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Rigidity of the harmonic map heat flow from the sphere to compact Kähler manifolds

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Abstract. A Łojasiewicz-type estimate is a powerful tool in studying the rigidity properties of the harmonic map heat flow. Topping proved such an estimate using the Riesz potential method, and established various uniformity properties of the harmonic map heat flow from $\mathbb{S}^2$ to $\mathbb{S}^2$ (*J. Differential Geom.* **45** (1997), 593–610). In this note, using an inequality due to Sobolev, we will derive the same estimate for maps from $\mathbb{S}^2$ to a compact Kähler manifold N with nonnegative holomorphic bisectional curvature, and use it to establish the uniformity properties of the harmonic map heat flow from $\mathbb{S}^2$ to N, which generalizes Topping's result.

1. Introduction

In the harmonic map heat flow, i.e. the negative gradient flow of the energy functional, the map ϕ between two Riemannian manifolds evolves according to

$$\frac{\partial}{\partial t}\phi=\mathcal{T}(\phi),\quad \phi(\,\cdot\,,0)=\phi_0, \tag{1.1}$$

where $\mathcal{T}(\phi)$ is the tension field of ϕ (cf. [2], [10]).

In [11] Topping established various uniformity properties of the harmonic map heat flow between 2-spheres. It is interesting to see if such kind of properties hold for other manifolds. In this note we show that his result can be generalized for flow from $\mathbb{S}^2$ to compact Kähler manifolds with nonnegative holomorphic bisectional curvature. By Mok [5], manifolds with positive holomorphic bisectional curvature are biholomorphic to the complex projective space or an irreducible Hermitian symmetric space of rank$\geq$2, provided they are simply connected and their second Betti number is one. $\mathbb{C}P^n$, in particular $\mathbb{S}^2$, and the torus are basic examples of manifolds with nonnegative holomorphic bisectional curvature.

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Recall that the key step in [11] is to get a Łojasiewicz-type estimate ensuring the exponential decay of the $\bar{\partial}$-energy. To derive the estimate he identified $\mathbb{S}^2$ with the complex plane $\mathbb{C}$ via the stereographic projection, so that the Riesz potential comes into play. This trick does not work generally, but we find that the same estimate still holds when the targets are Kähler manifolds with nonnegative holomorphic bisectional curvature.

An interesting point of this note is that we use an inequality due to Sobolev to derive the Łojasiewicz-type estimate (see Section 3 below), which implies the uniformity properties of the harmonic map heat flow according to Topping [11] (see Section 4 below).

So far we do not know if such an estimate still holds for maps between other Kähler manifolds. The following question may be instructive: does any harmonic map between Kähler manifolds with sufficiently small $\bar{\partial}$-energy have to be holomorphic? If one believes that the energies of harmonic maps between analytic manifolds are isolated, this must be true.

Topping derived another kind of uniformity properties of the harmonic map heat flow in [12]. How to find its analogy between other manifolds is another challenge.

As in [4], how to find holomorphic spheres in Kähler manifolds via the harmonic map heat flow remains an interesting problem. Note that the key step in [9] to prove the Frankel conjecture is to find a holomorphic sphere in a suitable homotopy class. Unfortunately our convergence result need to preassume the existence of a holomorphic sphere (see Corollary 4.4 below). A successful approach without preassuming the existence of a holomorphic sphere may lead to another proof of the Frankel conjecture.

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2. Notation and preliminaries

In this section, we clarify some notation used in this note. Let (Σ, g) be a closed Riemannian surface, and (N, h, J) be a compact Kähler manifold, where h is the Kähler metric, and J is the natural complex structure. Denote the Gaussian curvature of Σ by K_Σ, and the Riemannian curvature tensor of N by R.

