Ark. Mat., **48** (2010), 371–393
DOI: 10.1007/s11512-009-0096-2
© 2009 by Institut Mittag-Leffler. All rights reserved

The hitting distributions of a half real line for two-dimensional random walks

Kôhei Uchiyama

Abstract. For every two-dimensional random walk on the square lattice $\mathbf{Z}^2$ having zero mean and finite variance we obtain fine asymptotic estimates of the probability that the walk hits the negative real line for the first time at a site $(s,0)$, when it is started at a site far from both $(0,s)$ and the origin.

1. Introduction and results

Let S_n be a two-dimensional random walk on the square lattice $\mathbf{Z}^2$ whose increments are independent and identically distributed and have zero mean and finite variance. We embed $\mathbf{Z}^2$ in the complex plane $\mathbf{C}$ to use complex number notation. The walk S_n is supposed irreducible, namely for every $x+iy\in\mathbf{Z}^2$ there exists n such that $P_{x+iy}[S_n=0]>0$, where P_{x+iy} stands for the probability measure of the walk starting at $x+iy$. Denote by $H_{x+iy}(s)$ the probability that the first visit (after time 0) of the real axis by the walk starting at $x+iy$ takes place at $s\in\mathbf{Z}$:

$$H_{x+iy}(s)=P_{x+iy}[\text{there is } n\geq 1 \text{ such that } S_n=s \text{ and } S_k\notin\mathbf{Z} \text{ for } 1\leq k<n].$$

Similarly, denote by $H^+_{x+iy}(s)$ and $H^-_{x+iy}(-s)$, $s=1,2,3,\dots$, the distributions of the first visiting sites (after time 0) of the positive and negative real axes, respectively (the origin is excluded from both the half lines).

The present author [12] derived certain asymptotic expressions of $H_{x+iy}(s)$ as $|x-s+iy|\to\infty$ which are valid uniformly either in $x-s$ or in y. In this paper we shall obtain similar ones for H^-_{x+iy}. The significance of these results consist largely in their being of a fundamental nature. If the walk is started at a point far from the half line, Donsker's invariance principle says that the law H^-_{x+iy} if suitably normalized is approximated by the corresponding Brownian law whose density, say $h^-_{x+iy}(s)$, is explicitly written down in a simple form (see Appendix B). However, this approximation, being in the topology of weak convergence of probability laws, contains little information about, e.g., the probability that a point near the end of the half line is the first visited site. Our result provides a precise asymptotic form of such probabilities in terms of a pair of harmonic functions on the half line associated to H_0. A crude application of it shows that $H^-_{x+iy}(s)$ is uniformly comparable to $h^-_{x+iy}(s)$, namely there exists two positive numbers c_1 and c_2 such that for all $s\in\mathbf{Z}\cap(-\infty,-1]$ and all $x+iy\in\mathbf{Z}^2\setminus((-\infty,0]\times\{0\})$,

$$c_1 h^-_{x+iy}(s)\le H^-_{x+iy}(s)\le c_2 h^-_{x+iy}(s), \tag{1}$$

provided $E[|S^{(1)}_1|^2\log|S^{(1)}_1|]<\infty$ in addition. Our approach is based on a profound theory concerning one-dimensional walk on a half line (as given in the Spitzer's book [10]); owing to it we can use the powerful method of generating functions and Fourier analysis.

Let Q be the covariance matrix for the variable S_1 under P_0 and put $\sigma=(\det Q)^{1/4}$. Let $\phi(t)$ be the characteristic function of $H_0(s)$, namely

$$\phi(t)=\sum_{s=-\infty}^{\infty} e^{its}H_0(s),\quad -\pi\le t\le\pi,$$

and bring in the functions

$$\rho(t)=1-\phi(t)\quad\text{and}\quad\text{for } t\ne0,\ \theta(t)=\arg\rho(t),$$

and the constants

$$\theta_+=\frac{1}{\pi}\int_0^\pi\frac{\theta(t)}{2\tan t/2}\,dt\quad\text{and}\quad c=\exp\left(\frac{1}{\pi}\int_0^\pi\log|\rho(t)|\,dt\right),$$

where $\arg\rho$ denotes the argument $\in(-\pi/2,\pi/2)$ of a complex number ρ with $\mathrm{Re}\,\rho>0$ and the integrals are absolutely convergent (Lemma 2.1).

