $\bar\partial(\phi R)=0$.

The reduced case of course corresponds to $\mathcal{I}=\mathcal{I}_Z$.

Remark 1.3. If $f_1=(F_1,...,F_p)$ is a complete intersection, one can choose (1.2) as the Koszul complex, and then the residue current is precisely the Coleff–Herrera product μ^F, see, e.g., [3], Corollary 3.2. If $\mathcal{I}=\mathcal{I}_Z$ we thus get back Tsikh's theorem.

Let $\mathcal{I}$ be any ideal sheaf of codimension p and let (1.2) be a resolution of $\mathcal{O}/\mathcal{I}$. Let Z_k be the analytic set where f_k does not have optimal rank. These sets Z_k are independent of the choice of resolution, $\subset Z_{p+2}\subset Z_{p+1}\subset Z_{\text{sing}}\subset Z_p=...=Z_1=Z$, where Z is the zero set of $\mathcal{I}$, and codim $Z_k\geq k$ for all k. Moreover, $\mathcal{I}$ is pure if and only if codim $Z_k\geq k+1$ for all $k>p$, and $\mathcal{I}$ is Cohen–Macaulay if and only if $Z_k=\varnothing$ for $k>p$. All these facts are well known and can be found in, e.g., [11], Chapter 20.

For each meromorphic function ϕ on $Z\sim\mathcal{I}$ there is a smallest analytic subvariety P_ϕ, the pole set, outside which ϕ is strongly holomorphic. As an application of Theorem 1.2 we get the following theorem.

Theorem 1.4. *Assume that Z has pure codimension p. If ϕ is meromorphic and*

$$\operatorname{codim}(P_\phi\cap Z_k)\geq k+2, \quad k\geq p, \tag{1.4}$$

then ϕ is (*strongly*) *holomorphic.*

Assume now that Z is reduced. Recall that a function is called *weakly holomorphic* on Z if it is holomorphic on $Z_{\rm reg}$ and locally bounded at $Z_{\rm sing}$. It is well known that each weakly holomorphic function is meromorphic, see, e.g., [9]. If each germ of a weakly holomorphic function at $x\in Z$ is strongly holomorphic, then necessarily Z_x is irreducible and x is said to be a *normal point*. If ϕ is weakly holomorphic, then clearly P_ϕ is contained in $Z_{\rm sing}$. From Theorem 1.4 we therefore immediately get the following corollary.

Corollary 1.5. *Assume that Z is reduced with pure codimension p and let $\mathcal{I}_x$ be the corresponding local ideal at $x\in Z$. If*

$$\operatorname{codim} Z_{{\rm sing},x} \geq 2+p, \tag{1.5}$$

and

$$\operatorname{codim} Z_{k,x} \geq 2+k, \quad k>p, \tag{1.6}$$

then x is a normal point.

Conversely, conditions (1.5) and (1.6) are fulfilled if x is a normal point. In fact, these conditions are equivalent to Serre's criterion (conditions R1 and S2) for the ring $\mathcal{O}_{Z,x}$ to be normal, see, e.g., [11], pp. 255 and 462. (Condition (1.5) is precisely R1 and by an argument similar to the proof of Corollary 20.14 in [11] it follows that (1.6) is equivalent to the condition S2.) The normality of $\mathcal{O}_{Z,x}$ is equivalent to the fact that it is equal to its integral closure in $\mathcal{M}_{Z,x}$, which in turn is equivalent to the fact that x is a normal point, see also [1].

Remark 1.6. One can check that the sets $Z^0=Z_{\rm sing}$ and $Z^l=Z_{p+l}$ for $l>0$ are independent of the embedding and thus intrinsic analytic subsets of the analytic space Z. In this notation the Serre condition says that $\operatorname{codim} Z^l\geq 2+l$ for $l\geq 0$.

Example 1.7. If $\mathcal{I}_x$ is a Cohen–Macaulay ideal, then $Z_k=\emptyset$ for $k>p$ and hence (1.6) is trivially fulfilled. If $Z_{\rm sing}$ is just a point x, then (1.6) is fulfilled if Z_k avoids x for each $k>n-2$. This means that $\mathcal{O}_{Z,x}=\mathcal{O}_x/\mathcal{I}_x$ has depth at least 2.

