

Decomposition of D -modules over a hyperplane arrangement in the plane

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Abstract. Let $\alpha_1, \alpha_2, \dots, \alpha_m$ be linear forms defined on \mathbb{C}^n and $\mathcal{X} = \mathbb{C}^n \setminus \bigcup_{i=1}^m V(\alpha_i)$, where $V(\alpha_i) = \{p \in \mathbb{C}^n : \alpha_i(p) = 0\}$. The coordinate ring $O_{\mathcal{X}}$ of \mathcal{X} is a holonomic A_n -module, where A_n is the n th Weyl algebra and since holonomic A_n -modules have finite length, $O_{\mathcal{X}}$ has finite length. We consider a “twisted” variant of this A_n -module which is also holonomic. Define M_{α}^{β} to be the free rank-1 $\mathbb{C}[x]_{\alpha}$ -module on the generator α^{β} (thought of as a multivalued function), where $\alpha^{\beta} = \alpha_1^{\beta_1}, \dots, \alpha_m^{\beta_m}$ and the multi-index $\beta = (\beta_1, \dots, \beta_m) \in \mathbb{C}^m$. Our main result is the computation of the number of decomposition factors of M_{α}^{β} and their description when $n=2$.

1. Introduction

1.1. Definition of the module M_{α}^{β}

Let $\alpha_i : \mathbb{C}^n \rightarrow \mathbb{C}$, $i=1, 2, \dots, m$, be linear forms and $H_i = \{P \in \mathbb{C}^n : \alpha_i(P) = 0\}$ be the corresponding hyperplanes. If we let $X = \mathbb{C}^n \setminus \bigcup_{i=1}^m H_i$ be the complement of the central hyperplane arrangement that H_i , $i=1, \dots, m$, define, then the coordinate ring of X is the localization $\mathbb{C}[x_1, \dots, x_n]_{\alpha}$, where $\alpha = \prod_{i=1}^m \alpha_i$. This is a module over the Weyl algebra A_n . Consider now, for varying values of the complex parameters $\beta = (\beta_1, \dots, \beta_m) \in \mathbb{C}^m$, the multivalued function

$$\alpha^{\beta} = \alpha_1^{\beta_1} \dots \alpha_m^{\beta_m}.$$

(We will throughout this paper use the above multi-index notation. We will also write $\mathbb{C}[x] = \mathbb{C}[x_1, \dots, x_n]$.) The A_n -module generated by α^{β} as a $\mathbb{C}[x]_{\alpha}$ -module can be abstractly described in the following obvious way as a twisted version of $\mathbb{C}[x]_{\alpha}$.

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Definition 1.1. The module M_α^β , where $\beta=(\beta_1, \dots, \beta_m) \in \mathbb{C}^m$ is (as a $\mathbb{C}[x]$ -module) the free rank-1 $\mathbb{C}[x]_\alpha$ -module on the generator α^β . It is furthermore an A_n -module if we define

$$\partial_j \alpha^\beta = \sum_{i=1}^m \frac{\beta_i \partial_j(\alpha_i)}{\alpha_i} \alpha^\beta$$

for $j=1, 2, \dots, n$ and extend this to an action of the whole A_n on M_α^β .

We may think of this as the fibres of a flat family of $A_n[\beta_1, \dots, \beta_n]$ -modules, where now β_i is a free scalar variable (i.e. commuting with the derivations). The problem which we consider in this paper, and solve in the plane case, is to find the decomposition factors $DF(M_\alpha^\beta)$ of M_α^β , and the number $c(M_\alpha^\beta)$ of them. A starting point for us was the intriguing fact—see [5] and [6]—that $c(M_\alpha^\beta)$ is easily expressible in terms of combinatorial data of the hyperplane arrangement when $\beta=0$.

1.2. Motivating example

The case where $m=n=1$, and the A_1 -module $M_\alpha^\beta = \mathbb{C}[x]_x x^\beta$ is easy to analyze.

- Proposition 1.2.** (i) *If $\beta \in \mathbb{Z}$, then $c(M_\alpha^\beta) = 2$.*
 (ii) *If $\beta \in \mathbb{C} \setminus \mathbb{Z}$, then M_α^β is a simple A_1 -module, so $c(M_\alpha^\beta) = 1$.*

Proof. By definition $M_\alpha^\beta = \mathbb{C}[x]_x x^\beta \cong \bigoplus_{i \in \mathbb{Z}} \mathbb{C} x^{\beta+i}$. For the proof of (i), note that if $\beta \in \mathbb{Z}$, then clearly $M_\alpha^\beta \cong \mathbb{C}[x]_x$. It is an easy exercise to see that

$$\mathbb{C}[x] \hookrightarrow \mathbb{C}[x]_x \twoheadrightarrow \mathbb{C}[x]_x / \mathbb{C}[x]$$

is exact and that $\mathbb{C}[x]$ and $\mathbb{C}[x]_x / \mathbb{C}[x]$ are simple $\mathbb{C}\langle x, \partial_x \rangle$ -modules. For the proof of (ii), suppose $\beta \in \mathbb{C} \setminus \mathbb{Z}$. We have first that

$$(x\partial_x - (\beta+i))x^{\beta+j} = (j-i)x^{\beta+j}.$$

If $f = \sum_{i=0}^k \alpha_i x^{\beta+i} \in M_\alpha^\beta$, where $\alpha_k \neq 0$, then

$$\prod_{i=0}^{k-1} (x\partial_x - (\beta+i))f = \alpha_k k! x^{\beta+k}.$$

So some monomial $x^{\beta+k} \in A_1 f$. Now use the formulas

$$(1) \quad \partial_x^i x^{\beta+k} = (\beta+k) \dots (\beta+k-i) x^{\beta+k-i} \neq 0$$

by assumption, and

$$(2) \quad x^i x^{\beta+k} = x^{\beta+k+i},$$

to show that all monomials $x^{\beta+i} \in A_1 f$ for all $i \in \mathbb{Z}$, and so $M_\alpha^\beta \subset A_1 f$. Since f was an arbitrary element, this means that M_α^β is simple. \square

1.3. Main result

The goal of this paper is the following theorem that gives the number of decomposition factors when $n=2$. Note that the different possibilities are distinguished by linear conditions on β .

Theorem 1.3. *Assume that $n=2$. Let $\beta = (\beta_1, \dots, \beta_m) \in \mathbb{C}^m$ and k be the number of $\beta_i \in \mathbb{Z}$.*

- (i) *If $k=m$, then $c(M_\alpha^\beta) = 2m$.*
- (ii) (a) *If $k < m$ and $\sum_{i=1}^m \beta_i \in \mathbb{Z}$, then $c(M_\alpha^\beta) = m+k-1$.*
- (b) *If $k < m$ and $\sum_{i=1}^m \beta_i \in \mathbb{C} \setminus \mathbb{Z}$, then $c(M_\alpha^\beta) = k+1$.*

As a corollary, M_α^β is simple in exactly the following two cases: When $m \leq 2$ and $k=0$, or if $m \geq 3$ and $k=0$ and $\sum_{i=1}^m \beta_i \in \mathbb{C} \setminus \mathbb{Z}$.