Let X_n be the one-dimensional random walk with the transition probability $p^X(x,y)=H_0(y-x)$. It is natural to write P_0 also for the law of X_n if X_n starts at the origin: X_n may then be identified with the place on $\mathbf{Z}$ at which the walk $S.$ starting at the origin makes the nth return to the real axis. The walk S_n is recurrent and so is X_n. Let u_k and v_k, $k=0,1,2,...$, be two sequences determined via the equations

$$\sum_{k=0}^{\infty}u_kz^k=\frac{1}{\sqrt{c}}\exp\sum_{k=1}^{\infty}\frac{1}{k}E_0[z^{-X_k};X_k<0]; \tag{2a}$$

$$\sum_{k=0}^{\infty}v_kz^k=\frac{1}{\sqrt{c}}\exp\sum_{k=1}^{\infty}\frac{1}{k}E_0[z^{X_k};X_k>0]; \tag{2b}$$

$|z|<1$, (for well-definedness cf. [10], p. 202) and put for integers s

$$\mu(s)=\frac{\sqrt{\pi}\,e^{\theta_+}}{\sigma}\sum_{k=0}^{\infty}(v_0+\ldots+v_k)H_0(s-k)$$

and

$$\nu(s)=\frac{\sqrt{\pi}\,e^{-\theta_+}}{\sigma}\sum_{k=0}^{\infty}(u_0+\ldots+u_k)H_0(-s+k).$$

For two real numbers a and b write $a\wedge b=\min\{a,b\}$ and $a\vee b=\max\{a,b\}$. Denote by $S_n^{(1)}$ and $S_n^{(2)}$ the first and second components of S_n, respectively.

Theorem 1.1. *Let $s<0$. If $x\vee(-s)\to\infty$ with $x\geq 0$, then*

$$H_x^-(s)=\frac{\sigma^2}{2\pi}\frac{\nu(x)\mu(s)}{|x-s|}(1+o(1)). \tag{3}$$

(*Here $o(1)$ vanishes in the stated limit.*) *This formula holds true also in the case $x<0$ with $o(1)\to 0$ as $|s-x|\to\infty$ in such a manner that either $(\log|s|)\sqrt{s/x}$ or $(\log|x|)\sqrt{x/s}$ is bounded by any prescribed constant; if $E[|S_1^{(1)}|^2\log|S_1^{(1)}|]<\infty$ in addition, this constraint on the manner of $|s-x|$ tending to infinity may be removed. Moreover*

$$\sqrt{x}\,u_x\to\frac{e^{\theta_+}}{\sigma\sqrt{\pi}}\ \text{and}\ \frac{\nu(x)}{\sqrt{x}}\to\frac{2}{\sigma^2},\quad \text{as } x\to\infty, \tag{4}$$

and

$$\sqrt{-s}\,\mu(s)\to 1\ \text{and}\ \sqrt{-s}\,\nu(s)\to 1,\quad \text{as } s\to-\infty. \tag{5}$$

The proof of Theorem 1.1 (as that of Kesten [4] does for the corresponding result) depends on the following representation of the Green function for the walk X_n killed on $x\leq -1$:

$$g(x,y)=\sum_{0\leq n\leq x\wedge y}u_{x-n}v_{y-n},\quad x,y\geq 0, \tag{6}$$

(cf. Spitzer [10], Section 19, Proposition 3). In view of this formula the probability $H_x^-(s)$ is rather directly related to the functions μ and ν via the identity

$$H_x^-(s)=\sum_{y=0}^{\infty}g(x,y)H_0(s-y)$$

which is our real starting point for the proof of Theorem 1.1.

We make it explicit in the following corollary that the principal part of H_x^- has the same form (for large values of $x\wedge(-s)$) as the density of the corresponding distribution for the standard Brownian motion.