We also obtain a new proof of the following result due to Malgrange [13] and Spallek [16]. One says that a function ϕ on Z is in $C^k(Z)$ if it is (locally) the restriction to Z of a C^k-function in the ambient space.

Corollary 1.8. *Assume that Z has pure codimension and is reduced. There is a natural number m such that if $\phi\in C^m(Z)$ is holomorphic on $Z_{\rm reg}$ then ϕ is strongly holomorphic on Z.*

It is desirable to express the ideal $\mathcal{I}$ as

$$\mathcal{I} = \bigcap_{l=1}^{\nu} \operatorname{ann} \mu_l, \tag{1.7}$$

where μ_j are so-called Coleff–Herrera currents, $\mu_j \in \mathcal{CH}_Z$, on Z. In fact, (locally) a Coleff–Herrera current μ is just a meromorphic differential operator acting on the current of integration $[Z]$ (combined with contractions with holomorphic vector fields), see [7] (or [2]). Therefore $\phi\mu=0$ is an elegant intrinsic way to express that certain holomorphic differential operators applied to ϕ vanish on Z. If $\mathcal{I}$ has pure codimension then, see, e.g., (1.6) in [2], $\mathcal{I}$ is equal to the annihilator of the analytic sheaf

$$\mathcal{H}om(\mathcal{O}/\mathcal{I}, \mathcal{CH}_Z) = \{\mu \in \mathcal{CH}_Z; \mathcal{I}\mu = 0\}.$$

This sheaf turns out to be coherent, and therefore there is a finite family of global sections in a neighborhood X of the closed unit ball such that (1.7) holds. One can ask whether there is a criterion for strong holomorphicity expressed in terms of the μ_l.

Theorem 1.9. *Assume that $\mathcal{I}$ has pure codimension p and that μ_l, $l=1, ..., N$, generate $\mathcal{H}om(\mathcal{O}/\mathcal{I}, \mathcal{CH}_Z)$. Let ϕ be meromorphic and assume that*

$$\operatorname{codim}(P_\phi \cap Z_k) \geq k+2, \quad k > p. \tag{1.8}$$

Then ϕ is holomorphic if and only if $\phi\mu_l$ are $\bar\partial$-closed for all l.

If for instance $\mathcal{I}$ is Cohen–Macaulay, then Z_k is empty for $k>p$ so (1.8) is fulfilled for any meromorphic ϕ. If h is holomorphic and generically nonvanishing on Z, then $\bar\partial(1/h)\wedge\mu_l$ are Coleff–Herrera currents whose common annihilator is precisely the ideal $h+\mathcal{I}$, see Theorem 4.2 below.

2. Some residue theory

In [5] we introduced the sheaf of *pseudomeromorphic* currents $\mathcal{PM}$ in X. It is a module over the sheaf of smooth forms, and closed under $\bar\partial$. For any $T \in \mathcal{PM}$ and variety V there exists a restriction $T\mathbf{1}_V$ that is in $\mathcal{PM}$ and has support on V, and $T = T\mathbf{1}_V$ if and only if T has support on V. Moreover, $\mathbf{1}_V \mathbf{1}_{V'} T = \mathbf{1}_{V\cap V'} T$ and $\xi \mathbf{1}_V T = \mathbf{1}_V(\xi T)$ if ξ is smooth. If H is a holomorphic tuple such that $\{x; H(x) = 0\} = V$, then $|H|^{2\lambda} T$ has a current-valued analytic continuation to $\operatorname{Re}\lambda > -\varepsilon$ and

$$T\mathbf{1}_V = T - |H|^{2\lambda} T|_{\lambda=0}. \tag{2.1}$$

We say that a current T with support on a variety V has the standard extension property (SEP) (with respect to V) if $T\mathbf{1}_W=0$ for each $W\subset V$ with positive codimension. The following result (Corollary 2.4 in [5]) will be used frequently.

Proposition 2.1. *If $\mu\in\mathcal{PM}$ with bidegree $(*,p)$ has support on a variety V of codimension $k>p$ then $\mu=0$.*

Let Z be a variety of pure codimension p. The sheaf of $\bar\partial$-closed $\mathcal{PM}$ currents of bidegree $(0,p)$ with support on Z coincides with the so-called sheaf of Coleff–Herrera currents, $\mathcal{CH}_Z$; see Proposition 2.5 in [5].