To enlarge the list of examples of explicit calculations with D -modules has partly motivated this note. In addition D -modules on hyperplane configurations has been of interest to several authors, e.g. [11], [8] and [12], not to mention many works on the corresponding equivalent category of sheaves. Khoroshkin and Varchenko (see [8]) study a subcategory of holonomic D -modules with regular singularities along the stratification given by the intersections of the hyperplanes, and describe it in terms of quivers. This category however does not include our modules. By the Riemann–Hilbert correspondence and the known description of D -modules in terms of quivers, the problem studied in this note corresponds both to an assertion in a certain category of perverse sheaves and to an assertion on the number of decomposition factors of a quiver (see [2] and [1]), over a certain path algebra, and it would be illuminating to see a proof of our theorem using these tools. Another technique that we do not explicitly use are Bernstein–Sato polynomials. They have been calculated for hyperplane configurations in e.g. [12]. What we indirectly construct, corresponds to a multi-dimensional version (see [10]); it would be interesting to make this relation explicit.

The organization of the paper is as follows. In Section 2 we prove that M_α^β is holonomic and give a lemma on external products that we will need and some preliminary results. In the next section we consider the easy normal crossings case

when $m \leq n$ and also the case—our starting point—when all $\beta_i \in \mathbb{Z}$, which corresponds to $M_\alpha^\beta \cong \mathbb{C}[x]_\alpha$. In the last case there is a combinatorial description of the number of decomposition factors. Then follows in Section 4 the main part of the proof of Theorem 1.3. The main idea of the argument is to study the annihilator of $\alpha^\beta \in M_\alpha^\beta$. In Section 5 we conclude the proof of Theorem 1.3, and also describe the support of the decomposition factors.

2. Preliminaries

2.1. Some easy properties of M_α^β

It is immediate that M_α^β is holonomic, since it is the direct image of a connection on X . This is also easy to see directly, by copying the proof for the localization (which corresponds to the case $\beta=0$). The latter argument also provides an estimate of the multiplicity of the module, so we sketch it here.

- Proposition 2.1.** (i) $M_\alpha^\beta \cong M_\alpha^\gamma$, if $\beta \equiv \gamma \pmod{\mathbb{Z}^m}$.
 (ii) $M_\alpha^\beta \cong \mathbb{C}[x]_\alpha$, if $\beta \in \mathbb{Z}^m$.
 (iii) M_α^β is holonomic with multiplicity less than $(m+1)^n$.

Proof. (ii) is a special case of (i). Suppose that $\beta = \gamma + \tau$, $\tau \in \mathbb{Z}^m$. Define $\theta: M_\alpha^\beta \rightarrow M_\alpha^\gamma$ by

$$\theta\left(\frac{p}{\alpha^r} \alpha^\beta\right) = \frac{p}{\alpha^r} \alpha^\tau \alpha^\gamma.$$

Clearly this is a one-to-one, onto map and it is an easy exercise to show that it is an A_n -module homomorphism. This proves (ii).

To prove holonomicity, recall that m is the degree of α and for $k \geq 0$ set

$$\Gamma_i = \left\{ \frac{q}{\alpha^k} \alpha^\beta : q \in \mathbb{C}[x] \text{ and } \deg q \leq (m+1)i \right\}.$$

It is straightforward to prove that this filtration has the following properties:

- (i) $\Gamma_i \subset \Gamma_j$ if $i \leq j$;
- (ii) $\bigcup_{i \geq 0} \Gamma_i = M_\alpha^\beta$;
- (iii) $x_i \Gamma_j \subset \Gamma_{j+1}$ and $\partial_{x_i} \Gamma_j \subset \Gamma_{j+1}$.

Hence for every fixed i the submodule $A_n \Gamma_i$ is holonomic. The dimension of Γ_j cannot exceed the dimension of the vector space of polynomials of degree $(m+1)j$ and so

$$\dim_{\mathbb{C}}(A_n \Gamma_i) \cap \Gamma_j \leq \binom{(m+1)j+n}{n} \leq \frac{(m+1)^n j^n}{n!} + c j^{n-1}$$

for large enough values of j and some constant c . By [3, Chapter 10, Lemma 3.1], $A_n \Gamma_i$ is a holonomic module of multiplicity less than or equal to $(m+1)^n$. By this fixed upper bound on the multiplicity, and the additivity of the multiplicity, also M_α^β has to be holonomic (and the filtration is a good one, by the way). \square

2.2. External product of modules

Let A and B be \mathbb{C} -algebras. Suppose that M is a left A -module and N is a left B -module. Then the \mathbb{C} -vector space $M \otimes_{\mathbb{C}} N$ is an $A \otimes_{\mathbb{C}} B$ -module denoted by $M \widehat{\otimes} N$ and called the *external product* of M and N . The action of $a \otimes b \in A \widehat{\otimes} B$ on $u \otimes v \in M \otimes_{\mathbb{C}} N$ is given by the formula $(a \otimes b)(u \otimes v) = au \otimes bv$. If $A = A_n$ and $B = A_m$, then $M \widehat{\otimes} N$ is in this way an A_{n+m} -module (for this see [3]). We will use that the external product of two simple Weyl algebra modules is simple and include a proof for convenience, shown to us by Rolf Källström. We need the following well-known result [7, Theorem 2.6].

Lemma 2.2. *If M is a simple A_n -module then $\text{End}_{A_n} M = \mathbb{C}$.*

Lemma 2.3. *Let M be a simple A_n -module and N be a simple A_m -module. Then $M \widehat{\otimes} N$ is a simple A_{m+n} -module.*

Proof. Let $0 \neq f = \sum_{i=1}^k m_i \otimes n_i \in M \widehat{\otimes} N$ be an arbitrary element. We want to show that $A_{m+n} f = M \widehat{\otimes} N$. We will make an induction on k .

Step 1. Let $f = m_1 \otimes n_1$, $m_1 \neq 0$ and $n_1 \neq 0$. Let $g = \sum_{i=1}^k r_i \otimes s_i \in M \widehat{\otimes} N$ be an arbitrary element. We know that $A_n m_1 = M$ and $A_m n_1 = N$ and hence there are $a_i \in A_n$ and $b_i \in A_m$ such that $r_i = a_i m_1$ and $s_i = b_i n_1$, for $i = 1, \dots, k$. Then

$$g = \left(\sum_{i=1}^k a_i \otimes b_i \right) (m_1 \otimes n_1),$$

and hence $A_{m+n} f = M \widehat{\otimes} N$.

Step 2. Now let $f = \sum_{i=1}^k m_i \otimes n_i$, $m_i \in M$ and $n_i \in N$, where $k \geq 2$ and n_i , $i = 1, \dots, k$, are linearly independent (over \mathbb{C}). Let $J_i := \text{Ann}(m_i)$. Assume first that there are $1 \leq i, j \leq k$ and a such that $a \in J_i \setminus J_j$. Then $h := (a \otimes 1)f = \sum_{i=1}^k m_i \otimes n_i \neq 0$, is a sum of fewer terms than f and by induction $A_{m+n} h = M \widehat{\otimes} N$. This implies that $A_{m+n} f = M \widehat{\otimes} N$. Otherwise $J_i = J_j$ for all i and j . Then we have an isomorphism $\phi_1 : A_n/J_1 \rightarrow M$ defined by $a + J_1 \mapsto am_1$ and a similar isomorphism $\phi_2 : A_n/J_2 \rightarrow M$.