Let $\phi\colon \Sigma \to N$ be a smooth map. We write $d\phi$ for the real differential of ϕ and

$$\partial\phi\colon T^{1,0}\Sigma \longrightarrow T^{1,0}N \quad \text{and} \quad \bar{\partial}\phi\colon T^{1,0}\Sigma \longrightarrow T^{0,1}N$$

for the corresponding components of the complexification of $d\phi$.

Take holomorphic coordinate systems $z=x+iy$ near some point $p\in\Sigma$, and $w^1, w^2, ..., w^n$ near $\phi(p)\in N$. In these coordinates, g and h are represented by

$$g=2\lambda(z)\,dz\,d\bar z \quad \text{and} \quad h=2h_{\alpha\bar\beta}\,dw^\alpha\,d\overline{w}^\beta,$$

where $\lambda(z)$ is a real-valued function, $h_{\alpha\bar\beta}=h_{\bar\beta\alpha}$, and the conjugate of $h_{\alpha\bar\beta}$, namely $\overline{h_{\alpha\bar\beta}}=h_{\bar\alpha\beta}$. Here and in the sequel the summation convention is used. Locally one can write $\partial\phi$ and $\bar\partial\phi$ as

$$\partial\phi=w_z^\alpha\,dz\otimes\frac{\partial}{\partial w^\alpha} \quad \text{and} \quad \bar\partial\phi=\overline{w}_z^\alpha\,dz\otimes\frac{\partial}{\partial\overline{w}^\alpha}, \tag{2.1}$$

where $w_z^\alpha=\partial w^\alpha/\partial z$ and $\overline{w}_z^\alpha=\partial\overline{w}^\alpha/\partial z$, and similarly $\overline{w}_{\bar z}^\alpha=\partial\overline{w}^\alpha/\partial\bar z$ and $w_{\bar z}^\alpha=\partial w^\alpha/\partial\bar z$. A map $\phi\colon\Sigma\to N$ is called holomorphic (anti-holomorphic) if $\bar\partial\phi=0$ ($\partial\phi=0$).

The Hermitian inner products on $T^{1,0}\Sigma\times T^{0,1}\Sigma$ and $T^{1,0}N\times T^{0,1}N$ reads

$$\left\langle\frac{\partial}{\partial z},\frac{\partial}{\partial\bar z}\right\rangle=\lambda(z) \quad \text{and} \quad \left\langle\frac{\partial}{\partial w^\alpha},\frac{\partial}{\partial\overline{w}^\beta}\right\rangle=h_{\alpha\bar\beta},$$

respectively. We set

$$e_\partial(\phi)=|\partial\phi|^2=\lambda^{-1}(z)w_z^\alpha\overline{w}_{\bar z}^\beta h_{\alpha\bar\beta},$$
$$e_{\bar\partial}(\phi)=|\bar\partial\phi|^2=\lambda^{-1}(z)\overline{w}_z^\alpha w_{\bar z}^\beta h_{\bar\alpha\beta}.$$

Then the energy density of ϕ is given by

$$e(\phi)=\tfrac12|d\phi|^2=e_\partial(\phi)+e_{\bar\partial}(\phi),$$

and whence the energy of ϕ,

$$E(\phi)=\int_\Sigma e(\phi)\,dv_g=E_\partial(\phi)+E_{\bar\partial}(\phi),$$

where

$$E_\partial(\phi)=\int_\Sigma e_\partial(\phi)\,dv_g \quad \text{and} \quad E_{\bar\partial}(\phi)=\int_\Sigma e_{\bar\partial}(\phi)\,dv_g.$$

Let

$$\tau^\alpha(z)=\frac{2}{\lambda(z)}\big(w_{z\bar z}^\alpha+\Gamma_{\gamma\delta}^\alpha w_z^\delta w_{\bar z}^\gamma\big), \quad \alpha=1,2,...,n,$$

where $\Gamma_{\beta\gamma}^\alpha$ are the Christoffel symbols for the Levi-Civita connection on N. The associated geometric object is $\tau^\alpha\frac{\partial}{\partial w^\alpha}$, and the tension field of ϕ is represented by

$$\mathcal{T}(\phi)=\tau^\alpha\frac{\partial}{\partial w^\alpha}+\bar\tau^\alpha\frac{\partial}{\partial\overline{w}^\alpha}.$$