We have to recall the construction of a residue current associated with a complex of locally free sheaves in [4]. Let

$$0\longrightarrow E_N\xrightarrow{f_N}\dots\xrightarrow{f_3}E_2\xrightarrow{f_2}E_1\xrightarrow{f_1}E_0\longrightarrow 0 \tag{2.2}$$

be a generically exact complex of Hermitian vector bundles over X, where $E_0\simeq\mathbb{C}$ for simplicity, let

$$0\longrightarrow\mathcal{O}(E_N)\xrightarrow{f_N}\dots\xrightarrow{f_1}\mathcal{O}(E_0) \tag{2.3}$$

be the corresponding complex of locally free sheaves, and let $\mathcal{I}$ be the ideal sheaf $f_1\mathcal{O}(E_1)\subset\mathcal{O}$. Assume that (2.2) is pointwise exact outside the variety Z, and over $X\setminus Z$ let $\sigma_k\colon E_{k-1}\to E_k$ be the minimal inverses of f_k. Then $f\sigma+\sigma f=I$, where I is the identity on $E=\bigoplus_k E_k$, $f=\bigoplus_k f_k$ and $\sigma=\bigoplus_k\sigma_k$. The bundle E has a natural superbundle structure $E=E^+\oplus E^-$, where $E^+=\bigoplus_k E_{2k}$ and $E^-=\bigoplus_k E_{2k+1}$, and f and σ are odd mappings with respect to this structure, see, e.g., [4] for more details.

The operator $\nabla=f-\bar\partial$ acts as an odd mapping on $\mathcal{C}^{0,\bullet}(X,E)$, the space of $(0,*)$-currents with values in E, and extends to an odd mapping $\nabla_{\rm End}$ on $\mathcal{C}^{0,\bullet}(X,\operatorname{End}E)$, and $\nabla^2_{\rm End}=0$. If

$$u=\sigma+(\bar\partial\sigma)\sigma+(\bar\partial\sigma)^2\sigma+\dots,$$

then $\nabla_{\rm End}u=I$ in $X\setminus Z$. One can define a canonical current extension U of u across Z as the analytic continuation to $\lambda=0$ of $U^\lambda=|F|^{2\lambda}u$, where F is a holomorphic tuple that vanishes on Z; e.g., $F=f_1$ will do if (2.3) is a resolution. From [5] we know that U is in $\mathcal{PM}$. For further reference we notice that $\mathbf{1}_VU=0$ for any V with positive codimension. In fact, since U is smooth outside Z, $\mathbf{1}_VU$ must vanish there, and thus it has support on Z. However, from the definition of U it follows that $\mathbf{1}_ZU=0$. Therefore, $\mathbf{1}_VU=\mathbf{1}_Z\mathbf{1}_VU=\mathbf{1}_V\mathbf{1}_ZU=0$. Now

$$\nabla_{\rm End}U^\lambda=I-R^\lambda,$$

where

$$R^\lambda = (1-|F|^{2\lambda})I+\bar\partial|F|^{2\lambda}\wedge u. \tag{2.4}$$

Then the current

$$R=R^\lambda|_{\lambda=0}$$

is in $\mathcal{PM}$, has support on Z, and

$$\nabla_{\mathrm{End}}U=I-R. \tag{2.5}$$

More precisely,

$$R=\sum_{l\geq 0}R^l=\sum_{l,k\geq 0}R^l_k,$$

where R^l_k is a $\mathcal{PM}$-current of bidegree $(0,k-l)$ that takes values in $\mathrm{Hom}(E_l,E_k)$.

As before, let Z_k be the set where f_k does not have optimal rank. By the Buchsbaum–Eisenbud theorem, see [11], Chapter 20, (2.3) is a resolution of $\mathcal{O}/\mathcal{I}$ if and only if $\operatorname{codim} Z_k\geq k$ for all k. We also recall from [4] that if (2.3) is a resolution, then $R^l=0$ for all $l\geq 1$. In view of Proposition 2.1 then $R=R^0=R_p+R_{p+1}+\dots$. Since $E_0=\mathbb{C}$ we can consider $R=R^0$ as taking values in E rather than $\mathrm{Hom}(E_0,E)$, and since $\nabla_{\mathrm{End}}R=0$ thus $\nabla R=0$.

Below we will consider analogues of R and U obtained in a different way. The following proposition is proved precisely as Proposition 2.2 in [4].