By Lemma 2.2, $\eta(m) := \phi_2 \circ \phi_1^{-1}(m) = \alpha m$ for some $\alpha \in \mathbb{C}$ and all $m \in M$. This implies that $\eta(m_1) = \alpha m_1 = m_2$ and hence

$$f = m_1 \otimes (n_1 + \alpha n_2) + m_3 \otimes n_3 + \dots + m_k \otimes n_k.$$

Thus, again by induction $A_{m+n}f = M \widehat{\otimes} N$. This finishes the proof. \square

2.3. Decomposition factors

Let R be a ring and M be an R -module. If $0 = M_0 \subset M_1 \subset \dots \subset M_r = M$ is a composition series of M , then define the set of *decomposition factors* as

$$\text{DF}(M) := \{M_i/M_{i-1}\}_{i=1}^r,$$

and let $c(M)$ be the number of decomposition factors. We will use the following lemma on the decomposition factors of R -modules.

Lemma 2.4. *Let M be an R -module, and let N be a submodule of M . Then*

- (i) $\text{DF}(M) = \text{DF}(N) \cup \text{DF}(M/N)$,
- (ii) $c(M) = c(N) + c(M/N)$.

Corollary 2.5. *Let $0 = M_0 \subset M_1 \subset \dots \subset M_k = M$ be a composition series of an A_n -module M and $0 = N_0 \subset N_1 \subset \dots \subset N_l = N$ be a composition series of an A_m -module N . Then*

$$\text{DF}(M \widehat{\otimes} N) = \{M_i/M_{i-1} \widehat{\otimes} N_j/N_{j-1}\}_{i=1, j=1}^{k, l}$$

and hence $c(M \widehat{\otimes} N) = c(M)c(N)$.

Proof. It suffices to note that $M_i/M_{i-1} \widehat{\otimes} N_j/N_{j-1}$ is simple by Lemma 2.3 and that taking exterior tensor product is a flat functor. \square

3. Examples

3.1. Normal crossings

Consider first the case when $m \leq n$ and recall that we have a general assumption that $\alpha_1, \dots, \alpha_m$ are linearly independent. After a base change the linear forms may be taken to be $\alpha_1 = x_1, \dots, \alpha_m = x_m$. Recall that, in the introduction, we considered the case $m = n = 1$. Now using that, we are going to treat the general case.

Proposition 3.1. *Let $M_\alpha^\beta = \mathbb{C}[x]_\alpha \alpha^\beta$, $\alpha = x_1 \dots x_m$ and $m \leq n$. Assume that we have $\beta_1, \dots, \beta_k \in \mathbb{Z}$ and $\beta_{k+1}, \dots, \beta_m \in \mathbb{C} \setminus \mathbb{Z}$. Then $c(M_\alpha^\beta) = 2^k$. In particular M_α^β is simple if and only if $\beta_1, \dots, \beta_m \in \mathbb{C} \setminus \mathbb{Z}$.*

Proof. The multiplication map induces an isomorphism:

$$\mathbb{C}[x]_{x_1 \dots x_m} x_1^{\beta_1} \dots x_m^{\beta_m} \cong \left(\bigotimes_{i=1}^m \mathbb{C}[x_i]_{x_i} x_i^{\beta_i} \right) \widehat{\otimes} \left(\bigotimes_{i=m+1}^n \mathbb{C}[x_i] \right).$$

Now $c(\mathbb{C}[x_i]_{x_i} x_i^{\beta_i})$ is 2 or 1 depending on if β_i is an integer or not. The result then follows by Corollary 2.5. \square

3.2. The case of $\mathbb{C}[x]_\alpha$

By Proposition 2.1 this corresponds to $\beta \in \mathbb{Z}^m$. In this case there is a known combinatorial description of $c(M_\alpha^\beta)$ in terms of certain linearly independent subsets of $\Theta := \{\alpha_1, \dots, \alpha_m\}$. We will give a short description (for details see [6]).

Recall first that to each closed embedding $i: Y \rightarrow X$ between smooth varieties, such that Y has codimension d , there is, by Kashiwara’s theorem, associated a simple \mathcal{D}_Y -module $L_{Y/X} = i_* \mathcal{O}_Y = H_Y^d(\mathcal{O}_X)$. When $X = \mathbb{C}^n$ and $0 = Y = V(x_1, x_2, \dots, x_n)$ is a point, $L_{0/X}$ may be described as

$$L_{0/X} = \frac{\mathbb{C}[x_1, \dots, x_n]_{x_1 \dots x_n}}{\sum_{i=1}^n \mathbb{C}[x_1, \dots, x_n]_{x_1 \dots \hat{x}_i \dots x_n}} \cong \bigoplus_{\beta \in \mathbb{Z}_-^n} \mathbb{C} x^\beta,$$

where \hat{x}_i signifies that the corresponding variable is not present in the expression, and \mathbb{Z}_- is the set of strictly negative integers. More generally, when $Y = V(x_1, x_2, \dots, x_m)$, $L_{Y/X}$ may be described as the exterior tensor product of an A_m -module and an A_{n-m} -module:

$$L_{Y/X} \cong L_{0/\mathbb{C}^m} \widehat{\otimes} \mathbb{C}[x_{m+1}, \dots, x_n] \cong \bigoplus_{\substack{\beta_1 \leq -1 \\ \vdots \\ \beta_m \leq -1}} \mathbb{C} x^\beta.$$

Note that this corresponds to choosing a vector space splitting $\mathbb{C}^n = Y \times Z$, and describing $L_{Y/X} \cong L_{0/Z} \widehat{\otimes} L_{Y/Y}$, where $\mathbb{C}[x_{m+1}, \dots, x_n] = \mathcal{O}_Y = L_{Y/Y}$. We will now fix the terminology associated with a hyperplane arrangement.

Definition 3.2. Let $R := \mathbb{C}[x_1, \dots, x_n]$. Furthermore let $\Theta = \{\alpha_1, \dots, \alpha_m\}$ be forms, and if $S \subset \Theta$ let $\alpha_S = \prod_{\alpha \in S} \alpha$ and $R_S := R_{\alpha_S}$. The subset $S \subset \Theta$ defines the linear

subspace H_S where all the forms in S vanish, i.e. the flat $H_S = V(\alpha \in S)$. Furthermore let $i_S: H_S \rightarrow \mathbb{C}^n$ be the inclusion and $L_S = i_{S*} \mathcal{O}_{H_S}$ be the corresponding simple A_n -module as above.

The L_S are the modules that occur in a decomposition series of $\mathbb{C}[x]_\alpha$. Note that L_{S_1} and L_{S_2} are isomorphic exactly when $H_{S_1} = H_{S_2}$. First we need the *polar filtration*.