For some $T>0$, let $\phi\colon \Sigma\times(0,T]\to N$ be a smooth solution of (1.1), which is equivalent in holomorphic coordinates to

$$\frac{\partial w^\alpha}{\partial t}=\frac{2}{\lambda(z)}\big(w^\alpha_{z\bar z}+\Gamma^\alpha_{\gamma\delta}w^\delta_z w^\gamma_{\bar z}\big),\quad \alpha=1,2,...,n. \tag{2.2}$$

Let

$$R^\alpha_{\beta\gamma\bar\delta}=-\frac{\partial}{\partial\overline{w}^\delta}\Gamma^\alpha_{\gamma\beta}\quad\text{and}\quad R_{\alpha\bar\beta\gamma\bar\delta}=h_{\alpha\bar\varepsilon}R^{\bar\varepsilon}_{\bar\beta\gamma\bar\delta}.$$

Set

$$q_1(\phi)=\lambda^{-2}R_{\alpha\bar\beta\gamma\bar\delta}\big(w^\gamma_z\overline{w}^\delta_{\bar z}-w^\gamma_{\bar z}\overline{w}^\delta_z\big)\overline{w}^\beta_{\bar z}w^\alpha_z,$$
$$q_2(\phi)=\lambda^{-2}R_{\alpha\bar\beta\gamma\bar\delta}\big(w^\gamma_z\overline{w}^\delta_{\bar z}-w^\gamma_{\bar z}\overline{w}^\delta_z\big)w^\alpha_{\bar z}\overline{w}^\beta_z.$$

Then we have the following result (see the elliptic case in [8, p. 50]).

Lemma 2.1. *Let ϕ be a solution of (2.2). Then the partial energies $e_\partial(\phi)$ and $e_{\bar\partial}(\phi)$ evolve according to*

$$\left(\frac{\partial}{\partial t}-\Delta\right)e_\partial(\phi)=-2|\nabla\partial\phi|^2-2q_1(\phi)-2K_\Sigma e_\partial(\phi), \tag{2.3}$$

$$\left(\frac{\partial}{\partial t}-\Delta\right)e_{\bar\partial}(\phi)=-2|\nabla\bar\partial\phi|^2-2q_2(\phi)-2K_\Sigma e_{\bar\partial}(\phi). \tag{2.4}$$

Proof. For simplicity, we choose holomorphic normal coordinates near $p\in\Sigma$ and $\phi(p,t)\in N$ such that, at the point p,

$$\lambda=1,\quad \lambda_z=0\quad\text{and}\quad \lambda_{z\bar z}=-K_\Sigma,$$

and at $\phi(p,t)$,

$$h_{\alpha\bar\beta}=\delta_{\alpha\beta},\qquad\qquad h_{\alpha\bar\beta,\gamma}=\Gamma^\beta_{\alpha\gamma}=0,$$
$$h_{\alpha\bar\beta,\gamma\bar\delta}=\Gamma^\beta_{\alpha\gamma,\bar\delta}=-R^\beta_{\alpha\gamma\bar\delta}=R_{\alpha\bar\beta\gamma\bar\delta},\quad h_{\alpha\bar\beta,\bar\gamma\delta}=\overline{\Gamma^\alpha_{\beta\gamma,\bar\delta}}=R_{\alpha\bar\beta\delta\bar\gamma}.$$