Corollary 1.2.

$$H_x^-(s)=\frac{1}{\pi}\frac{1}{x-s}\sqrt{\frac{x}{-s}}(1+o(1)),$$

where $o(1)\to 0$ *as* $x\wedge(-s)\to\infty$.

It holds that $c=\exp(-\sum_{k=1}^{\infty}P[X_k=0]/k)$ (see (21) of Section 2); in particular the sequences u_k and v_k are the same as those introduced in [10] (D18.2). The constant θ_+ has a simple probabilistic expression:

$$\theta_+=\sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{1}{2}-P[X_k>0]\right)+\frac{1}{2}\log c=\frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{k}(P_0[X_k<0]-P_0[X_k>0]). \tag{7}$$

The first infinite sum in (7) appears in [10] (Section 18, Proposition 5) but in the case when the second moment of the walk X_n is finite ((7) will be verified in Appendix C of this paper although not used for proofs of theorems). (The constants u_k and v_k also have simple probabilistic meanings; see [10], p. 203.) The restrictions of μ and ν to the nonnegative integers are positive solutions of the Wiener–Hopf integral equations (associated with the kernels $H_0(\pm x)$, $x\geq 0$) to the effect that

$$\mu(x)=\frac{\sqrt{\pi}\,e^{\theta_+}}{\sigma}(v_0+\ldots+v_x)\quad\text{and}\quad\nu(x)=\frac{\sqrt{\pi}\,e^{-\theta_+}}{\sigma}(u_0+\ldots+u_x)$$

for $x=0,1,2,\ldots$ (cf. [10], p. 212), which may be conveniently used to deduce the second relation of (4) from the first.

The error estimates in the formula (3) may be improved under the stronger moment conditions. We here state results only in the case when $E_0|S_1|^{2+\delta}<\infty$ for some $\delta>\frac{1}{2}$ (see Theorem 4.1 in Section 4 for the case $\delta\leq\frac{1}{2}$); also the initial site x is restricted to $[0,\infty)$.

Theorem 1.3. *Suppose* $E_0|S_1|^{2+\delta}<\infty$ *for some* $\delta>\frac{1}{2}$. *Then uniformly for* $x\geq 0$ *and* $s<0$,

$$H_x^-(s)=\frac{1}{\pi}\frac{1}{x-s}\sqrt{\frac{x\vee 1}{-s}}\left[1+O\left(\frac{1}{\sqrt{-s}}\right)+O\left(\frac{1}{\sqrt{x\vee 1}}\right)\right]; \tag{8}$$

moreover

$$H_x^-(s)=\frac{\sigma^2}{2\pi}\frac{\nu(x)}{(x-s)\sqrt{-s}}\left[1+O\left(\frac{1}{\sqrt{-s}}\right)+O\left(\frac{x}{-s}\right)\right],\quad \text{as } \frac{x\vee 1}{-s}\to +0, \tag{9}$$

and

$$H_x^-(s)=\frac{\sqrt{x}\mu(s)}{\pi(x-s)}+O\left(\frac{1}{x}\right),\quad \text{as } \frac{x}{-s}\to\infty. \tag{10}$$

((9) *and* (10) *are valid (uniformly) in the ranges* $-s>x\geq 0$ *and* $x\geq -s$, *respectively, but 'sharp' only in the case as indicated in the limits.*)

Due to the present method of proof the error estimates in the formulae above do not depend on δ (i.e. do not improve as δ increases) (see remark on the proof of Theorem 4.1 in Section 4).

From the result on H^-_x given above together with those on H_{x+iy} obtained in [12] one can derive asymptotic formulae for H^-_{x+iy} as we shall exhibit at the end of this section in the case when $|x-s+iy|\wedge|x+iy|\to\infty$.