Definition 3.3. Define

$$R_i := \sum_{|S|=i} R_S \subset \mathbb{C}[x]_\alpha$$

(and $R_{-1} := 0$). Clearly R_S is an A_n -submodule of $\mathbb{C}[x]_\alpha$, and $\mathbb{C}[x] = R_0 \subset R_1 \subset \dots$ gives an A_n -module filtration of $\mathbb{C}[x]_\alpha$.

Theorem 3.4. ([6]) (i) $\dim \text{supp } R_i/R_{i-1} = n - i$, for $i = 0, \dots, n$.

(ii) R_i/R_{i-1} is a semi-simple A_n -module.

(iii) $R_i/R_{i-1} \cong \bigoplus_S L_S$, where the sum is taken over a certain set, of all so-called no-broken (see [6]) linearly independent subsets of Θ of cardinality i . The image of L_S in R_i/R_{i-1} is generated by α_S^{-1} .

Remark 3.5. The polar filtration is the ordinary dimension filtration of a module over a Noetherian commutative ring, only that the commutative ring in this case is the subalgebra $\mathbb{C}[\partial]$ of constant differential operators. Since the module is holonomic, its characteristic varieties have dimension n over the symbol algebra. Hence this filtration by increasing dimension induces a filtration over $\mathbb{C}[x]$ where the graded pieces have decreasing dimension.

If $n = 2$ we have the following sequence of A_2 -modules

$$0 \longrightarrow R_0 (= \mathbb{C}[x, y]) \subset R_1 \subset R_2 = \mathbb{C}[x]_\alpha.$$

If the forms are $\Theta = \{\alpha_1, \dots, \alpha_m\}$, then the set of no-broken linearly independent subsets is

$$\{\emptyset, \{\alpha_1\}, \dots, \{\alpha_m\}, \{\alpha_1, \alpha_2\}, \dots, \{\alpha_1, \alpha_m\}\}.$$

So in the plane case $\mathbb{C}[x]_\alpha$ decomposes in the following way.

Corollary 3.6. (Plane case) (i) *There is one unique simple factor $\mathbb{C}[x, y]$ with support equal to \mathbb{C}^2 .*

(ii) *The simple factors with support on lines are given by the m non-isomorphic modules in the direct sum decomposition*

$$R_1/R_0 = \bigoplus_{i=1}^m L_{\alpha_i}.$$

(iii) *There are $m-1$ isomorphic simple decomposition factors with support at the origin:*

(3)
$$R_2/R_1 = \bigoplus_{i=2}^m L_{\{\alpha_1, \alpha_i\}}.$$

(iv) $c(\mathbb{C}[x]_{\alpha}) = 2m.$

For example, the image of the decomposition factor L_S corresponding to $S = \{\alpha_1, \alpha_2\}$ in R_2/R_1 is just

$$\text{Im } L_{\{\alpha_1, \alpha_2\}} = \bigoplus_{\substack{\beta_1 \leq -1 \\ \beta_2 \leq -1}} \mathbb{C} \alpha_1^{\beta_1} \alpha_2^{\beta_2}.$$

We will need to know the image of L_{α_i} . It may be described as follows. Choose for each α_i either $\alpha_i^c = x$ or $\alpha_i^c = y$, so that α_i and α_i^c are linearly independent. Then the image of L_{α_i} in R_1/R_0 is the vector space

(4)
$$\text{Im } L_{\alpha_i} = \bigoplus_{\substack{k \geq 0 \\ l \leq -1}} \mathbb{C} (\alpha_i^c)^k \alpha_i^l.$$

4. Proof of Theorem 1.3 when all $\beta_i \in \mathbb{C} \setminus \mathbb{Z}$

This section consists of the proof of a special case of Theorem 1.3, to which the general case may be reduced. By a change of coordinates, assume in the sequel that

$$M_{\alpha}^{\beta} = \mathbb{C}[x, y]_{\alpha} \alpha^{\beta},$$

where $\alpha = xy \prod_{i=3}^m (c_i x + y)$, and $c_i \neq c_j \neq 0$ for $i \neq j$. We will continue to call the forms $\alpha_1 = x, \alpha_2 = y, \dots$, and extend the definition of $c_i = \partial_x(\alpha_i)$ (so $c_0 = 1$ and $c_1 = 0$). Note that the case when $m \leq 2$ was treated in Proposition 3.1. With no β_i integers, it turns out that whether $|\beta| := \sum_{i=1}^m \beta_i$ is an integer or not, determines whether the module is simple.

Theorem 4.1. *Assume that $\beta_i \in \mathbb{C} \setminus \mathbb{Z}$, $i=1, \dots, m$.*

- (i) *If $|\beta| := \sum_{i=1}^m \beta_i \in \mathbb{Z}$, then $c(M_\alpha^\beta) = m - 1$.*
- (ii) *If $|\beta| \in \mathbb{C} \setminus \mathbb{Z}$, then $c(M_\alpha^\beta) = 1$ and M_α^β is simple.*

Studying the annihilator of α^β is the crucial ingredient of the proof.

4.1. Normal form algorithm

The Euler derivation $P = P(\beta) = x\partial_x + y\partial_y - |\beta|$ belongs to the annihilator of α^β . Another annihilator is found by considering the action of ∂_y :

$$\left(\partial_y - \sum_{i=2}^m \frac{\beta_i}{\alpha_i} \right) \alpha^\beta = 0$$

and clearing denominators. This gives

$$Q = Q(\beta) := \left(\prod_{j=2}^m \alpha_j \right) \partial_y - \sum_{i=2}^m \beta_i \prod_{\substack{j=2 \\ j \neq i}}^m \alpha_j$$

(the similarly constructed annihilator involving ∂_x is contained in the ideal generated by P and Q). We will next describe a normal form for elements modulo P and Q . For this use the *graded reverse lexicographic order*, ordering the variables by $y > x > \partial_x > \partial_y$, and letting

$$y^{i_1} x^{j_1} \partial_x^{k_1} \partial_y^{l_1} > y^{i_2} x^{j_2} \partial_x^{k_2} \partial_y^{l_2}$$

if

$$i_1 + j_1 + k_1 + l_1 > i_2 + j_2 + k_2 + l_2$$

or

$$i_1 + j_1 + k_1 + l_1 = i_2 + j_2 + k_2 + l_2$$

and the last non-zero coordinate of $(i_1 - i_2, j_1 - j_2, k_1 - k_2, l_1 - l_2)$ is negative. The *normal form* algorithm, see [11, Chapter 1] and [4, Chapter 2], with respect to $\{P, Q\}$, inputs an element F of the Weyl algebra and outputs an element R such that there exist S_1 and S_2 in the Weyl algebra with $F = S_1 P + S_2 Q + R$ and where the initial term of R is not divisible by the initial terms of P and Q . Since the initial term of P is $\underline{x\partial_x}$ and the initial term of Q is $\underline{y^{m-1}\partial_y}$, it follows that

$$A_2 = A_2 P + A_2 Q + N,$$

where

$$N = \bigoplus_{(i,j,k,l) \in M} \mathbb{C}y^i x^j \partial_x^k \partial_y^l$$

and $M \subset \mathbb{Z}_{\geq 0}^4$, is the set

$$(5) \quad M = \{(i, j, k, l) : jk = 0, \text{ and } i \leq m - 2 \text{ if } l \neq 0\}.$$

We will use this in a slightly different form. Give the variables $x, y \in A_2$ weight 1, and the derivations ∂_x and ∂_y weight -1 . Then both P and Q are homogeneous elements. Denote by $(A_2)_0 \subset A_2$ the space of homogeneous differential operators of weight 0.

