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Spherical spectral synthesis and two-radius theorems on Damek–Ricci spaces

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Abstract. We prove that spherical spectral analysis and synthesis hold in Damek–Ricci spaces and derive two-radius theorems.

1. Introduction

D. Pompeiu investigated the following problem in 1929: Given a closed set $K \subset \mathbb{R}^2$ of positive volume and a continuous function f satisfying

$$\int_{\sigma(K)} f\, dy = 0 \tag{1}$$

for all rigid motions σ of the plane. Does this imply that $f \equiv 0$?

The question has a negative answer in the case of a closed disk. In fact, for every radius $r>0$ there are vectors $\xi \in \mathbb{R}^2$ such that the functions

$$f(x) = e^{i\langle x, \xi\rangle}$$

have vanishing integrals over all closed disks of radius $r>0$. A longstanding question is whether disks are essentially the only simply-connected bounded counterexamples (Pompeiu's problem). This is closely related to Schiffer's conjecture (for more details see, e.g., [23, Problem 80] and [7], [8], [15] and [22]).

However, if (1) holds true for two different disks $K = B_{r_1}$ and $K = B_{r_2}$ with radii $r_1, r_2 > 0$ whose quotient avoids a particular exceptional set, then $f \equiv 0$. This exceptional set is given by the quotient of any two different positive zeros of a particular Bessel function expression. For more details and more general two-radius theorems

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we refer to [24]. Two-radius problems have also been considered in more general geometries. The paper [6] generalizes the above result to all rank-one symmetric spaces. In fact, for compact symmetric spaces of rank one it is sufficient to check vanishing of integrals over all balls of a single radius to conclude that $f\equiv 0$, as long as this radius is not a zero of a particular Jacobi polynomial expression.

In this paper we prove two-radius results for Damek–Ricci spaces. Damek–Ricci spaces are harmonic manifolds and comprise the rank-one symmetric spaces of non-compact type. Recently, Heber [13] proved that the non-flat simply-connected *homogeneous* harmonic spaces are precisely the symmetric spaces of rank one and the non-symmetric Damek–Ricci spaces.

To state our main results we first have to introduce some notation. For a detailed discussion of the geometry and analysis of Damek–Ricci spaces we refer the reader to the exposition [18]. A Damek–Ricci space is a semidirect product $X=\mathbb{R}\ltimes N$ of a generalized Heisenberg group N with $\mathbb{R}$. Let $\mathfrak{n}$ be the Lie algebra of N with $\mathfrak{z}=[\mathfrak{n},\mathfrak{n}]$ and $\mathfrak{v}=\mathfrak{z}^\perp$ with respect to the inner product of $\mathfrak{n}$. We denote the dimensions of $\mathfrak{v}$ and $\mathfrak{z}$ by p and q. Since the pair (p,q) plays an important role in our results we often write $X^{(p,q)}$ for the Damek–Ricci space. In this context, we have $\mathbb{R}=X^{(0,0)}$. A Damek–Ricci space X is a solvable group carrying a non-positively curved left-invariant Riemannian metric.

In the sequel the *spherical functions* $\varphi_\lambda^{(p,q)}$, defined in terms of the hypergeometric function F by

$$\varphi_\lambda^{(p,q)}(r)=F\big(\rho-i\lambda,\rho+i\lambda,\tfrac{1}{2}n,-\sinh^2\tfrac{1}{2}r\big), \tag{2}$$

will be crucial. Here $n=p+q+1$ and $\rho=\frac{1}{4}p+\frac{1}{2}q$. These are the radial eigenfunctions φ_λ of the Laplacian of $X=X^{(p,q)}$, see (6).

The space of smooth radial functions on X endowed with the topology of uniform convergence on compact sets is denoted by $\mathcal{E}_0(X)$. A variety V is a proper closed subspace of $\mathcal{E}_0(X)$ which is invariant under convolution with radial distributions of compact support.

Now we can state the two-radius theorems for Damek–Ricci spaces. In the sequel integration will always be with respect to the left-invariant metric on X. Let $B_r(x)$ and $S_r(x)$ denote the geodesic ball and geodesic sphere around $x\in X$ of radius r, respectively. Spheres and balls of radius r around the identity element $e\in X$ will be denoted by S_r and B_r.

Theorem 1.1. *Let $X=X^{(p,q)}$ be a Damek–Ricci space and let $r_1,r_2>0$ be such that the equations*

$$\varphi_\lambda^{(p,q+2)}(r_j)=0,\quad j=1,2,$$

have no common solution $\lambda\in\mathbb{C}$.

Suppose $f \in C(X)$ *and*

$$\int_{B_r(x)} f\,dy = 0$$

for $r = r_1, r_2$ *and all* $x \in X$. *Then* $f \equiv 0$.

Similarly, for spherical averages, we have the following result.

Theorem 1.2. *Let* $X = X^{(p,q)}$ *be a Damek–Ricci space and let* $r_1, r_2 > 0$ *be such that the equations*

$$\varphi_\lambda^{(p,q)}(r_j) = 0, \quad j = 1, 2,$$

have no common solution $\lambda \in \mathbb{C}$.

Suppose $f \in C(X)$ *and*

$$\int_{S_r(x)} f\,dy = 0$$

for $r = r_1, r_2$ *and all* $x \in X$. *Then* $f \equiv 0$.