We obtain at p,

$$\begin{aligned}\Delta e_\partial(\phi)&=\frac{2}{\lambda}\left(\frac{1}{\lambda}h_{\alpha\bar\beta}w^\alpha_z\overline{w}^\beta_{\bar z}\right)_{z\bar z}\\&=2h_{\alpha\bar\beta,z\bar z}w^\alpha_z\overline{w}^\beta_{\bar z}+2h_{\alpha\bar\beta}w^\alpha_{zz\bar z}\overline{w}^\alpha_{\bar z}+2w^\alpha_{zz}\overline{w}^\alpha_{\bar z\bar z}\\&\quad+2w^\alpha_{z\bar z}\overline{w}^\alpha_{\bar zz}+2w^\alpha_z\overline{w}^\alpha_{\bar zz\bar z}+2K_\Sigma w^\alpha_z\overline{w}^\alpha_{\bar z}.\end{aligned} \tag{2.5}$$

Using the chain rule, one has

$$h_{\alpha\bar\beta,z\bar z}=h_{\alpha\bar\beta,\gamma\bar\delta}w^\gamma_z\overline{w}^\delta_{\bar z}+h_{\alpha\bar\beta,\bar\gamma\delta}\overline{w}^\gamma_z w^\delta_{\bar z}. \tag{2.6}$$

Applying the harmonic map heat equation (2.2),

$$w^\alpha_{zz\bar z}=w^\alpha_{z\bar z z}=\tfrac12 w^\alpha_{tz}-\left(\Gamma^\alpha_{\gamma\delta}w^\delta_z w^\gamma_{\bar z}\right)_z=\tfrac12 w^\alpha_{tz}-R_{\delta\bar\alpha\gamma\bar\mu}\overline{w}^\mu_z w^\delta_z w^\gamma_{\bar z}, \tag{2.7}$$

and whence

$$\overline{w}^\alpha_{\bar z z\bar z}=\overline{w^\alpha_{z\bar z z}}=\tfrac12\overline{w}^\alpha_{t\bar z}-R_{\alpha\bar\delta\mu\bar\gamma}w^\mu_{\bar z}\overline{w}^\delta_{\bar z}\overline{w}^\gamma_z. \tag{2.8}$$

The covariant differential on $(T^{1,0}\Sigma)^*\otimes\phi^*(TN)$ is denoted by ∇. Then we have at p,

$$|\nabla\partial\phi|^2=w^\alpha_{zz}\overline{w}^\alpha_{\bar z\bar z}+w^\alpha_{z\bar z}\overline{w}^\alpha_{\bar z z}. \tag{2.9}$$

Obviously we have at p,

$$\frac{\partial}{\partial t}e_\partial(\phi)=w^\alpha_{tz}\overline{w}^\alpha_{\bar z}+\overline{w}^\alpha_{t\bar z}w^\alpha_z. \tag{2.10}$$

Combining (2.5)–(2.10), we obtain (2.3) by noting that $R_{\alpha\bar\beta\gamma\bar\delta}=R_{\alpha\bar\delta\gamma\bar\beta}=R_{\gamma\bar\beta\alpha\bar\delta}$.
Similarly we derive (2.4). □

Given two J-invariant planes σ and σ' in T_xN, recall that the holomorphic bisectional curvature $H(\sigma,\sigma')$ is defined by

$$H(\sigma,\sigma')=R(X,JX,Y,JY),$$

where X is a unit vector in σ and Y a unit vector in σ'. The following lemma is well known (see for example [8, p. 142]).

Lemma 2.2. *If the holomorphic bisectional curvature of (N,h) is nonnegative, i.e., $H(\sigma,\sigma')\ge 0$ for all J-invariant planes σ and σ', then in local holomorphic coordinates $w^1,w^2,...,w^n$,*

$$R_{\alpha\bar\beta\gamma\bar\delta}\xi^\alpha\bar\xi^\beta\eta^\gamma\bar\eta^\delta\le 0$$

for all $(\xi^\alpha),(\eta^\beta)\in\mathbb{C}^n$.