Lemma 4.2.

$$(A_2)_0 \subset N_0 + A_2P + A_2Q,$$

where N_0 is the vector space

$$(6) \quad L = \bigoplus_{k \geq 1} \mathbb{C}(y\partial_x)^k \oplus \bigoplus_{\substack{k,l \geq 1 \\ k+l \leq m-2}} \mathbb{C}(y\partial_x)^k (y\partial_y)^l \oplus \bigoplus_{\substack{k \geq 0 \\ l \leq m-2}} \mathbb{C}(y\partial_y)^l (x\partial_y)^k.$$

Proof. Since P and Q are homogeneous elements

$$(A_2)_0 = (A_2P)_0 + (A_2Q)_0 + N_0,$$

where $N_0 = N \cap (A_2)_0$ differs from N , by the added condition that $i + j - k - l = 0$ in (5). Using that $(A_2)_0$ is generated by monomials in $x\partial_y, x\partial_x, y\partial_y$ and $y\partial_x$, and changing the presentation of the monomials in N_0 gives the description of the monomials in the lemma. \square

4.2. The annihilator of α^β

Now we can get some information on the annihilator of α^β .

Lemma 4.3. *Assume that $\beta_i \in \mathbb{C} \setminus \mathbb{Z}$, $i = 1, \dots, m$. Let $\text{Ann} = \text{Ann}_{A_2} \alpha^\beta$. Then*

- (i) $A_2P + A_2Q \subset \text{Ann}$;
- (ii) *the homogeneous part Ann_0 of Ann satisfies*

$$\text{Ann}_0 \subset A_2P + A_2Q.$$

Proof. Note that (i) does not need the hypothesis and is a routine verification. We will have use for the *valuation* with respect to the form $L=\alpha_i, i=1, 2, \dots, m$. If $m \in M_\alpha^\beta$, define $O_L(m)$ to be the greatest $k \in \mathbb{Z}$ such that

$$m = r(x, y)L^k \alpha^\beta,$$

where $r(x, y) \in \mathbb{C}(x, y)$ is a quotient of two polynomials that are not divisible by L (using that the ring of polynomials is a unique factorization domain). Furthermore define

$$V_{L,k} = \{m \in M_\alpha^\beta : O_L(m) \geq k\}.$$

Clearly $\dots \supset V_{L,-1} \supset V_{L,0} \supset V_{L,1} \supset \dots$. To avoid confusion, we emphasize that when we below speak of the *order* of a differential operator in A_2 and the *order filtration* of A_2 , it will be in the ordinary sense of differential operators. Note that if $S \in A_2$ has order s , then

$$(7) \quad SV_{L,k} \subset V_{L,k-s},$$

and that

$$(8) \quad \partial_y V_{x,k} \subset V_{x,k} \quad \text{and} \quad \partial_x V_{y,k} \subset V_{y,k}.$$

After these preliminaries, suppose now that $B \in \text{Ann}$ is homogeneous. Using the previous Lemma 4.2, $B = U + S_1P + S_2Q$, where $U \in L$ and $S_i \in A_2, i=1, 2$. Hence, by (1), $U \in \text{Ann}$. We will show that U must be 0.

Let r be the order of U . If $r \geq m-1$, then the leading term, with respect to the order filtration, of U will look like

$$\gamma(y\partial_x)^r + \sum_{\substack{l \leq m-2 \\ n=r-l}} \alpha_{l,n}(y\partial_y)^l(x\partial_y)^n = \gamma y^r \partial_x^r + R(x, y)x^{r-(m-2)}\partial_y^r,$$

where the degree of $R(x, y)$ is equal to $m-2$. Consider first valuation with respect to x . Evaluate modulo $V_{x,-r+1}$, using (8), gives

$$0 = U\alpha^\beta = \beta_1(\beta_1-1)\dots(\beta_1-r+1)\gamma \frac{y^r}{x^r},$$

and hence $\gamma=0$ (since $\beta_1 \notin \mathbb{Z}$). Next let $L=\alpha_i, i=2, \dots, m$. Evaluate again modulo $V_{L,-r+1}$, using (7), gives

$$0 = U\alpha^\beta = R(x, y)\beta_i(\beta_i-1)\dots(\beta_i-r+1)x^{r-(m-2)}L^{-r} + V_{L,-r+1}.$$

Since $\beta_i \notin \mathbb{Z}$, L divides $R(x, y)$. Repeating this for all $\alpha_i, i=2, \dots, m$, implies that $\alpha_2L\alpha_3\dots\alpha_m$ divides $R(x, y)$. This contradicts that $\text{deg}_y R(x, y) \leq m-2$. Hence $R(x, y)=0$, and so we are reduced to the second case, when $r \leq m-2$. Clearly

we may assume that $r > 0$, since otherwise U would be a constant annihilating α^β and so $U = 0$. Then the leading term of U will look like

$$\sum_{k=0}^r \alpha_k (y\partial_y)^k (y\partial_x)^{r-k}.$$

Consider now again the different valuations. With respect to $L = x$, and in view of (8), we have that

$$0 = U\alpha^\beta = \frac{\alpha_0\beta_1(\beta_1-1)\dots(\beta_1-r+1)y^r}{x^r} \pmod{V_{x,-r+1}}.$$

This implies that $\alpha_0 = 0$. Now if $L = \alpha_i = c_i x + y$, $i = 3, \dots, m$,

$$U\alpha^\beta = \left(\sum_{k=0}^r \alpha_k c_i^{r-k} \right) \frac{\beta_i(\beta_i-1)\dots(\beta_i-r+1)y^r}{L^r} + V_{L,-r+1}.$$

This gives $m - 2$ equations $p(c_i) = \sum_{k=1}^r \alpha_k c_i^{r-k} = 0$, where $p(x) = \sum_{k=1}^r \alpha_k x^{r-k}$, and since the degree of p is $r - 1 \leq m - 3$, we have $p(x) = 0$. This finally implies that $U = 0$, and finishes the proof of the lemma. \square

4.3. The structure of $A_2 / (A_2P + A_2Q)$

We have a surjective map

$$A_2 / (A_2P + A_2Q) \twoheadrightarrow A_2 / \text{Ann} \subset M_\alpha^\beta,$$

and we will find the decomposition factors of M_α^β by analyzing the first module.