Some upper bounds of H^-_{x+iy} are obtained in [7] by a method that is quite different from and more flexible than ours, while the results derived in this paper short cut some arguments made there. For simple random walk some evaluation of $H^-_{x+iy}(s)$ was given by [4] and [1]. Kesten [4] verifies among others the second relation of (4) and the upper bound $\sum_{s<-2r} H^-_x(s)=O(1/\sqrt{(-x+1)r})$ uniformly valid for $-r<x\le 0$. The time reversed version of (14) shows that $1/\sqrt{(-x+1)r}$ is the correct order in this bound. The method of [2] also gives an estimate of $H^-_x(s)$ (for simple walk), which is better than (8) and that of [4] but not sharp near the edge (cf. [13]) as in (9) or (10). Bousquet-Mélou and Schaeffer [1] compute the number of walks of length n that avoid the half line (for a class of walks admitting nonnearest neighbor transitions) by algebraic arguments and apply the result to compute the generating function $\sum_{k=1}^{\infty} H_{x+iy}(-k)z^k$ explicitly in a sense: it gives e.g. $H^-_{(-1,1)}(-1)=\frac{1}{2}$, $H^-_{(1,-1)}(-1)=2-\sqrt{2}$ (these results are due to Kenyon according to that paper) and $H^-_0(-s)=[(\sqrt{2}-1)/2\pi]^{-1/2}(-s)^{-3/2}(1+o(1))$; they also study the walk starting at the end point $(-1,0)$ and killed on the negative half line and compute the hitting probability and the mean sojourn time of a lattice point outside the half line and also prove the local limit theorem for the conditional process given that the walk has not been killed. The upper bounds of H^-_{x+iy} are used in [5] and [6] to obtain upper bounds of the growth rate of the diffusion-limited aggregation model. Fukai [3] studies the first hitting time of the negative half line for the walk similar to ours (with $\delta>0$) but started at $(-1,0)$ and shows that the tail of its distribution function is asymptotic to a constant times $t^{1/4}$. The hitting distribution of long linear segments is computed by using the results of the present paper [13].

The essential part of the proofs is done in Section 2, where certain analytic properties of the generating function of v_k (and of u_k) are obtained by means of Fourier analytic methods. In Section 3 the results obtained in Section 2 will be used to find asymptotic estimates for v_k. With Spitzer's representation of the Green function (6) together with the estimates of its constituents the proofs of the theorems are performed by rather elementary calculus as will be carried out in Section 4.

General starting points $x+iy$. We conclude this section by exhibiting estimates of $H^-_{x+iy}(s)$ (for $x+iy \notin L^+$) which are deduced from theorems given above combined with the estimates of $H_{iy}(s)$ in [12] in view of the decomposition

$$H^-_{x+iy}(s) = H_{iy}(s-x) + \sum_{\xi=0}^{\infty} H_{iy}(\xi - x) H^-_{\xi}(s). \tag{11}$$

Define $a^*(y) = 1 + \sum_{n=1}^{\infty} (E_0[S_n^{(2)} = 0] - E_0[S_n^{(2)} = -y])$: $a^*(y)$ is equal to the potential function of the one-dimensional walk $S_n^{(2)}$ if $y \neq 0$ (cf. [10]) and $a^*(0) = 1$. Let σ_{kj} be the entries of the covariance matrix Q and bring in two constants

$$\lambda = \frac{\sigma^2}{\sigma_{22}} \quad \text{and} \quad \omega = \frac{\sigma_{12}}{\sigma_{22}}. \tag{12}$$

For simplicity we here suppose that $E_0[|S_1^{(1)}|^2 \log |S_1^{(1)}|] < \infty$. It then follows that

$$H_{x+iy}(s) = \frac{\sigma^2 a^*(y)}{\pi[(s-x+\omega y)^2 + (\lambda y)^2]}(1 + o(1)), \tag{13}$$

where $o(1)$ approaches zero as $|x-s+iy| \to \infty$ (Theorem 2 of [12]). Using this together with (3) (with $x \geq 0$) the Brownian version of (11) (see (50)) yields not only the formula (1) but also that for $s<0$, as $|x-s+iy| \wedge |x+iy| \to \infty$,

$$H^-_{x+iy}(s) = \frac{\sigma_{22} a^*(y) \mu(s) \sqrt{-s}}{|y|} h^-_{x-\omega y + i\lambda y}(s), \quad y \neq 0, \tag{14}$$

where $h^-_{x+iy}(s)$ is the density of the first hitting distribution of the negative real line for the standard Brownian motion. Using an explicit form of $h^-_{x+iy}(s)$ in (14) (see Appendix B) we obtain the following result.