3. The key estimate

We would like to show how the following inequality of Sobolev can be used to derive the Łojasiewicz-type estimate for Kähler manifolds with nonnegative holomorphic bisectional curvature.

Lemma 3.1. (Cf. [6, p. 385].) *Let f be any smooth function with compact support in $\mathbb{R}^2$. Then*

$$\int_{\mathbb{R}^2} f^4\,dx \le \frac{1}{2}\int_{\mathbb{R}^2} f^2\,dx \int_{\mathbb{R}^2} |\nabla f|^2\,dx,$$

where dx is the volume element of $\mathbb{R}^2$.

Using a covering argument we immediately have the following result.

Lemma 3.2. *Let (Σ, g) be a compact Riemannian surface. Then there is a constant C depending only on (Σ, g) such that for any smooth function f on Σ there holds*

$$\int_\Sigma f^4\,dv_g \le C\int_\Sigma f^2\,dv_g \int_\Sigma \left(f^2+|\nabla f|^2\right)dv_g, \tag{3.1}$$

where dv_g is the standard volume element on (Σ, g).

The following is the *key* estimate (the Łojasiewicz-type estimate).

Lemma 3.3. *Suppose N is a Kähler manifold of complex dimension n with nonnegative bisectional curvature. Then there exist $\varepsilon_0>0$ and $C>0$ such that*

(i) *if $\phi\colon \mathbb{S}^2\to N$ satisfies $E_\partial(\phi)<\varepsilon_0$, we have the estimate*

$$E_\partial(\phi)\le C\|\mathcal{T}(\phi)\|^2_{L^2(\mathbb{S}^2)}; \tag{3.2}$$

(ii) *if $\phi\colon \mathbb{S}^2\to N$ satisfies $E_{\bar\partial}(\phi)<\varepsilon_0$, we have the estimate*

$$E_{\bar\partial}(\phi)\le C\|\mathcal{T}(\phi)\|^2_{L^2(\mathbb{S}^2)}. \tag{3.3}$$

Proof. To begin with, we observe that

$$\frac{1}{4}\int_{\mathbb{S}^2}|\mathcal{T}(\phi)|^2\,dv_g = \int_{\mathbb{S}^2}|\nabla\partial\phi|^2\,dv_g + E_\partial(\phi) + \int_{\mathbb{S}^2} q_1(\phi)\,dv_g, \tag{3.4}$$

where we have used the notation in Section 2. Locally

$$q_1(\phi)=\lambda^{-2}R_{\alpha\bar\beta\gamma\bar\delta}\left(w^\gamma_z\overline{w}^\delta_{\bar z}-w^\gamma_{\bar z}\overline{w}^\delta_z\right)\overline{w}^\beta_{\bar z}w^\alpha_z.$$

To derive (3.4) one may integrate by parts, but we prefer the following approach. Let us view ϕ as the initial data of the harmonic map heat flow $\phi(\cdot,t)$ (see also [4]). Then $e_\partial(\phi)$ evolves according to (2.3) (see Lemma 2.1 above). Integrating (2.3), and noting that $E_\partial(v)-E_{\bar\partial}(v)$ is homotopic invariant,

$$\frac{d}{dt}E_\partial(\phi)=\frac{d}{dt}E_{\bar\partial}(\phi)=\frac{1}{2}\frac{d}{dt}E(\phi)=-\frac{1}{2}\int_{\mathbb{S}^2}|\mathcal{T}(\phi)|^2\,dv_g, \tag{3.5}$$

we immediately get (3.4).