Proposition 2.2. *Consider the generically exact complex (2.2) and let U and R be any currents such that (2.5) holds. If $R^1=0$ then $\operatorname{ann} R=\mathcal{I}$. If $R^l=0$ for all $l\geq 1$ then the associated sheaf complex (2.3) is exact, i.e., a resolution of $\mathcal{O}/\mathcal{I}$.*

3. Multiplication by meromorphic functions

For any pseudomeromorphic current T and holomorphic function h, the product $(1/h)T$ is defined in [5] (Proposition 2.1) as the value at $\lambda=0$ of $|h|^{2\lambda}T$. It is again a pseudomeromorphic current and it is clear that $\alpha(1/h)T=(1/h)\alpha T$ if α is smooth. However, in general it is *not* true that $f(1/fg)T=(1/g)T$. One can verify, cf. the proof of Proposition 5.1 in [3], that $(1/h)T$ is equal to the limit of $\chi(|h|/\varepsilon)T/h$ when $\varepsilon\to 0$, cf. (1.1) above. Moreover, if we define $\bar\partial(1/h)\wedge T$ as the value at $\lambda=0$ of $\bar\partial|h|^{2\lambda}\wedge(1/h)T$, then the Leibniz rule $\bar\partial[(1/h)T]=\bar\partial(1/h)\wedge T+(1/h)\bar\partial T$ holds.

Lemma 3.1. *Suppose that $Z\sim\mathcal{I}$ has pure codimension p and let R be the residue current associated with a resolution (1.2). If h is generically nonvanishing on Z, then $(1/h)R$ has the SEP on Z.*

Proof. Assume that $V\subset Z$ has positive codimension. Then $((1/h)R_p)\mathbf{1}_V=0$ in view of Proposition 2.1. Outside the variety Z_{p+1} we have that $R_{p+1}=\alpha_{p+1}R_p$ where $\alpha_{p+1}=\bar\partial\sigma_{p+1}$ is smooth, and hence

$$\Big(\frac{1}{h}R_{p+1}\Big)\mathbf{1}_V=\Big(\frac{1}{h}\alpha_{p+1}R_p\Big)\mathbf{1}_V=\Big(\alpha_{p+1}\frac{1}{h}R_p\Big)\mathbf{1}_V=\alpha_{p+1}\Big(\frac{1}{h}R_p\Big)\mathbf{1}_V=0.$$

It follows that $((1/h)R_{p+1})\mathbf{1}_V$ has support on Z_{p+1} which has codimension $\geq p+2$, and hence it vanishes by virtue of Proposition 2.1. Now $R_{p+2}=\alpha_{p+2}R_{p+1}$ outside Z_{p+2} which has codimension $\geq p+3$, and so $(g(1/h)R_{p+2})\mathbf{1}_V=0$ by a similar argument. Continuing in this way the lemma follows. $\square$

Given a meromorphic function ϕ on Z we can define ϕR as $g(1/h)R$ if g/h represents ϕ. Since $(1/h)R$ has the SEP also $g(1/h)R$ has. Since the difference of two representations of ϕ lies in $\mathcal{I}$ outside some $V\subset Z$ of positive codimension and $\mathcal{I}R=0$, it follows from the SEP that ϕR is well defined. Moreover, if $\psi\in\mathcal{O}_Z$, it follows that

$$\psi(\phi R)=(\psi\phi)R=\phi(\psi R).$$

Since ϕR is well defined, we also have a well-defined current $\bar\partial\phi\wedge R$, and by the Leibniz rule,

$$\bar\partial\phi\wedge R=-\nabla(\phi R)=g\bar\partial\frac{1}{h}\wedge R. \tag{3.1}$$

The proof of Theorem 1.2 follows the outline of the proof of Tsikh's theorem in the introduction, and the following result is crucial.

Theorem 3.2. *Assume that $\mathcal{I}$ has pure codimension and let R be the residue current associated with a resolution. If h is generically nonvanishing on Z, then the annihilator of*

$$\bar\partial\frac{1}{h}\wedge R.$$

is precisely $h+\mathcal{I}$.

Theorem 3.2 is a special case of a more general result for product complexes, Theorem 4.2, that we obtain without too much extra effort.