Theorem 1.4. *Suppose $E_0[|S_1^{(1)}|^2 \log |S_1^{(1)}|] < \infty$ in addition. Then*

$$H^-_{x+iy}(s) = \frac{\sigma^2 a^*(y) \mu(s)(r(x,y) - s)}{\pi [r(x-s,y)]^2 \sqrt{2(r(x,y) - x + \omega y)}}, \quad \text{as } |x-s+iy| \wedge |x+iy| \to \infty$$

for $s<0$ and $x+iy \notin [0,\infty) \times \{0\}$, where $r(x,y) = \sqrt{(x-\omega y)^2 + (\lambda y)^2}$.

If the limit is taken in such a way that the ratio $(r+|s|)/\sqrt{(r+1-x+\omega y)|s|}$ remains bounded, then the first term $H_{x+iy}(s)$ in the decomposition (11) contributes to the sum significantly and vice versa; and if so is taken the limit, without the extra condition $E_0[|S_1^{(1)}|^2 \log |S_1^{(1)}|] < \infty$, both (1) and the formula of Theorem 1.4 can break down in view of Theorem 1.3 of [12].

2. The generating function of v_n

Recall that X_n in (2) is the one-dimensional random walk starting at 0 with the transition probability $p^X(x,x')=H_0(x'-x)$. Define $H_0^0(s)=\delta_{0,s}$ (Kronecker's symbol) and $H_0^k(s)=\sum_{j=-\infty}^{\infty} H_0(j)H_0^{k-1}(s-j)$. Then for $|z|<1$,

$$\sum_{n=0}^{\infty} v_n z^n = \frac{1}{\sqrt{c}} \exp \sum_{n=1}^{\infty} z^n \sum_{k=1}^{\infty} \frac{1}{k} H_0^k(n) \tag{15}$$

and analogously for $\sum_{n=0}^{\infty} u_n z^n$ (with $H_0^k(-n)$ in place of $H_0^k(n)$). The inner infinite sum in the right-hand side is the (nth) Fourier coefficient of $-\log(1-\phi(t))$ (see (21) below) and Fourier analysis will reveal a certain analytic nature of the power series in the exponent and hence that of the left-hand side.

Recall that $\rho(t)=1-\phi(t)$. We introduce a function $F(t)$ defined by $F(0)=0$ and

$$F(t) = -\log\left[\frac{\rho(t)/\sigma^2}{|2\sin(t/2)|}\right], \quad 0<|t|\le\pi,$$

(the logarithm is understood to be the principal branch) and denote by $F^\vee(n)$ its Fourier coefficient:

$$F^\vee(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(t)\, e^{-int}\, dt. \tag{16}$$

Then the fact mentioned just after (15) may be expressed in the form

$$\sum_{k=1}^{\infty} \frac{H_0^k(n)}{k} = F^\vee(n) + \frac{1}{2|n|}, \quad n\ne 0, \tag{17}$$

(see (22) below) and (15) and its analogue for $\{u_n\}_{n=0}^{\infty}$ accordingly become

$$\sum_{n=0}^{\infty} v_n z^n = \frac{1}{\sqrt{c}\sqrt{1-z}}\, e^{m_+(z)} \quad \text{and} \quad \sum_{n=0}^{\infty} u_n z^n = \frac{1}{\sqrt{c}\sqrt{1-z}}\, e^{m_-(z)}, \tag{18}$$

where

$$m_\pm(z) = \sum_{n=1}^{\infty} z^n F^\vee(\pm n), \quad |z|<1. \tag{19}$$

The formula (17) is derived in the following preliminary discussion.