Since N has nonnegative holomorphic bisectional curvature, we have by Lemma 2.2,

$$q_1(\phi)\ge -Ce_\partial^2(\phi)$$

for some positive constant C depending only on $(\mathbb{S}^2,g)$, and thus by (3.4) there holds

$$\begin{aligned}\frac{1}{4}\int_{\mathbb{S}^2}|\mathcal{T}(\phi)|^2\,dv_g &\ge \int_{\mathbb{S}^2}|\nabla\partial\phi|^2\,dv_g+E_\partial(\phi)-C\int_{\mathbb{S}^2}e_\partial^2(\phi)\,dv_g\\ &\ge \int_{\mathbb{S}^2}\big|\nabla|\partial\phi|\big|^2\,dv_g+E_\partial(\phi)-C\int_{\mathbb{S}^2}e_\partial^2(\phi)\,dv_g.\end{aligned} \tag{3.6}$$

Here we have used the fact that

$$\int_{\mathbb{S}^2}|\nabla\partial\phi|^2\,dv_g\ge\int_{\mathbb{S}^2}\big|\nabla|\partial\phi|\big|^2\,dv_g, \tag{3.7}$$

where $\nabla|\partial\phi|$ is the weak gradient of the function $|\partial\phi|$. In fact, setting

$$\psi(x)=\begin{cases}\nabla|\partial\phi|(x), & \partial\phi(x)\neq 0,\\ 0, & \partial\phi(x)=0,\end{cases}$$

where ∇ is the covariant derivative on $\mathbb{S}^2$, one can easily check that $\psi(x)$ is the weak gradient of the function $|\partial\phi|$ (cf. [3, p. 150, Theorem 7.4]). If $|\partial\phi|(x)\neq 0$, then using holomorphic normal coordinates near x and $\phi(x)$ as in the beginning of the proof of Lemma 2.1, we have $\big|\nabla|\partial\phi|(x)\big|^2\le|\nabla\partial\phi(x)|^2$. Hence (3.7) holds.

Notice that Lemma 3.2 also holds for all functions belonging to the Sobolev space $H^{1,2}(\mathbb{S}^2)$, applying it to $f=|\partial\phi|$, we have

$$\int_{\mathbb{S}^2}e_\partial^2(\phi)\,dv_g\le CE_\partial(\phi)\left(E_\partial(\phi)+\int_{\mathbb{S}^2}\big|\nabla|\partial\phi|\big|^2\,dv_g\right). \tag{3.8}$$

This together with (3.6) gives the inequality (3.2) provided that $E_\partial(\phi)$ is sufficiently small, i.e. (i) holds.

In the same way we can derive (ii). □

4. Applications of the key estimate

In this section, we give several applications of Lemma 3.3. Let (N,h,J) be a compact Kähler manifold of complex dimension n with nonnegative holomorphic bisectional curvature. One way to study the analytic aspects of the harmonic maps from $\mathbb{S}^2$ to N is to embed N into $\mathbb{R}^K$ for a sufficiently large integer K. This embedding is denoted by $N\hookrightarrow\mathbb{R}^K$. Let $v(\,\cdot\,,t)\colon \mathbb{S}^2\times[0,\infty)\to N\hookrightarrow\mathbb{R}^K$ be the harmonic map heat flow evolving according to (1.1), which is equivalent to

$$\frac{\partial v}{\partial t}=\Delta_{\mathbb{S}^2}v+A(v)(\nabla v,\nabla v), \tag{4.1}$$

where $\Delta_{\mathbb{S}^2}$ is the Laplace–Beltrami operator on $\mathbb{S}^2$, and $A(v)(X,Y)\in(T_vN)^\perp$ is the second fundamental form of N. We will denote the right-hand side of (4.1) also by $\mathcal{T}$, without confusion.

According to Struwe [10], (4.1) has a global weak solution and is smooth except for finitely many points in space-time. The following theorem is about the bubbling phenomenon in the harmonic map flow, and now well known (cf. [1], [7] and [13]).