Remark 3.3. Let ϕ be holomorphic in $Z\setminus V$, where $V\subset Z$ has positive codimension and contains $Z_{\rm sing}$. If ϕ is meromorphic on Z, then we have seen that ϕR has a natural current extension from $X\setminus V$ across V. Also the converse holds. In fact, one can always find a holomorphic form α with values in $\mathrm{Hom}(E_p,E_0)$ such that $R_p\cdot\alpha=[Z]$, see [2], Example 1. Therefore, if ϕR has an extension across V also $\phi[Z]$ has, and it then follows from [12] that ϕ is meromorphic.

4. Tensor products of resolutions

Assume that $\mathcal{O}(E_k^g)$, g_k and $\mathcal{O}(E_l^h)$, h_l are resolutions of $\mathcal{O}/\mathcal{I}$ and $\mathcal{O}/\mathcal{J}$, respectively. We can define a complex (2.3), where

$$E_k = \bigoplus_{i+j=k} E_i^g \otimes E_j^h, \tag{4.1}$$

$f=g+h$, or more formally, $f=g\otimes I_{E^h}+I_{E^g}\otimes h$, such that

$$f(\xi\otimes\eta)=g\xi\otimes\eta+(-1)^{\deg\xi}\xi\otimes h\eta.$$

Notice that $E_0=E_0^g\otimes E_0^h=\mathbb{C}$ and that $f_1\mathcal{O}(E_1)=\mathcal{I}+\mathcal{J}$. One extends (4.1) to current-valued sections ξ and η and $\deg\xi$ then means total degree. It is natural to write $\xi\wedge\eta$ rather than $\xi\otimes\eta$, and of course we can define $\eta\wedge\xi$ as $(-1)^{\deg\xi\deg\eta}\xi\wedge\eta$. Notice that

$$\nabla(\xi\otimes\eta)=\nabla^g\xi\otimes\eta+(-1)^{\deg\xi}\xi\otimes\nabla^h\eta. \tag{4.2}$$

Let u^g and u^h be the corresponding $\mathrm{Hom}(E^g)$-valued and $\mathrm{Hom}(E^h)$-valued forms, cf. Section 2. Then $u=u^h\wedge u^g$ is a $\mathrm{Hom}(E)$-valued form outside $Z^g\cup Z^h$. Following the proof of Proposition 2.1 in [5] we can define $\mathrm{Hom}(E)$-valued pseudomeromorphic currents

$$R^h\wedge R^g=R^{h,\lambda}\wedge R^g|_{\lambda=0} \quad \text{and} \quad R^g\wedge R^h=R^{g,\lambda}\wedge R^h|_{\lambda=0}.$$

Remark 4.1. It is important here that $R^{h,\lambda}=\bar\partial|H|^{2\lambda}\wedge u^h$ with $H=h_1$. If we use a tuple H that vanish on a larger set than Z^h, the result may be affected. It is also important to notice that even if a certain component $(R^h)_k^l$ vanishes, it might very well happen that $(R^h)_k^l\wedge R^g$ is nonvanishing. In particular, notice that $(R^h)_l^l\wedge R^g=\mathbf{1}_{Z^h}I_{E_l^h}\wedge R^g$, cf. (2.4) and (2.1), which is nonvanishing if $Z^h\supset Z^g$.

We can now state our main result of this section.

Theorem 4.2. *Assume that $\mathcal{I}$ and $\mathcal{J}$ are ideal sheaves such that*

$$\operatorname{codim}(Z_k^{\mathcal{I}}\cap Z_l^{\mathcal{J}})\geq k+l, \quad k,l\geq 1. \tag{4.3}$$

Then

$$R^h\wedge R^g=R^g\wedge R^h \tag{4.4}$$

and the annihilator of $R^h\wedge R^g$ is equal to $\mathcal{I}+\mathcal{J}$.

In case both sheaves are Cohen–Macaulay and both resolutions have minimal lengths, $R^h \wedge R^g$ coincides with the current obtained from the tensor product of the resolutions.