Note that in Damek–Ricci spaces a function is harmonic if and only if it satisfies the mean-value property for all radii. In fact, it suffices to have the mean-value property for only two suitably chosen radii in order to conclude harmonicity of a function:

Theorem 1.3. *Let* $X = X^{(p,q)}$ *be a Damek–Ricci space and let* $r_1, r_2 > 0$ *be such that the equations*

$$\varphi_\lambda^{(p,q)}(r_j) = 1, \quad j = 1, 2,$$

have no common solution $\lambda \in \mathbb{C} \setminus \{\pm i\rho\}$.

Then $f \in C^\infty(X)$ *is harmonic if and only if*

$$\frac{1}{\operatorname{vol}(S_r(x))} \int_{S_r(x)} f\,dy = f(x)$$

for $r = r_1, r_2$ *and all* $x \in X$.

Two-radius theorems are closely related to spectral analysis and synthesis of the underlying space (see, e.g., [7]). L. Schwartz [19] proved that spectral synthesis holds on the real line. Theorem 1.4 below carries over this result to radial functions in Damek–Ricci spaces. For symmetric spaces of rank one, spherical spectral synthesis was proved in [3]. For further results on spectral synthesis in symmetric spaces see, e.g., [4], [5] and [21].

Theorem 1.4. (Spherical spectral synthesis) *Let X be a Damek–Ricci space and V be a variety of radial functions. Then V is the closure of the span of all functions $\varphi_{\lambda,k}=(d^k/d\lambda^k)\varphi_\lambda$ contained in V.*

This article is organized as follows: In Section 2 we introduce some basic properties and notions which are used throughout this paper and discuss Schwartz's fundamental result. In Section 3, we introduce the Abel transform, prove some useful properties and obtain a Paley–Wiener theorem. These properties will be used in Section 4 to derive spherical spectral synthesis for Damek–Ricci spaces. Finally, Section 5 is devoted to the proofs of the above two-radius results.

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2. Preliminaries

2.1. The spaces $\mathcal{D}(X), \mathcal{E}(X), \mathcal{D}_0(X)$ and $\mathcal{E}_0(X)$

A Damek–Ricci space $X:=X^{(p,q)}$ of dimension $n=p+q+1$ is a semidirect product $\mathbb{R}\ltimes N$, where N is a generalized Heisenberg group of dimension $p+q$ with q-dimensional center. We may thus write elements of X as pairs $x=(t(x),n(x))=t(x)\cdot n(x)$. Note that $t\colon X\to\mathbb{R}$ is a group homomorphism.

By $\mathcal{E}(X)$ we denote the space of all smooth functions on X with the topology determined by the seminorms

$$\|f\|_{D,K}=\sup_{x\in K}|Df(x)|,$$

where D is an arbitrary differential operator on X and $K\subset X$ is an arbitrary compact subset. $\mathcal{E}(X)$ is a Fréchet space.

Definition 2.1. Let $r(x)=d(x,e)$ denote the distance of $x\in X$ from the identity e. The *averaging projector* $\pi\colon\mathcal{E}(X)\to\mathcal{E}(X)$ is defined by

$$\pi f(x)=\frac{1}{\operatorname{vol}(S_{r(x)})}\int_{S_{r(x)}} f\,dy.$$

Let $\mathcal{E}_0(X)=\pi\mathcal{E}(X)$ denote the space of all smooth radial functions on X, equipped with the relative topology. The spaces $\mathcal{E}(\mathbb{R})$ and $\mathcal{E}_0(\mathbb{R})$ are analogously defined. Note that $Q\colon\mathcal{E}_0(\mathbb{R})\to\mathcal{E}_0(X)$, $Qf(x)=f(r(x))$ is a topological isomorphism.

Let $\mathcal{D}(X)$ denote the space of all smooth functions on X with compact support, and $\mathcal{D}_K(M)$, for $K\subset X$ compact, the subspace of $\mathcal{D}(X)$ of functions with support in K. The topology of $\mathcal{D}(X)$ is the inductive limit topology of the spaces $\mathcal{D}_K(X)$, and $\mathcal{D}_K(X)$ has the induced topology of $\mathcal{E}(X)$ (see, e.g., [14]). Again, we have $\mathcal{D}_0(X)=\pi\mathcal{D}(X)$.

The convolution of $f\in\mathcal{E}(X)$ and $g\in\mathcal{D}(X)$ (or $f\in\mathcal{D}(X)$ and $g\in\mathcal{E}(X)$) is defined as

$$f*g(x)=\int_X f(y)g(y^{-1}x)\,dy=\langle f,(\check{g})_{x^{-1}}\rangle,$$

where $\check{g}(x):=g(x^{-1})$ and $g_x(y):=g(xy)$.

We quote some useful properties of π from [9]:

$$\pi^2=\pi, \tag{3}$$

$$\langle \pi f,g\rangle=\langle f,\pi g\rangle, \tag{4}$$

$$\pi(f*\pi g)=\pi f*\pi g. \tag{5}$$

The convolution in $\mathcal{E}(X)$ is associative but not commutative. However, property (5) yields commutativity of the convolution for radial functions and $\mathcal{D}_0(X)*\mathcal{E}_0(X)=\mathcal{E}_0(X)*\mathcal{D}_0(X)\subset\mathcal{E}_0(X)$.