Theorem 4.1. *Let v be a solution of (4.1). Then there exist finitely many points $\{\hat{x}^k\}_{k=0}^m$, and finitely many nonconstant harmonic maps $\{\widehat{\omega}_k\}_{k=0}^s$ from $\mathbb{S}^2$ to $N\hookrightarrow\mathbb{R}^K$ together with*

(i) *sequences $\{t_i\}_{i=1}^\infty$ with $t_i\to\infty$;*

(ii) *$v(\,\cdot\,,t_i)\rightharpoonup\widehat{\omega}_0$, weakly in $W^{1,2}(\mathbb{S}^2,\mathbb{R}^K)$ and strongly in $W^{1,2}_{\mathrm{loc}}(\mathbb{S}^2\setminus\{\hat{x}^1,\dots,\hat{x}^m\},\mathbb{R}^K)$ as $i\to\infty$;*

(iii)

$$\lim_{t\to\infty}E(v(\,\cdot\,,t))=\sum_{k=0}^{s}E(\omega_k).$$

The map $\widehat{\omega}_0\colon \mathbb{S}^2\to N$ is called *body* map, and the maps $\widehat{\omega}^i\colon \mathbb{S}^2\to N$ are called *bubble* maps.

Once the key estimate (Lemma 3.3) is established, as pointed out by Topping, one can get the uniformity properties of the harmonic map heat flow [11, Theorems 2 and 3].

Theorem 4.2. *Suppose we have a solution v of the harmonic map heat flow from $\mathbb{S}^2\times[0,\infty)\to N\hookrightarrow\mathbb{R}^K$, where N is a compact Kähler manifold with nonnegative holomorphic bisectional curvature. Suppose moreover that at infinite time, the bubble maps and the body map defined in Theorem* 4.1 *are all holomorphic or anti-holomorphic. Then*

(i) *$v(\,\cdot\,,t)\rightharpoonup\widehat{\omega}_0$, as $t\to\infty$, weakly in $W^{1,2}(\mathbb{S}^2,\mathbb{R}^K)$, and hence strongly in $L^p(\mathbb{S}^2,\mathbb{R}^K)$ for any $p\in[1,\infty)$;*

(ii) $v(\cdot,t)\to\widehat{\omega}_0$ *uniformly as* $t\to\infty$ *in* $C^k_{\mathrm{loc}}(\mathbb{S}^2\setminus\{\hat{x}^1,...,\hat{x}^m\})$ *for any* $k\in\mathbb{N}$;
(iii) *for any* $r>0$ *sufficiently small and* $k\in\{1,...,m\}$, *the quantity*

$$E_{(\hat{x}^k,r)}(v(\cdot,t))=\int_{B_r(\hat{x}^k)} e(v(\cdot,t))\,dv_g$$

converges to a limit $F_{k,r}$ *uniformly as* $t\to\infty$, *where* $B_r(\hat{x}^k)$ *is the geodesic ball centered at* $\hat{x}^k\in\mathbb{S}^2$ *with radius* r.

Theorem 4.3. *Let* N *be a compact Kähler manifold with nonnegative bisectional curvature. Suppose we have two solutions of the heat equation* (4.1), *which we write as maps* v *and* w *from* $\mathbb{S}^2\times[0,\infty)$ *to* $N\hookrightarrow\mathbb{R}^K$, *with initial maps* v_0 *and* w_0 *from* $\mathbb{S}^2$ *to* $N\hookrightarrow\mathbb{R}^K$. *Suppose moreover that for the flow* v *there are no bubble maps at finite time and that the bubble maps and the body map at infinite time are all holomorphic or all anti-holomorphic.*

Then with x^k *being the blow-up points of the flow* v, *we have that for all* $\varepsilon>0$, $\Omega\Subset\mathbb{S}^2\setminus\{\hat{x}^1,...,\hat{x}^m\}$ *and* $r>0$ *sufficiently small, there exists* $\delta>0$ *independent of* w *such that if*

$$\|v_0-w_0\|_{W^{1,2}(\mathbb{S}^2)}<\delta,$$

then

(i) $\|v(\cdot,t)-w(\cdot,t)\|_{L^2(\mathbb{S}^2)}<\varepsilon$ *for all* $t>0$;
(ii) $\|v(\cdot,t)-w(\cdot,t)\|_{W^{1,2}(\Omega)}<\varepsilon$ *for all* $t>0$.