Proof of Theorem 3.2. Let $\mathcal{I}$ be the sheaf associated with Z and let $\mathcal{J}=(h)$. Then $0\to\mathcal{O}(E_1^h)\to\mathcal{O}(E_0^h)$ is a resolution of $\mathcal{O}/\mathcal{J}$ if $E_1^h\simeq E_0^h\simeq\mathbb{C}$ and the mapping is multiplication by h. Thus $Z^h=Z_1^h=\{z;h(z)=0\}$ and $Z_l^h=\varnothing$ for $l>1$. Since Z has pure codimension, $\operatorname{codim} Z_k\geq k+1$ for all k. Thus $\operatorname{codim} Z_k\cap Z_l^h\geq k+l$. As $R^h\wedge R=\bar\partial(1/h)\wedge R$, Theorem 3.2 follows from Theorem 4.2. □

Remark 4.3. Let $\mathcal{I}=(g_1)$ and $\mathcal{J}=(h_1)$ be complete intersections, and choose the Koszul complexes as resolutions. Then, see [4], R^g and R^h are the Bochner–Martinelli type residues introduced in [15]. Moreover, the tensor product of these resolutions is the Koszul complex generated by (g_1,h_1), and so the last statement in the theorem means that this product coincides with the Bochner–Martinelli residue associated with the ideal (g_1,h_1). This fact is proved already in [18].

Remark 4.4. Theorem 4.2 extends in a natural way to any finite number of ideal sheaves.

Analogously we can define currents

$$U^h\wedge R^g=U^{h,\lambda}\wedge R^g|_{\lambda=0}\quad\text{and}\quad R^g\wedge U^h=R^{g,\lambda}\wedge U^h|_{\lambda=0},$$

etc. From (4.2) we get that

$$\nabla_{\mathrm{End}}(U^h\wedge R^g)=I^h\wedge R^g-R^h\wedge R^g. \tag{4.5}$$

In fact, $\nabla_{\mathrm{End}}(U^{h,\lambda}\wedge R^g)=(I^h-R^{\lambda,h})\wedge R^g$ since $\nabla^g_{\mathrm{End}}R^g=0$ and so (4.5) follows. In the same way

$$\nabla_{\mathrm{End}}(R^g\wedge U^h)=R^g\wedge I^h-R^g\wedge R^h. \tag{4.6}$$

If we define

$$U=I^h\wedge U^g+U^h\wedge R^g,\quad R=R^h\wedge R^g\quad\text{and}\quad I=I_E,$$

then

$$\nabla_{\mathrm{End}}U=I-R. \tag{4.7}$$

Lemma 4.5. *If the hypothesis in Theorem 4.2 holds, we have that*

$$U^h\wedge R^g=R^g\wedge U^h. \tag{4.8}$$

Proof. We have to prove that

$$(U^h)^r_l (R^g)^s_k - (R^g)^s_k (U^h)^r_l \tag{4.9}$$

vanishes for $l>r\geq 0$ and $k\geq s\geq 0$. Since U^h is smooth outside $Z^h=Z^h_1$, (4.9) vanishes there. On the other hand, both terms have support on $Z^h=Z^h_1$. Thus (4.9) has support on $Z^h_1\cap Z^g_1$. Let us first consider the case when $r=s=0$. If $k=0$, then (4.9) is

$$0-I^g_{E^g_0}\mathbf{1}_{Z^g}(U^h)^0_l,$$

which vanishes since Z^g has positive codimension, cf. Section 2 above. Next assume that $l=k=1$. Then (4.9) has bidegree $(0,1)$ and support on $Z^h_1\cap Z^g_1$, which by the hypothesis has codimension at least 2. Thus (4.9) must vanish in view of Proposition 2.1. We now proceed by induction. Assume that we have proved that (4.9) vanishes whenever $l+k<m$, and assume that $l+k=m$. If $l\geq 2$ we know from the induction hypothesis that

$$(U^h)^0_{l-1}(R^g)^0_k - (R^g)^0_k (U^h)^0_{l-1} = 0. \tag{4.10}$$

Outside Z^h_l we can apply the smooth form $\alpha^h_l=\bar\partial\sigma^h_l$ to (4.10), cf. the proof of Lemma 3.1 above, and conclude that

$$(U^h)^0_l (R^g)^0_k - (R^g)^0_k (U^h)^0_l \tag{4.11}$$

vanishes there, i.e., its support is contained in Z^h_l. If $k\geq 2$ we find in a similar way that (4.11) must have support on Z^g_k. In any case, we find that (4.9) has bidegree $(0,m-1)$ and has support on $Z^h_l\cap Z^g_k$, which has codimension at least $l+k=m$, so (4.9) must vanish. The case when $r+s>0$ is handled in a similar way. □