Let $\Delta=\operatorname{div}\operatorname{grad}$ denote the Laplacian on X. For every $\lambda\in\mathbb{C}$ there exists a unique radial function $\varphi_\lambda\in\mathcal{E}_0(X)$ satisfying

$$\Delta\varphi_\lambda=-(\lambda^2+\rho^2)\varphi_\lambda \quad\text{and}\quad \varphi_\lambda(e)=1,$$

where $\rho=\frac{1}{4}p+\frac{1}{2}q$ (see [18]). We have

$$\varphi_\lambda(x)=\varphi_\lambda^{(p,q)}(r(x)), \tag{6}$$

where $\varphi_\lambda^{(p,q)}$ was defined in (2).

Remark 2.2. The parameter $\rho=\frac{1}{4}p+\frac{1}{2}q$ of the Damek–Ricci space $X=X^{(p,q)}$ can be interpreted isoperimetrically, asymptotically and in terms of the spectrum:

(i) We have

$$h(X):=\inf_{K\subset X\text{ compact}}\frac{\operatorname{area}(\partial K)}{\operatorname{vol}(K)}=2\rho.$$

This follows from [18, p. 66] and the explicit *isoperimetric Cheeger constant* calculation in [17].

(ii) Using

$$\mathrm{vol}(S_r) = \frac{2^n \pi^{n/2}}{\Gamma\left(\frac{1}{2}n\right)} \sinh^{p+q}\left(\tfrac{1}{2}r\right) \cosh^q\left(\tfrac{1}{2}r\right),$$

one easily verifies the identity

$$\lim_{r\to\infty} \frac{\log \mathrm{vol}(S_r)}{r} = 2\rho$$

for the *exponential volume growth* of spheres.

(iii) We have

$$\sigma(\Delta) = (-\infty, -\rho^2]$$

for the *spectrum of the Laplacian* in the Hilbert space $L^2(X,\mu)$. This follows from the fact that the spherical Fourier transform $\mathcal{F}f(\lambda)=\langle f, \varphi_\lambda\rangle$ on $\mathcal{D}_0(X)$ extends to a Hilbert space isomorphism (see [18, Theorem 15]) and that Δ transforms under this isomorphism into the multiplication operator $g \mapsto -(\lambda^2+\rho^2)g$.

2.2. The spaces $\mathcal{D}'(X), \mathcal{E}'(X), \mathcal{D}'_0(X)$ and $\mathcal{E}'_0(X)$

We denote by $\mathcal{E}'(X)$ the dual of $\mathcal{E}(X)$, endowed with the strong dual topology. $\mathcal{E}'(X)$ is the space of distributions of compact support on X. The spaces $\mathcal{D}'(X), \mathcal{D}'_0(X)$ and $\mathcal{E}'_0(X)$ are defined analogously.

The convolution of two distributions $S\in\mathcal{D}'(X)$ and $T\in\mathcal{E}'(X)$ (or $T\in\mathcal{D}'(X)$ and $S\in\mathcal{E}'(X)$) can be calculated as follows:

$$\langle S*T, f\rangle = \langle S, x\mapsto \langle T, y\mapsto f(xy)\rangle\rangle.$$

The space $\mathcal{D}(X)$ is contained in $\mathcal{E}'(X)$ via $f\mapsto T_f$, $\langle T_f, g\rangle := \langle f, g\rangle$, and we have $T*T_f = T_{T*f}$ with $(T*f)(x) = \langle T, (\check{f})_{x^{-1}}\rangle$. Using a Dirac sequence $\rho_\varepsilon\in\mathcal{D}_0(X)$ we have, for $T\in\mathcal{E}'(X)$, $T*\rho_\varepsilon\to T$ as $\varepsilon\to 0$, which shows that $\mathcal{D}(X)$ is dense in $\mathcal{E}'(X)$. Therefore, all the above properties for functions carry over to distributions.

The spherical Fourier transform of a distribution $T\in\mathcal{E}'_0(X)$ is defined as

$$\mathcal{F}T(\lambda) = \langle T, \varphi_\lambda\rangle.$$

If $T\in\mathcal{E}'_0(\mathbb{R})$, we have for the classical Fourier transform

$$\widehat{T}(\lambda) = \langle T, \phi_\lambda\rangle = \langle T, \psi_\lambda\rangle,$$

where $\phi_\lambda(t) = e^{i\lambda t}$ and $\psi_\lambda(t) = \frac{1}{2}(e^{i\lambda t} + e^{-i\lambda t})$.

2.3. Schwartz's result

Mean-periodic functions were first introduced and studied by Delsarte in a paper of 1935. L. Schwartz [19] proposed the following intrinsic definition of mean-periodic functions: a function $f\in\mathcal{E}(\mathbb{R})$ is called *mean periodic* if not every function in $\mathcal{E}(\mathbb{R})$ can be obtained as a limit of finite linear combinations of translates $f_x(y)=f(x+y)$ of f. The vector space of functions obtained as such limits is called the variety V^f of f. Equivalently, V^f can be defined as the closure of all functions of the type $T*f$ with $T\in\mathcal{E}'(\mathbb{R})$. Thus, a non-zero function $f\in\mathcal{E}(\mathbb{R})$ is mean periodic if V^f is a proper subspace of $\mathcal{E}(\mathbb{R})$. We denote by

$$\phi_{\lambda,k}(t):=\frac{d^k}{d\lambda^k}e^{i\lambda t}=i^k t^k e^{i\lambda t}.$$

The spectrum $\operatorname{spec} f$ of f is defined as follows:

$$\operatorname{spec} f:=\{\phi_{\lambda,k}\in V^f \mid k\in\mathbb{N}_0 \text{ and } \lambda\in\mathbb{C}\}.$$

L. Schwartz proved the following fundamental result.

Theorem 2.3. *Let $f\in\mathcal{E}(\mathbb{R})$ be a non-zero mean-periodic function. Then f is the limit of finite linear combinations of functions in* $\operatorname{spec} f$.