The hypothesis that all bubbles are holomorphic (anti-holomorphic) is to guarantee that

$$E_{\bar\partial}(v(\cdot,t_i))\to 0\ (E_{\partial}(v(\cdot,t_i))\to 0),\quad \text{as } i\to\infty,$$

which can be easily derived from Ding–Tian's description of bubbles in Theorem 4.1 (cf. [1]) instead of Theorem 1 in [11]. Now Topping's argument in [11] is easily seen to be valid if we use Lemma 3.3 instead of Lemma 1 in [11]. Since no new idea comes out, we omit the details of the proofs of Theorems 4.2 and 4.3 here but refer the reader to Section 2 in [11].

Finally we state a special case of Theorem 4.3.

Corollary 4.4. *Let* $N\hookrightarrow\mathbb{R}^K$ *be a compact Kähler manifold with nonnegative holomorphic bisectional curvature. Suppose* u *is a holomorphic map from* $\mathbb{S}^2$ *to* N. *For any* $\varepsilon>0$ *there exists* $\delta>0$ *such that if* v_0 *is another map from* $\mathbb{S}^2$ *to* N *with*

$$\|v_0-u\|_{W^{1,2}(\mathbb{S}^2)}<\delta, \tag{4.2}$$

then $v(\,\cdot\,,t)$, *the solution of the harmonic map heat flow* (4.1) *with initial map* v_0, *exists globally and converges to a holomorphic map* v_∞. *Moreover, we have*

$$\|v(\,\cdot\,,t)-u\|_{W^{1,2}(\mathbb{S}^2)}<\varepsilon \tag{4.3}$$

for all $t>0$.

Again we refer the reader to [11] for the proof of Corollary 4.4.

References

1. Ding, W. and Tian, G., Energy identity for approximate harmonic maps from surfaces, *Comm. Anal. Geom.* **3** (1995), 543–554.
2. Eells, J. and Sampson, J., Harmonic maps of Riemann manifolds, *Amer. J. Math.* **86** (1964), 109–160.
3. Gilbarg, D. and Trudinger, N., *Elliptic Partial Differential Equations of Second Order*, Springer, Berlin, 2001.
4. Liu, Q., Rigidity of the harmonic map heat flow, *Preprint*, 2007.
5. Mok, N., The uniformization theorem for compact Kähler manifolds of nonnegative holomorphic bisectional curvature, *J. Differential Geom.* **27** (1988), 179–214.
6. Pachpatte, B., *Mathematical Inequalities*, North-Holland Mathematical Library **67**, Elsevier, Amsterdam, 2005.
7. Qing, J. and Tian, G., Bubbling of the heat flows for harmonic maps from surfaces, *Comm. Pure Appl. Math.* **50** (1997), 295–310.
8. Schoen, R. and Yau, S. T., *Lectures on Harmonic Maps*, Conference Proceedings and Lecture Notes in Geometry and Topology **II**, International Press, Cambridge, 1997.
9. Siu, Y. T. and Yau, S. T., Compact Kähler manifolds of positive bisectional curvature, *Invent. Math.* **59** (1980), 189–204.
10. Struwe, M., On the evolution of harmonic mappings of Riemannian surfaces, *Comment. Math. Helv.* **60** (1985), 558–581.
11. Topping, P., Rigidity in the harmonic map heat flow, *J. Differential Geom.* **45** (1997), 593–610.
12. Topping, P., Repulsion and quantization in almost-harmonic maps, and asymptotics of the harmonic map flow, *Ann. of Math.* **159** (2004), 465–534.
13. Wang, C., Bubble phenomenon of certain Palais–Smale sequences from surfaces to general targets, *Houston J. Math.* **22** (1996), 559–590.

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