Proof of Theorem 4.2. Applying ∇_{End} to (4.8) we get by (4.5) and (4.6) that

$$(I^h-R^h)\wedge R^g = R^g\wedge(I^h-R^h)$$

which is precisely (4.4). Since $(R^g)^s=0$ for $s\geq 1$ we have that

$$R=\sum_{s,r\geq 0}(R^h)^r\wedge(R^g)^s=\sum_{r\geq 0}(R^h)^r\wedge(R^g)^0.$$

In view of (4.4) we thus have that $R=(R^h)^0\wedge(R^g)^0=R^0$ i.e., $R^m=0$ for $m\geq 1$. From Proposition 2.2 we now conclude that $\mathcal{O}(E), f$ is a resolution and $\operatorname{ann} R=\mathcal{I}+\mathcal{J}$.

Finally, assume that $\mathcal{I}$ and $\mathcal{J}$ are Cohen–Macaulay sheaves and the resolutions $\mathcal{O}(E^g)$, g and $\mathcal{O}(E^h)$, h have minimal lengths $\operatorname{codim}\mathcal{I}$ and $\operatorname{codim}\mathcal{J}$, respectively. Then the product resolution $\mathcal{O}(E), f$ has (minimal) length $p=\operatorname{codim}\mathcal{I}+\operatorname{codim}\mathcal{J}$. Let U^f and R^f denote the currents associated with this complex. Then R^f as well as

$R^h\wedge R^g$ are $\bar\partial$-closed pseudomeromorphic currents of bidegree $(0,p)$ with support on $Z=Z^g\cap Z^h$ which has codimension p, and hence they are Coleff–Herrera currents, according to Proposition 2.1. Moreover, cf. (4.7),

$$\nabla_{\rm End}(U-U^f)=R^f-R=R^f-R^h\wedge R^g.$$

It follows from Lemma 3.1 in [2] that $R^f-R^h\wedge R^g=0$. $\square$

Remark 4.6. If $\mathcal{O}(E^g)$, g and $\mathcal{O}(E^h)$, h are resolutions one can verify (without residue calculus) that the product complex is a resolution as well if and only if (4.3) holds. Since this should be well known we just sketch an argument: It is not too hard to see that (for each fixed point x)

$$H^m(E^h\otimes E^g)=\bigoplus_{l+k=m}H^l(E^h)\otimes H^k(E^g). \tag{4.12}$$

In fact, choose Hermitian metrics on E^g and E^h. If g^* and h^* and $f^*=g^*+h^*$ are the induced adjoint mappings and $\Delta^f=ff^*+f^*f$, etc., then $\Delta^f=\Delta^g+\Delta^h$. As usual each class in $H^m(E^h\otimes E^g)$ has a unique harmonic representative

$$v=\sum_{l+k=m}\xi_l\wedge\eta_k.$$

However, it is easily verified that $\Delta^f v=0$ if and only if $\Delta^g\xi_l=0=\Delta^h\eta_k$ for all l and k. Thus (4.12) follows.

Let $Z_k^{\mathcal{I}}$ and $Z_l^{\mathcal{J}}$ be the varieties associated to the sheaves $\mathcal{I}$ and $\mathcal{J}$. Since $\mathcal{O}(E^g)$, g is exact, it follows that $H^k(E^g)=0$ at a given point x if and only if $x\notin Z_k^{\mathcal{I}}$, and similarly for E^h. In view of (4.12), therefore $H^m(E)\neq 0$ at x if and only if

$$x\in\bigcup_{l+k=m}Z_k^{\mathcal{I}}\cap Z_l^{\mathcal{J}}.$$

Thus $\operatorname{codim} Z_m\geq m$ for all m if and only if (4.3) holds, and according to the Buchsbaum–Eisenbud theorem therefore $\mathcal{O}(E)$, f is a resolution if and only if (4.3) holds.

5. Proofs of the main results

Proof of Theorem 1.2. If ϕ is strongly holomorphic, then it is represented by a function Φ that is holomorphic in a neighborhood of Z. Thus $\nabla(\phi R)=\nabla(\Phi R)=\Phi\nabla R=0$.