Schwartz's result actually means that *spectral synthesis* holds in $\mathcal{E}(\mathbb{R})$. *Spectral analysis* is the weaker statement that $\operatorname{spec} V^f$ is non-empty for every non-zero mean-periodic function f.

We now adapt the above theorem to $\mathcal{E}_0(\mathbb{R})$, the subspace of $\mathcal{E}(\mathbb{R})$ of even functions on $\mathbb{R}$. The averaging projector $\pi\colon \mathcal{E}(\mathbb{R})\to\mathcal{E}_0(\mathbb{R})$ is the canonical projection $(\pi f)(x):=\frac{1}{2}(f(x)+f(-x))$. Let

$$\psi_{\lambda,k}(t):=\tfrac{1}{2}(\phi_{\lambda,k}(t)+\phi_{\lambda,k}(-t)).$$

Theorem 2.4. *Let $f\in\mathcal{E}_0(\mathbb{R})$ be a non-zero mean-periodic function, i.e.,*

$$\{0\}\neq V_0^f:=\overline{\{T*f \mid T\in\mathcal{E}_0'(\mathbb{R})\}}\neq\mathcal{E}_0(\mathbb{R}).$$

Then f is the limit of finite linear combinations of functions in

$$\operatorname{spec}_0 f:=\{\psi_{\lambda,k}\in V_0^f\setminus\{0\} \mid k\in\mathbb{N}_0 \textit{ and } \lambda\in\mathbb{C}\}.$$

Proof. As a consequence of (5) and the continuity of π, we obtain for a mean-periodic function $f\in\mathcal{E}_0(\mathbb{R})$ that

$$V^f=\overline{\{T*f \mid T\in\mathcal{E}'(\mathbb{R})\}}\subset\pi^{-1}(V_0^f)\neq\mathcal{E}(\mathbb{R}). \tag{7}$$

Consequently, f is also mean periodic in $\mathcal{E}(\mathbb{R})$. Now using Theorem 2.3, f is the limit of functions f_j which are finite linear combinations of functions in $\operatorname{spec} f$. Then we also have that

$$\pi(f_j) \to \pi(f) = f.$$

Since (7) implies that $\pi(\operatorname{spec} f) \subset \operatorname{spec}_0 f \cup \{0\}$, $\pi(f_j)$ is a finite linear combination of functions in $\operatorname{spec}_0 f$, finishing the proof. □

3. The Abel transform on distributions

3.1. The Abel transform

The Abel transform will be of great importance in our considerations.

Definition 3.1. Let j and a be the maps

$$j\colon \mathcal{E}_0(\mathbb{R}) \longrightarrow \mathcal{E}(X) \quad \text{with } jf(x) = e^{\rho t(x)} f(t(x))$$

and

$$a\colon \mathcal{E}_0(\mathbb{R}) \longrightarrow \mathcal{E}_0(X) \quad \text{with } a = \pi \circ j,$$

i.e.,

$$af(x) = \frac{1}{\operatorname{vol}(S_{r(x)})} \int_{S_{r(x)}} e^{\rho t(y)} f(t(y))\, dy.$$

The *Abel transform* $\mathcal{A}$ is then defined as the dual of a, i.e., as the map

$$\mathcal{A} \longrightarrow \mathcal{E}_0'(X) \longrightarrow \mathcal{E}_0'(\mathbb{R}) \quad \text{with } \langle \mathcal{A}T, f\rangle = \langle T, af\rangle$$

for distributions $T \in \mathcal{E}_0'(X)$ of compact support and smooth functions $f \in \mathcal{E}_0(\mathbb{R})$.

Remark 3.2. (i) The restriction of $\mathcal{A}$ to $\mathcal{D}_0(X) \subset \mathcal{E}_0'(X)$ is explicitly given by (see [18]):

$$\mathcal{A}f(t) = e^{\rho t} \int_N f(tn)\, dn.$$

(ii) We have

$$a\psi_\lambda = \varphi_\lambda \quad \text{and} \quad a\psi_{\lambda,k} = \varphi_{\lambda,k}. \tag{8}$$

For the first equation, see [18, p. 80]. The second equation is obtained by differentiating this with respect to λ. The spherical Fourier transform can be expressed in terms of the Abel transforms as

$$\mathcal{F}T(\lambda) = \langle T, \varphi_\lambda\rangle = \langle T, a\psi_\lambda\rangle = \langle \mathcal{A}T, \psi_\lambda\rangle = \widehat{\mathcal{A}T}(\lambda) \tag{9}$$

for $T \in \mathcal{E}_0'(X)$.

3.2. Properties of the Abel transform

A key property of the Abel transform is that it preserves convolution.