Lemma 4.4. *Let $\tilde{\beta} = \beta + N$, where $N \in \mathbb{Z}^m$, and $\alpha^{\tilde{\beta}} = \alpha^N \alpha^\beta \in M_\alpha^\beta$. Then*

$$J = J(\tilde{\beta}) := A_2x + A_2P(\tilde{\beta}) + A_2Q(\tilde{\beta}) = A_2x + A_2(y\partial_y - (|\tilde{\beta}| + 1)) + A_2(y^{m-2}).$$

Proof. Recall that $P(\tilde{\beta}) = x\partial_x + y\partial_y - |\tilde{\beta}| = \partial_x x + y\partial_y - (|\tilde{\beta}| + 1)$. This shows that $y\partial_y - (|\tilde{\beta}| + 1) \in J$. Secondly,

$$Q(\tilde{\beta}) = \left(\prod_{j=2}^m \alpha_j \right) \partial_y - \sum_{i=2}^m \tilde{\beta}_i \prod_{\substack{j=2 \\ j \neq i}}^m \alpha_j = Gx + \left(y^{m-1} \partial_y - \sum_{i=2}^m \tilde{\beta}_i y^{m-2} \right)$$

for some $G \in A_2$. Hence

$$J = A_2x + A_2(y\partial_y - (|\tilde{\beta}| + 1)) + A_2 \left(y^{m-1} \partial_y - \sum_{i=2}^m \tilde{\beta}_i y^{m-2} \right).$$

But $y^{m-1}\partial_y - \sum_{i=2}^m \tilde{\beta}_i y^{m-2} - y^{m-2}(y\partial_y - (|\tilde{\beta}|+1)) = (\tilde{\beta}_1+1)y^{m-2} \in J$. Since $\tilde{\beta}_1+1 \neq 0$, by assumption, $y^{m-2} \in J$, and

$$J = A_2x + A_2(y\partial_y - (|\tilde{\beta}|+1)) + A_2(y^{m-2}). \quad \square$$

Lemma 4.5. *Let $A_1 = \mathbb{C}\langle y, \partial_y \rangle$. Let $J = A_1(y\partial_y - \gamma) + A_1y^k$ for $k \geq 0$. Then the following are true.*

- (i) *If $\gamma \notin \{-1, \dots, -k\}$, then $J = A_1$.*
- (ii) *If $-k \leq \gamma \leq -1$, then $J = A_1(y\partial_y - \gamma) + A_1y^{|\gamma|}$. Furthermore,*

$$A_1/J \cong \mathbb{C}[y]_y/\mathbb{C}[y]$$

and is hence simple.

Proof. (i) If $\gamma \notin \{-1, \dots, -k\}$, then $j+\gamma \neq 0$, for $j \in \{1, \dots, k\}$. Now,

$$\partial_y y^k - y^{k-1}(y\partial_y - \gamma) = (k+\gamma)y^{k-1} \in J.$$

Since $k+\gamma \neq 0$, $y^{k-1} \in J$. Iterating we find that $1 \in J$, as by assumption $k+\gamma \neq 0$, $k-1+\gamma \neq 0, \dots, 1+\gamma \neq 0$, and hence $J = A_1$.

(ii) If $-k \leq \gamma \leq -1$, the same argument still gives that $J = A_1(y\partial_y - \gamma) + A_1y^{|\gamma|}$. Let $\theta: A_1 \rightarrow \mathbb{C}[y]_y/\mathbb{C}[y]$ be the map defined by $\theta(P) = P(\bar{y}^\gamma)$. Clearly $J \subset \text{Ker } \theta$ and θ is surjective. Now,

$$A_1 = J + \bigoplus_{i \geq 0} \mathbb{C}\partial_y^i \oplus \bigoplus_{j=1}^{|\gamma|-1} \mathbb{C}y^j,$$

so $J = \text{Ker } \theta$. This concludes the proof. \square

Lemma 4.6.

$$A_2/(A_2x + A_2P(\tilde{\beta}) + A_2Q(\tilde{\beta})) \cong A_2/(A_2x + \text{Ann}(\tilde{\beta})) \cong A_2\alpha^{\tilde{\beta}}/A_2x\alpha^{\tilde{\beta}}$$

is a non-trivial simple A_2 -module if and only if $-(m-2) \leq |\tilde{\beta}|+1 \leq -1$ and zero otherwise.

Proof. The last isomorphism is obvious. By Lemma 4.4,

$$A_2/(A_2x + A_2P(\tilde{\beta}) + A_2Q(\tilde{\beta})) \cong A_2/(A_2x + A_2(y\partial_y - (|\tilde{\beta}|+1)) + A_2(y^{m-2})).$$

So this module is the external product

$$(\mathbb{C}\langle x, \partial_x \rangle/\mathbb{C}\langle x, \partial_x \rangle x) \widehat{\otimes} (\mathbb{C}\langle y, \partial_y \rangle/\mathbb{C}\langle y, \partial_y \rangle \langle y\partial_y - (|\tilde{\beta}|+1), y^{m-2} \rangle).$$

Hence the result on simplicity follows by Lemma 4.5(ii) and Lemma 2.3. It remains to prove the first isomorphism. There is a canonical surjection

$$\theta: A_2/(A_2x+A_2P(\tilde{\beta})+A_2Q(\tilde{\beta})) \longrightarrow A_2/A_2x+\text{Ann}(\tilde{\beta}).$$

If the second module is zero, then there are $B \in A_2$ and $C \in \text{Ann}(\tilde{\beta})$ such that $1 = Bx + C$ if and only if $C = 1 - Bx \in \text{Ann}(\tilde{\beta})$. Take the homogeneous weight 0 part: $1 = B_{-1}x + C_0$. Now, by Lemma 4.2, $C_0 \in A_2P(\tilde{\beta}) + A_2Q(\tilde{\beta})$ and so $1 \in A_2x + A_2P(\tilde{\beta}) + A_2Q(\tilde{\beta})$. Hence the first module is zero. Since it is always simple, this proves that the surjection θ is always an isomorphism and finishes the proof of the lemma. \square

- Lemma 4.7.** (i) *There exists $N_1 \in \mathbb{N}^m$ such that $\alpha^{\beta-N_1}$ generates M_α^β .*
 (ii) *There is $N_2 \in \mathbb{N}^m$ such that $A_2\alpha^{\beta+N_3}$ is a simple submodule if $N_3 \in N_2 + \mathbb{N}^m$.*

Proof. (i) follows directly from the fact that $M = M_\alpha^\beta$ is a holonomic module and hence Noetherian, see [3]. Let $e = (1, \dots, 1) \in \mathbb{N}^m$. By the Noetherian property the ascending sequence of submodules $A_2\alpha^{\beta-ne}$, $n = 1, 2, \dots$, stabilizes at some $n = n_1$. But, for large enough n any element in M is contained in $A_2\alpha^{\beta-ne}$. This holds in particular for a finite set of generators of M , and so M is generated by $\alpha^{\beta-n_1e}$.