Now assume that $\nabla(\phi R)=0$ and ϕ is represented by g/h. Then by (3.1), we have that

$$0=\nabla\left(g\frac{1}{h}R\right)=-g\bar{\partial}\frac{1}{h}\wedge R.$$

This means that g annihilates the current $\bar{\partial}(1/h)\wedge R$, and by Corollary 3.2 therefore $g=\alpha h+\psi$, where $\psi\in\mathcal{I}$. It follows that ϕ is represented by α and thus $\phi\in\mathcal{O}_Z$. □

Proof of Theorem 1.4. Assume that ϕ is meromorphic and (1.4) is fulfilled. Clearly, $\bar{\partial}\phi\wedge R$ has support on $P_\phi\cap Z$, so $\bar{\partial}\phi\wedge R_p$ must vanish for degree reasons. If now $\bar{\partial}\phi\wedge R_k=0$, then it follows that $\bar{\partial}\phi\wedge R_{k+1}$ has support in $P_\phi\cap Z_{k+1}$, and so it must vanish for degree reasons. □

Proof of Corollary 1.8. First assume that ϕ is (strongly) smooth and holomorphic on Z_{reg}. It is well known that each weakly holomorphic function on Z (i.e., ϕ holomorphic on Z_{reg} and locally bounded at Z_{sing}) is meromorphic, see, e.g., [9]. Therefore, we have a priori two definitions of ϕR; either as multiplication of smooth function times R or as multiplication by the meromorphic function ϕ. However, they coincide on Z_{reg} and by the SEP therefore they coincide even across Z_{sing}. Therefore also the two possible definitions of $\nabla(\phi R)=-\bar{\partial}\phi\wedge R$ coincide. Since ϕ is holomorphic on Z_{reg} it follows that $\bar{\partial}\phi\wedge R$ has support on Z_{sing}. On the other hand,

$$(\bar{\partial}\phi\wedge R)\mathbf{1}_{Z_{\text{sing}}}=\bar{\partial}\phi\wedge R\mathbf{1}_{Z_{\text{sing}}}=0$$

by Lemma 3.1, and hence $\nabla(\phi R)=-\bar{\partial}\phi\wedge R=0$. Now the corollary follows from Theorem 1.2 with $m=\infty$. A careful inspection of all arguments reveals that only a finite number of derivatives (not depending on ϕ) come into play but we omit the details. □

Proof of Theorem 1.9. By hypothesis, $0=\bar{\partial}(\phi\mu)$ for all $\mu\in\mathcal{H}om(\mathcal{O}/\mathcal{I},\mathcal{CH}_Z)$. It is proved in [2] (Theorem 1.5) that each current μ in $\mathcal{H}om(\mathcal{O}/\mathcal{I},\mathcal{CH}_Z)$ can be written as $\mu=\xi R_p$ for some $\xi\in\mathcal{O}(E^*)$ such that $f^*_{p+1}\xi=0$ and conversely for each such ξ the current $\mu=\xi R_p$ is in $\mathcal{H}om(\mathcal{O}/\mathcal{I},\mathcal{CH}_Z)$. Here f^*_k are the induced mapping(s) on the dual complex $\mathcal{O}(E^*_k)$. Thus

$$0=\bar{\partial}\phi\wedge\xi R_p$$

for each such ξ. At a given stalk outside Z_{p+1}, the ideal $\mathcal{I}_x$ is Cohen–Macaulay, so if we choose a minimal resolution $\mathcal{O}(\widetilde{E})$, $\tilde{f}$ there it will have length p. If $\widetilde{R}_p$ denotes the resulting (germ of a) residue current, then the hypothesis implies that

$$0=\bar{\partial}\phi\wedge\widetilde{R}_p$$

as then trivially $\tilde{f}^*_{p+1}\xi=0$ for each $\xi\in\mathcal{O}(\widetilde{E}^*_p)$. However, $R_p=\alpha\widetilde{R}_p$, where α is smooth (Theorem 4.4 in [4]). It follows that $\bar\partial\phi\wedge R_p$ vanishes outside Z_{p+1}. Since $R_{p+1}=\alpha_{p+1}R_p$ outside Z_{p+1} it follows that also $\bar\partial\phi\wedge R_{p+1}$ has support on Z_{p+1}. However, it is clear that $\bar\partial\phi\wedge R$ must have support on P_ϕ. Using the hypothesis codim $(P_\phi\cap Z_k)\geq k+2$ for $k>p$, it follows by induction that $\bar\partial\phi\wedge R=0$. Thus ϕ is strongly holomorphic according to Theorem 1.2. $\square$

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Received January 21, 2008
in revised form February 23, 2009
published online June 16, 2009