Proposition 3.3. *For* $T, S \in \mathcal{E}_0'(X)$ *and* $f \in \mathcal{E}_0(\mathbb{R})$ *we have*

$$\mathcal{A}(T *_X S) = \mathcal{A}T *_{\mathbb{R}} \mathcal{A}S, \tag{10}$$

$$a(\mathcal{A}T *_{\mathbb{R}} f) = T *_X af. \tag{11}$$

Proof. Recall that for radial distributions,

$$\langle T * S, \phi \rangle = \langle T, x \mapsto \langle S, y \mapsto \phi(xy) \rangle \rangle,$$

and that $t\colon X \to \mathbb{R}$ is a homomorphism. We thus compute for all $\phi \in \mathcal{E}_0(X)$,

$$\begin{aligned}\langle \mathcal{A}T * \mathcal{A}S, \phi \rangle &= \langle \mathcal{A}T, r \mapsto \langle \mathcal{A}S, s \mapsto \phi(r+s) \rangle \rangle \\ &= \langle T, x \mapsto \langle S, y \mapsto \phi(t(x)+t(y)) e^{\rho(t(x)+t(y))} \rangle \rangle \\ &= \langle T, x \mapsto \langle S, y \mapsto \phi(t(xy)) e^{\rho(t(xy))} \rangle \rangle \\ &= \langle \mathcal{A}(T * S), \phi \rangle\end{aligned}$$

which proves (10). Next, we prove the second claim. Using $\langle g, h \rangle = (g * h)(e)$ and commutativity of the convolution of radial functions, we have for all $g, \phi \in \mathcal{D}_0(X)$,

$$\langle g * af, \phi \rangle = \langle af, g * \phi \rangle = \langle f, \mathcal{A}g * \mathcal{A}\phi \rangle = \langle \mathcal{A}g * f, \mathcal{A}\phi \rangle = \langle a(\mathcal{A}g * f), \phi \rangle$$

(see (10) for the second equality). Now, from the continuity of a and $\mathcal{A}$, and the density of $\mathcal{D}_0(X)$ in $\mathcal{E}_0'(X)$ we get (11). $\square$

We will prove, in the next subsection, that a is bijective. Assuming this for the moment we have the following result.

Proposition 3.4. *The following diagram commutes*:

$$\begin{array}{ccc} \mathcal{E}_0'(X) \times \mathcal{E}_0(X) & \xrightarrow{\ *_X\ } & \mathcal{E}_0(X) \\ \Big\downarrow{\scriptstyle \mathcal{A}\times\mathcal{B}} & & \Big\downarrow{\scriptstyle \mathcal{B}} \\ \mathcal{E}_0'(\mathbb{R}) \times \mathcal{E}_0(\mathbb{R}) & \xrightarrow{\ *_{\mathbb{R}}\ } & \mathcal{E}_0(\mathbb{R}), \end{array}$$

where

$$\mathcal{B} := a^{-1}\colon \mathcal{E}_0(X) \longrightarrow \mathcal{E}_0(\mathbb{R}).$$

3.3. Bijectivity of the dual Abel transform

In this subsection we show the following result.

Proposition 3.5. *The maps* $a\colon \mathcal{E}_0(\mathbb{R})\to\mathcal{E}_0(X)$ *and* $\mathcal{A}=a'$ *are topological isomorphisms.*

Proof. The proposition is essentially a consequence of the bijectivity of $\mathcal{A}\colon \mathcal{D}_0(X)\to\mathcal{D}_0(\mathbb{R})$ (see [2]). For injectivity of a, assume that $aw=0$ for some $w\in\mathcal{E}_0(\mathbb{R})$. Then

$$\langle u, aw\rangle = \langle \mathcal{A}u, w\rangle = 0$$

for all $u\in\mathcal{D}_0(X)$. Since $\mathcal{A}$ is surjective it follows that $w=0$. Surjectivity of a follows from the explicit calculation of $a^{-1}\colon \mathcal{E}_0(X)\to\mathcal{E}_0(\mathbb{R})$. Introducing the bijective map $\Phi\colon C^\infty([1,\infty))\to\mathcal{E}_0(\mathbb{R})$,

$$(\Phi f)(r) = f(\cosh r),$$

one first observes that

$$\Phi^{-1}\frac{d}{d(\cosh r)}\Phi = \frac{d}{dt} \quad \text{and} \quad \Phi^{-1}\frac{d}{d\big(\cosh \frac{1}{2}r\big)}\Phi = 2\sqrt{2}(t+1)^{1/2}\frac{d}{dt}.$$

Using the explicit formulas for $\mathcal{A}^{-1}\colon \mathcal{D}_0(\mathbb{R})\to\mathcal{D}_0(X)$ in [2], lengthy (but straightforward) computations yields in the case $p=2k$ and $q=2l$,

$$(\Phi^{-1}a^{-1}\Phi u)(t) = C_{p,q}(t+1)^{1/2}(t-1)^{1/2} \times\left(\frac{d}{dt}(t+1)^{1/2}\right)^k\left(\frac{d}{dt}\right)^l(t+1)^{l-1/2}(t-1)^{l+k-1/2}u(t),$$

and in the case $p=2k$ and $q=2l-1$,

$$(\Phi^{-1}a^{-1}\Phi u)(t) = C_{p,q}(t+1)^{1/2}(t-1)^{1/2} \times\left(\frac{d}{dt}(t+1)^{1/2}\right)^k\left(\frac{d}{dt}\right)^l(t+1)^{l-1/2}(t-1)^{l+k-1/2}(R_{1/2}^{(k+l-1,l-1)}u)(t),$$

with suitable constants $C_{p,q}$, and where $R_{1/2}^{(\alpha,\beta)}$ is defined in [16]. Hence $a\colon \mathcal{E}_0(\mathbb{R})\to \mathcal{E}_0(X)$ is a bijective linear continuous map. By the open mapping theorem (see, e.g., [20, Theorem 17.1]), a is a topological isomorphism. From the corollary of Proposition 19.5 in [20], we conclude that $\mathcal{A}\colon\mathcal{E}_0'(X)\to\mathcal{E}_0'(\mathbb{R})$ is also a topological isomorphism. $\square$