For (ii), note similarly that the descending sequence of submodules $A_2\alpha^{\beta+ne}$, $n = 1, 2, \dots$, stabilizes if n is larger than some n_2 . As a further consequence of holonomicity, M contains a simple submodule M_1 . If f is a generator of M_1 , then also $\alpha^{ne}f$, $n \geq 0$, is a generator, and if we then choose n big enough we may assume that $f \in A_2\alpha^{\beta+n_2e}$. Hence $M_1 \subset A_2\alpha^{\beta+n_2e}$. Furthermore, all decomposition factors of M have support on hyperplane intersections, and M has rank 1 as a $\mathbb{C}(x, y)$ -module, so any element in $A_2\alpha^{\beta+n_2e}/M_1$ is annihilated by a large enough power of α . In particular, there is $n_4 \geq n_2$ such that $\alpha^{\beta+n_4e} \in M_1$, and it follows that $A_2\alpha^{\beta+n_2e} = A_2\alpha^{\beta+n_4e} = M_1$. \square

4.4. Proof of Theorem 4.1

We are now in a position to prove Theorem 4.1. Assume first that $|\tilde{\beta}| \in \mathbb{C} \setminus \mathbb{Z}$. By Lemma 4.6, $A_2\alpha^{\tilde{\beta}}/A_2x\alpha^{\tilde{\beta}} = 0$ and hence $A_2\alpha^{\tilde{\beta}} = A_2x\alpha^{\tilde{\beta}} = \dots = A_2x^r\alpha^{\tilde{\beta}_1}$ for any $r \in \mathbb{Z}$ and this is true exchanging x for any of the other linear forms. Then by Lemma 4.7, $A_2\alpha^{\beta+N_3} = A_2\alpha^{\beta+N_1} = M_\alpha^\beta$. Therefore M_α^β is simple.

Assume then that $|\tilde{\beta}| \in \mathbb{Z}$. Again using the notation of the preceding lemma, put $\tilde{\beta} = \beta + N_1$, and consider $A_2\alpha^{\beta+N_1}$. Since if $\alpha^{\beta+N}$ generates M_α^β also $x^{-n}\alpha^{\beta+N}$ generates M_α^β if $n \geq 0$, we may assume that $|\tilde{\beta}| \leq -(m-1)$. By Lemma 4.6, if $|\tilde{\beta}|$ is not one of $-(m-1), \dots, -2$, we have that $A_2\tilde{\beta}/A_2x\tilde{\beta} = 0$. Hence $A_2\alpha^{\tilde{\beta}} = A_2x\alpha^{\tilde{\beta}} =$

$\dots = A_2 \alpha^{\tilde{\beta}_1}$, where $\alpha^{\tilde{\beta}_1} = x^r \alpha^{\tilde{\beta}}$ such that $|\tilde{\beta}_1| = -(m-1)$. Then by Lemma 4.6

$$A_2 \alpha^{\tilde{\beta}_1} \supset A_2 x \alpha^{\tilde{\beta}_1} \supset \dots \supset A_2 x^{m-2} \alpha^{\tilde{\beta}_1},$$

is a chain of strict submodules such that each quotient is simple and has support at $(0, 0)$. The last submodule, $A_2 x^{m-2} \alpha^{\tilde{\beta}_1}$, has the property (by again applying Lemma 4.6 to each α_i) that it equals $A_2 \alpha^N x^{m-2} \alpha^{\tilde{\beta}}$ for all $N \in \mathbb{N}^m$, and hence by Lemma 4.7 is simple.

Hence M_α^β has $m-2$ decomposition factors with support at the origin, and one with support on \mathbb{C}^2 . This concludes the proof.

5. The general plane case

In this section, we are going to consider the remaining case when $1 \leq k < m$ of the β 's are in \mathbb{Z} and the rest are in $\mathbb{C} \setminus \mathbb{Z}$. Let then $\beta_1, \dots, \beta_k \in \mathbb{Z}$, so they may be taken to be 0, by Lemma 2.1. Define $\tilde{\beta} := (\beta_{k+1}, \dots, \beta_m) \in (\mathbb{C} \setminus \mathbb{Z})^{m-k}$, and set $\tilde{\alpha} = \alpha_{k+1} \dots \alpha_m$. We then want to study the module

$$M_\alpha^\beta = \mathbb{C}[x, y]_\alpha \tilde{\alpha}^{\tilde{\beta}}.$$

First, by Section 3.2 the module $\mathbb{C}[x, y]_{\alpha_1 \dots \alpha_r}$ has a filtration:

$$R_0 = \mathbb{C}[x, y] \subset R_1 \subset R_2 = \mathbb{C}[x, y]_{\alpha_1 \dots \alpha_k}.$$

(If $k=1$, then $R_1=R_2$.) By localization and multiplication by $\tilde{\alpha}^{\tilde{\beta}}$ this induces a filtration by A_2 -modules

$$(9) \quad \mathbb{C}[x, y]_{\tilde{\alpha}} \tilde{\alpha}^{\tilde{\beta}} \subset (R_1)_{\tilde{\alpha}} \tilde{\alpha}^{\tilde{\beta}} = (R_2)_{\tilde{\alpha}} \tilde{\alpha}^{\tilde{\beta}} = \mathbb{C}[x, y]_\alpha \tilde{\alpha}^{\tilde{\beta}},$$

where the equality is a consequence of the fact that R_2/R_1 has support in the complement of $\tilde{\alpha} \neq 0$, and so vanishes when localized. By (3) the quotient

$$\mathbb{C}[x, y]_\alpha \tilde{\alpha}^{\tilde{\beta}} / \mathbb{C}[x, y]_{\tilde{\alpha}} \tilde{\alpha}^{\tilde{\beta}} = \bigoplus_{i=1}^k (L_{\alpha_i})_{\tilde{\alpha}} \tilde{\alpha}^{\tilde{\beta}},$$

as A_2 -modules. We can describe the factors $K_i := (L_{\alpha_i})_{\tilde{\alpha}} \tilde{\alpha}^{\tilde{\beta}}$ more precisely. Denote by i_{H_i} the inclusion $H_i = V(\alpha_i) \rightarrow \mathbb{C}^2$. Since K_i has support on H_i , it is equal to $i_{H_i*}(\text{Ker}_{K_i} \alpha_i)$, by Kashiwara's theorem. Note that α^c is a parameter on H_i . The restriction of a form α_j , $j=k+1, \dots, m$, to H_i is a non-zero multiple of α^c , and hence intuitively the restriction of $\tilde{\alpha}^{\tilde{\beta}}$ is a multiple of $(\alpha^c)^{\beta_{k+1} + \dots + \beta_m}$.

Lemma 5.1. *Let $i_H: H=V(\alpha_i)\subset\mathbb{C}^2$, put $t=\alpha_i^c$, and let $\beta_H=\beta_{k+1}+\dots+\beta_m$. Then*

- (i) $\text{Ker}_{K_i}\alpha_i\cong\mathbb{C}[t]_t t^{\beta_H}=\mathbb{M}_t^{\beta_H}$ as $\mathbb{C}\langle t, \partial_t \rangle$ -modules.
- (ii) $K_i=i_{H*}\mathbb{M}_t^{\beta_H}$, where i_H is the inclusion $\mathbb{C}\cong H\subset\mathbb{C}^2$.