3.4. The Paley–Wiener theorem for the spherical Fourier transform on distributions

Let $\mathbf{E}_0'$ denote the Fourier transform of the space $\mathcal{E}_0'(\mathbb{R})$. The classical Paley–Wiener theorem for distributions (see, e.g., [10]) states that $\mathbf{E}_0'$ consists of all even entire functions $f\colon \mathbb{C}\to\mathbb{C}$ of exponential type which are polynomially bounded on $\mathbb{R}$, i.e., there are constants $C, R\geq 0$ and $m\geq 0$ such that

$$|f(\lambda)|\leq C(1+|\lambda|)^m e^{|\mathrm{Im}\,\lambda|R}.$$

We topologize $\mathbf{E}_0'$ by choosing the subsets $U_a\subset\mathbf{E}_0'$ as a fundamental system of neighborhoods of 0, where

$$U_a := \{f\in\mathbf{E}_0' \mid |f(\lambda)|\leq a(\lambda)\}$$

and a is any continuous positive function of the form $a(\lambda)=a_1(\mathrm{Re}\,\lambda)a_2(\mathrm{Im}\,\lambda)$, where a_1 dominates all polynomials and a_2 dominates all linear exponentials. Then the Fourier transform is a topological isomorphism $\mathcal{E}_0'(\mathbb{R})\to\mathbf{E}_0'$, by [11, Theorem 5.19].

As a direct consequence of this fact, Proposition 3.5 and formulas (9) and (10), we obtain the following result (see also [12, Theorem 4]).

Theorem 3.6. *The spherical Fourier transform*

$$\mathcal{F}T(\lambda)=\langle T,\phi_\lambda\rangle$$

defines a topological isomorphism

$$\mathcal{F}\colon \mathcal{E}_0'(X)\longrightarrow \mathbf{E}_0'.$$

Furthermore, for distributions $T, S\in\mathcal{E}_0'(X)$, we have

$$\mathcal{F}(T*S)=\mathcal{F}T\cdot\mathcal{F}S.$$

4. Spherical spectral synthesis in Damek–Ricci spaces

In this section we prove spherical spectral synthesis in $\mathcal{E}_0(X)$. We begin with two applications of Proposition 3.4.

Lemma 4.1. *Let $T\in\mathcal{E}_0'(X)$. Then*

$$\langle T,\varphi_\lambda\rangle=0 \quad\Longleftrightarrow\quad T*\varphi_\lambda=0.$$

Proof. The implication $T*\varphi_\lambda=0\Rightarrow\langle T,\varphi_\lambda\rangle=0$ is obvious. Now, assume that $\langle T,\varphi_\lambda\rangle=0$. Using (8) and Proposition 3.4 we obtain that

$$\langle \mathcal{A}T,\psi_\lambda\rangle=\langle \mathcal{A}T,\mathcal{B}\varphi_\lambda\rangle=\langle T,\varphi_\lambda\rangle=0.$$

Using, again, Proposition 3.4, we also obtain that

$$\mathcal{B}(T*\varphi_\lambda)=(\mathcal{A}T)*(\mathcal{B}\varphi_\lambda)=(\mathcal{A}T)*\psi_\lambda.$$

Moreover,

$$\begin{aligned}(\mathcal{A}T*\psi_\lambda)(t)&=\langle \mathcal{A}T,s\mapsto\psi_\lambda(t-s)\rangle\\&=\tfrac{1}{2}(e^{i\lambda t}\langle\mathcal{A}T,\phi_\lambda\rangle+e^{-i\lambda t}\langle\mathcal{A}T,\phi_\lambda\rangle)=\psi_\lambda(t)\langle\mathcal{A}T,\psi_\lambda\rangle=0.\end{aligned}$$

Since $\mathcal{B}$ is an isomorphism, we conclude that $T*\varphi_\lambda=0$. □

Recall that a *variety* $V\subset\mathcal{E}_0(X)$ is a proper closed subspace satisfying the inclusion $\mathcal{E}_0'(X)*V\subset V$.

Lemma 4.2. *Let $V\subset\mathcal{E}_0(X)$ be a variety. If $\varphi_{\lambda,k}\in V\setminus\{0\}$ then also $\varphi_{\lambda,l}\in V$ for all $0\le l\le k$.*

Proof. Let $W:=\mathcal{B}(V)$. By Proposition 3.4, $W\subset\mathcal{E}_0(\mathbb{R})$ is also a variety. From (8) we have $\psi_{\lambda,k}=\mathcal{B}(\varphi_{\lambda,k})$. So it remains to prove that

$$\psi_{\lambda,k}\in W\setminus\{0\}\implies\psi_{\lambda,l}\in W\quad\text{for all }0\le l\le k.$$

In the case $\lambda\neq0$, we restrict our considerations to $k=1$ (the case $k\ge2$ is proved similarly.) Note that $f\in W$ implies that $f_s+f_{-s}\in W$, where $f_s(t)=f(s+t)$. Therefore, we have

$$(\psi_{\lambda,1})_s+(\psi_{\lambda,1})_{-s}-2\cos(s\lambda)\psi_{\lambda,1}=-2s\sin(\lambda s)\psi_\lambda(x).$$

Consequently, we have $\psi_\lambda\in W$. If $\lambda=0$, then $\psi_{0,k}=0$ if k is odd, and $\psi_{0,k}$ are monomials if k is even. Here,

$$(\psi_{0,2})_s+(\psi_{0,2})_{-s}-2\psi_{0,2}=-2s^2\psi_0.$$

(The case $k\ge4$ is treated similarly.) □

Next, we prove an equivalent formulation of *spherical spectral synthesis* in Damek–Ricci spaces (see Theorem 1.4).

Theorem 4.3. *Let $f\in\mathcal{E}_0(X)$ be a non-zero mean-periodic function, i.e.,*

$$\{0\}\neq V_0^f:=\overline{\{T*f\,|\,T\in\mathcal{E}_0'(X)\}}\neq\mathcal{E}_0(X).$$

Then f is the limit of finite linear combinations of functions in

$$\operatorname{spec}_0 f:=\{\varphi_{\lambda,k}\in V_0^f\setminus\{0\}\,|\,k\in\mathbb{N}_0 \text{ and } \lambda\in\mathbb{C}\}.$$

Proof. Proposition 3.4 implies that

$$\mathcal{B}(V_0^f)=\mathcal{B}(\overline{\{T*f\,|\,T\in\mathcal{E}_0'(X)\}})=\overline{\{S*\mathcal{B}f\,|\,S\in\mathcal{E}_0'(\mathbb{R})\}}=V_0^{\mathcal{B}(f)},$$

and, by (8),

$$\mathcal{B}(\operatorname{spec}_0 V_0^f)=\operatorname{spec}_0 V_0^{\mathcal{B}(f)}.$$

Now, the theorem follows immediately from Theorem 2.4. $\square$

Remark 4.4. Theorem 1.4 is a direct consequence of the above theorem since every function in a variety $V\subset\mathcal{E}_0(X)$ is mean periodic.

Corollary 4.5. (Spherical spectral analysis) *If $f\in\mathcal{E}_0(X)$ is mean periodic and $\operatorname{spec}_0 f$ is empty, then $f=0$.*

The following corollary will be used in the next section.

Corollary 4.6. *Let $\mathcal{P}$ be a non-empty set of distributions in $\mathcal{E}_0'(X)$. Then the following two statements are equivalent*:

(a) *There exists a non-zero function $f\in\mathcal{E}_0(X)$ such that $T*f=0$ for all $T\in\mathcal{P}$;*

(b) *There exists $\lambda\in\mathbb{C}$ such that*

$$\mathcal{F}T(\lambda)=0 \quad \text{for all } T\in\mathcal{P}.$$

Proof. We first prove (b) $\Rightarrow$ (a): If there exists $\lambda\in\mathbb{C}$ with $\mathcal{F}T(\lambda)=\langle T,\varphi_\lambda\rangle=0$ for all $T\in\mathcal{P}$, Lemma 4.1 yields that $T*\varphi_\lambda=0$ for all $T\in\mathcal{P}$. Thus, (a) is satisfied with $f=\varphi_\lambda$.

In the proof of (a) $\Rightarrow$ (b) we assume that $\mathcal{P}\neq\{0\}$. (The case $\mathcal{P}=\{0\}$ is trivial.) Since $\mathcal{P}$ contains at least one non-trivial distribution, we have $V_0^f\neq\mathcal{E}_0(X)$. From Corollary 4.5 we conclude that $\operatorname{spec}_0 f\neq\varnothing$. Then there exists a non-zero $\varphi_{\lambda,k}\in V_0^f$. Using Lemma 4.2 we conclude that $\varphi_\lambda\in V_0^f$, which implies that $T*\varphi_\lambda=0$ for all $T\in\mathcal{P}$. Consequently, we have

$$\mathcal{F}T(\lambda)=\langle T,\varphi_\lambda\rangle=(T*\breve{\varphi}_\lambda)(e)=0$$

for all $T\in\mathcal{P}$. $\square$

5. Applications: two-radius theorems in Damek–Ricci spaces

The following lemma is needed for the proofs of the two-radius theorems.

Lemma 5.1. *Let $T\in\mathcal{E}_0'(X)$ and $f\in\mathcal{E}(X)$. If $T*\check{f}=0$ then $T*(\pi f)=0$.*

Proof. Let $\rho_\varepsilon\in\mathcal{D}_0(X)$ be a Dirac sequence. Then $T*\rho_\varepsilon=:g_\varepsilon\to T$, as $\varepsilon\to 0$. Using (5) and $f*g=(\check{g}*\check{f})^\vee$, we conclude that

$$\begin{aligned} T*(\pi f) &= \lim_{\varepsilon\to 0} g_\varepsilon*(\pi f) = \lim_{\varepsilon\to 0} (\pi f)*g_\varepsilon = \lim_{\varepsilon\to 0} \pi(f*g_\varepsilon) \\ &= \lim_{\varepsilon\to 0} \pi((g_\varepsilon*\check{f})^\vee) = \pi((T*\check{f})^\vee) = 0. \quad \square \end{aligned}$$

Proof of Theorems 1.1 and 1.2. It suffices to prove the theorems for smooth functions only. This is because the averaging operators are continuous with respect to uniform convergence on compact sets and $\mathcal{E}(X)\subset C(X)$ is dense.