Proof. It suffices to prove (i). We may without loss of generality assume that a basis is chosen so that $\alpha_i=x$ and $t=\alpha_i^c=y$, where the decomposition $\alpha_s=b_sx+c_sy$, $s=k+1, \dots, m$, has the property that $c_s\neq 0$, $s=k+1, \dots, m$. (If some $c_s=0$, then x and α_s are linearly dependent.) Since kernels localize well, $\text{Ker}_{K_i}=(\text{Ker}_{L_i}x)_{\tilde{\alpha}}\tilde{\alpha}^{\tilde{\beta}}$. Hence, by the description of $\text{Im } L_{\alpha_i}$ in (4), vector space generators of $\text{Ker}_{K_i}x$ are given by

$$\text{Ker}_{K_i}x = \bigoplus_{j\geq 0} \mathbb{C} \frac{y^j}{x} \tilde{\alpha}^{\tilde{\beta}} + \sum_{\substack{j\in\mathbb{Z} \\ s=k+1, \dots, m}} \mathbb{C} \frac{1}{x\alpha_s^j} \tilde{\alpha}^{\tilde{\beta}}.$$

We may eliminate some generators, letting $\alpha_s=b_sx+c_sy$, and using that

$$\frac{c_s}{x\alpha_s} - \frac{c_r}{x\alpha_r} = 0,$$

(in K_i) and by an inductive argument getting

$$\frac{c_s}{x\alpha_s^j} - \frac{c_r}{x\alpha_r^j} = 0.$$

This gives

$$(10) \quad \text{Ker}_{K_i}x = \bigoplus_{j\geq 0} \mathbb{C} \frac{y^j}{x} \tilde{\alpha}^{\tilde{\beta}} \oplus \bigoplus_{j\in\mathbb{Z}} \mathbb{C} \frac{1}{x\alpha_{k+1}^j} \tilde{\alpha}^{\tilde{\beta}}.$$

Let $\theta: \mathbb{C}[x, y]\rightarrow\mathbb{C}[y]$ be the surjection corresponding to the injection $i_H: H=V(\alpha_i)\subset\mathbb{C}^2$. Since $\alpha_s=b_sx+c_sy$ we have that $\theta(\alpha_s)=c_sy$, $c_s\in\mathbb{C}$. Choose a branch of the logarithm, so that $c=\prod_{s=k}^m c_s^{\beta_s}$ is defined. Define the $\mathbb{C}[y]$ -homomorphism

$$\theta: \text{Ker}_{K_i}x \longrightarrow \mathbb{C}[y]_y y^{\beta_H} = \mathbb{M}_y^{\beta_H},$$

by

$$\theta\left(\frac{y^j}{x}\tilde{\alpha}^{\tilde{\beta}}\right) = cy^j y^{\beta_H} \quad \text{and} \quad \theta\left(\frac{1}{x\alpha_s^j}\tilde{\alpha}^{\tilde{\beta}}\right) = c(c_sy)^{-j} y^{\beta_H}.$$

The identity

$$\frac{y}{x\alpha_s^j} = \frac{1}{c_s} \frac{b_sx+c_sy}{x\alpha_s^j} = \frac{1}{c_s} \frac{1}{x\alpha_s^{j-1}},$$

in K_i , implies that $\theta(y/x\alpha_s^j)=y\theta(1/x\alpha_s^j)$. Furthermore the identity

$$\theta\left(\partial_y\left(\frac{1}{x}\tilde{\alpha}^{\tilde{\beta}}\right)\right)=\theta\left(\frac{1}{x}\sum_{s=k+1}^m\frac{\beta_s c_s}{\alpha_s}\tilde{\alpha}^{\tilde{\beta}}\right)=c\left(\sum_{s=k+1}^m\frac{\beta_s c_s}{c_s y}y_H^\beta\right)=\partial_y\left(\theta\left(\frac{1}{x}\tilde{\alpha}^{\tilde{\beta}}\right)\right),$$

shows part of the fact that θ commutes with the action of ∂_y and the rest follows similarly. Clearly θ is onto, and injective by the description (10). This finishes the proof. \square

This finally enables us to complete the proof of a more precise version of Theorem 1.3.

Theorem 5.2. *Assume that $\beta_1, \dots, \beta_k \in \mathbb{Z}$ and $\beta_{k+1}, \dots, \beta_m \in \mathbb{C} \setminus \mathbb{Z}$.*

(i) *If $\sum_{i=1}^m \beta_i \in \mathbb{C} \setminus \mathbb{Z}$, then $c(M_\alpha^\beta)=k+1$. There is one decomposition factor with support on the whole space, one with support on each H_s , $s=1, \dots, k$, and none with support at the origin.*

(ii) *If $\sum_{i=1}^m \beta_i \in \mathbb{Z}$, then $m-k \geq 2$ and $c(M_\alpha^\beta)=m+k-1$. There is one decomposition factor with support on the whole space, one with support on each H_s , $s=1, \dots, k$, and $m-k-2$ with support at the origin.*

Proof. By (9) and Lemma 2.4,

$$(M_\alpha^\beta)=c(\mathbb{C}[x, y]_{\tilde{\alpha}}\tilde{\alpha}^{\tilde{\beta}})+c(M_\alpha^\beta/\mathbb{C}[x, y]_{\tilde{\alpha}}\tilde{\alpha}^{\tilde{\beta}})=c(\mathbb{C}[x, y]_{\tilde{\alpha}}\tilde{\alpha}^{\tilde{\beta}})+\sum_{s=1}^k c(\mathbb{C}[y]_y y^{\beta_H}).$$

Clearly $\beta_H \in \mathbb{Z}$ if and only if $|\beta|=\sum_{i=1}^m \beta_i \in \mathbb{Z}$, so $c(\mathbb{C}[y]_y y^{\beta_H})$ is 2 or 1, depending on if $\sum_{i=1}^m \beta_i \in \mathbb{Z}$, or not, by the example in the introduction. Hence the contribution of the last part of the sum is that there is always exactly one decomposition factor with support on each H_i , $i=1, \dots, k$. An additional k decomposition factors with support at the origin exist if $|\beta| \in \mathbb{Z}$, while if $|\beta| \in \mathbb{C} \setminus \mathbb{Z}$ there are none.

On the other hand, by the previous section, $c(\mathbb{C}[x, y]_{\tilde{\alpha}}\tilde{\alpha}^{\tilde{\beta}})$ is $m-k-1$ or 1, depending on if $\sum_{s=k}^m \beta_s \in \mathbb{Z}$, which is equivalent to $\sum_{i=1}^m \beta_i \in \mathbb{Z}$, or not. There is always one decomposition factor with support on \mathbb{C}^2 , and $m-k-2$ with support at the origin if $\sum_{i=1}^m \beta_i \in \mathbb{Z}$. Combining these descriptions, if $|\beta| \in \mathbb{Z}$ there are $2k+(m-k-1)=m+k-1$ decomposition factors, and if $\sum_{i=1}^m \beta_i \in \mathbb{C} \setminus \mathbb{Z}$, there are $k+1$ decomposition factors. \square

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