The proof proceeds by contradiction: Let $r_1,r_2>0$ avoid the set described in the theorem and assume that

$$\mathcal{V}=\{f\in\mathcal{E}(X)\mid \langle T_{r_1},f_x\rangle=0 \text{ and } \langle T_{r_2},f_x\rangle=0 \text{ for all } x\in X\}\neq\{0\}, \tag{12}$$

where the corresponding families of distributions are in each case

$$\langle T_r,f\rangle = \int_{B_r} f\,dy, \tag{13}$$

$$\langle T_r,f\rangle = \int_{S_r} f\,dy. \tag{14}$$

Obviously, $\mathcal{V}$ is invariant under left-translations (isometries) in X (i.e., $f\in\mathcal{V}\Rightarrow f_x\in\mathcal{V}$ for all $x\in X$). Therefore we can find a function $f\in\mathcal{V}$ with $f(e)\neq 0$. Then $\pi f\neq 0$ and, using $\langle T,f_x\rangle=(T*\check{f})(x^{-1})$, Lemma 5.1 shows that $T_{r_1}*\pi f=T_{r_2}*\pi f=0$.

Now, for $\mathcal{P}=\{T_{r_1},T_{r_2}\}$, Corollary 4.6 implies that there exists a $\lambda\in\mathbb{C}$ with

$$\mathcal{F}T_{r_1}(\lambda)=\mathcal{F}T_{r_2}(\lambda)=0.$$

By the following lemma, we obtain a contradiction to the choice of the radii r_1 and r_2 at the beginning of the proof. $\square$

Lemma 5.2. *Let $X=X^{(p,q)}$ be a Damek–Ricci space of dimension $n+1=p+q+1$.*

(a) *Let T_r be defined as in (13). Then*

$$\mathcal{F}T_r(\lambda)=\frac{2^n\pi^{n/2}}{\Gamma\left(1+\frac{1}{2}n\right)}\sinh^n\left(\tfrac{1}{2}r\right)\cosh^{q-1}\left(\tfrac{1}{2}r\right)\varphi_\lambda^{(p,q+2)}(r).$$

(b) *Let T_r be defined as in (14). Then*

$$\mathcal{F}T_r(\lambda) = \langle T_r, \varphi_\lambda \rangle = \operatorname{vol}(S_r)\, \varphi_\lambda^{(p,q)}(r). \tag{15}$$

Proof. (b) is obvious. For the proof of (a), note that

$$\varphi_\lambda^{(p,q)}(r) = F\big(\rho - i\lambda, \rho + i\lambda, \tfrac{1}{2}n, -\sinh^2 \tfrac{1}{2}r\big)$$

with $\rho = \frac{1}{4}p + \frac{1}{2}q$. Choosing

$$z = -\sinh^2 \tfrac{1}{2}r, \quad a = \tfrac{1}{4}p + \tfrac{1}{2}q - i\lambda, \quad b = \tfrac{1}{4}p + \tfrac{1}{2}q + i\lambda \quad \text{and} \quad c = \tfrac{1}{2}n,$$

and using (see [1, formula 15.2.9])

$$\frac{d}{dz}(z^c(1-z)^{a+b+1-c} F(a+1, b+1, c+1, z)) = cz^{c-1}(1-z)^{a+b-c} F(a, b, c, z),$$

a straightforward calculation yields (a). □

Proof of Theorem 1.3. Let $r_1, r_2 > 0$ avoid the set described in the theorem, T_r be defined by

$$\langle T_r, f \rangle = \frac{1}{\operatorname{vol}(S_r)} \left(\int_{S_r} f\, dy \right) - f(e),$$

and $\mathcal{V}$ be as in (12). Observe that $\ker \Delta \subset \mathcal{V}$ and that $\ker \Delta \cap \mathcal{E}_0(X)$ is spanned by the constant function $\varphi_{i\rho} = \varphi_{-i\rho} = 1$. Assume that there is a function $f \in \mathcal{V}$ with $\Delta f \neq 0$. Since $\mathcal{V}$ and Δ are invariant under left-translations, we can assume that $\Delta f(e) \neq 0$. Let $g = \pi f$. Since Δ and π commute, we have $\Delta g(e) \neq 0$ and Lemma 5.1 implies that $g \in \mathcal{V} \cap \mathcal{E}_0(X)$. Note that g is a non-zero mean-periodic function (since all functions h in V_0^g satisfy $T_{r_j} * h = 0$ and thus $h(r_j) = h(0)$ for $j = 1, 2$).

Next, we show that $\operatorname{spec}_0 g = \{1\}$. Let $\varphi_{\lambda,k} \in V_0^g \setminus \{0\}$. We will show that $\lambda = \pm i\rho$ and $k = 0$. By Lemma 4.2, we also have $\varphi_\lambda \in V_0^g$ and, therefore, $\varphi_\lambda(r_1) = \varphi_\lambda(r_2) = \varphi_\lambda(0) = 1$. This implies that $\lambda = \pm i\rho$. If $k \geq 1$, then we also must have $\varphi_{i\rho,1} \in V_0^g$ and thus $\varphi_{i\rho,1}(r_j) = \varphi_{i\rho,1}(0)$. We have $\varphi_{i\rho,1} = a\psi_{i\rho,1}$ and

$$\psi_{i\rho,1}(r) = \frac{d}{d\lambda}\bigg|_{\lambda = i\rho} \cos \lambda r = -r \sin i\rho r = -ir \sinh \rho r$$

is $-i$ times a positive function for $r > 0$. Since the dual Abel transform is multiplication by a positive real function followed by an averaging operator, it preserves positivity. Thus $\varphi_{i\rho,1}(r)$ does not vanish for $r > 0$. But $\varphi_{i\rho,1}(e) = (a\psi_{i\rho,1})(0) = 0$ and, consequently, we must have $k = 0$.

By Theorem 4.3, g is a constant function, contradicting that $\Delta g \neq 0$. □

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