Let $\psi(t,l)$ be the characteristic function of one step transition of the walk S_n: $\psi(t,l)=E_0[e^{itX+ilY}]$, where X and Y denote the first and second components of S_1, respectively. It is not hard to see that

$$\frac{1}{\rho(t)} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{dl}{1-\psi(t,l)}, \quad t\ne 0,$$

and that if $E_0[|S_1|^{2+\delta}]<\infty$ for some $0\le\delta<1$, then as $t\to\pm 0$,

$$\rho(t)=\sigma^2|t|+o(|t|^{1+\delta}),\quad \rho'(t)=\pm\sigma^2+o(|t|^{\delta})\quad\text{and}\quad \rho''(t)=o(|t|^{\delta-1}) \tag{20}$$

(cf. [12], Lemmas 2.1 and 4.2). We have $H_0^k(s)=(2\pi)^{-1}\int_{-\pi}^{\pi}(1-\rho(t))^k e^{-ist}\,dt$, $s\in\mathbf{Z}$. Hence by dominated convergence

$$\sum_{k=1}^{\infty}\frac{1}{k}H_0^k(s)=-\frac{1}{2\pi}\int_{-\pi}^{\pi}\log\rho(t)\,e^{-ist}\,dt. \tag{21}$$

(In fact this identity holds generally for every nondegenerate random walk.) Here (as above) the logarithm is the principal branch, so that

$$\log\rho(t)=\log|\rho(t)|+i\theta(t).$$

Notice that $|\rho(t)|$ is even and $\theta(t)=\arg\rho(t)$ is odd; also $\theta(t)=\arg(\rho(t)/|\rho(t)|)\to 0$ as $t\to 0$. We are to separate from $\log\rho(t)$ its logarithmic singularity at 0. To this end it is convenient to represent it not by $\log|t|$ but by $\log|2\sin(t/2)|=-\sum_{k=1}^{\infty}k^{-1}\cos kt$, for which

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\log\left|2\sin\frac{t}{2}\right|e^{-int}\,dt=\begin{cases}-1/2|n|, & \text{if } n=\pm1,\pm2,\ldots,\\ 0, & \text{if } n=0.\end{cases} \tag{22}$$

Thus

$$-\frac{1}{2\pi}\int_{-\pi}^{\pi}\log\rho(t)\,e^{-int}\,dt=\frac{1}{2|n|}-\frac{1}{2\pi}\int_{-\pi}^{\pi}\log\left[\frac{\rho(t)/\sigma^2}{|2\sin(t/2)|}\right]e^{-int}\,dt,\quad n\neq 0, \tag{23}$$

which combined with (21) gives (17). An application of the Riemann–Lebesgue lemma shows that the last integral converges to zero as $n\to\infty$, where the speed of convergence depends on the regularity of ρ at 0, which in turn depends on moment conditions of S_1. In the typical case of simple random walk it is $O(1/n^N)$ for every $N>0$ (see Appendix A), while it is $o(1/n)$ in general (see Lemma 2.2).

Now we proceed to analysis of the function $m_\pm(z)$ defined by the power series (19): our interest is mainly in its boundary behavior at $z=1$. Clearly F is periodic of period 2π. From (20) it follows that as $t\to 0$,

$$\begin{aligned}&F(t)=o(|t|^{\delta}),\\ &F'(t)=-\frac{\rho'}{\rho}+\frac{1}{2}\cot\frac{t}{2}=o(|t|^{\delta-1})\quad\text{and}\quad F''(t)=o(|t|^{\delta-2});\end{aligned} \tag{24}$$

also by (22)

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}F(t)\,dt=\sum_{k=1}^{\infty}\frac{1}{k}H_0^k(0)+\log\sigma^2=-\log\frac{c}{\sigma^2}. \tag{25}$$

Hence

$$m_\pm(z) = \frac{1}{2}\log\frac{c}{\sigma^2} + \frac{1}{4\pi}\int_{-\pi}^{\pi}\frac{1+ze^{\mp it}}{1-ze^{\mp it}}F(t)\,dt.$$

We shall verify below that $m_\pm(1)$ exist,

$$\lim_{N\to\infty}\sum_{s=-N}^{N}F^\vee(s)=0, \tag{26}$$

and

$$m_+(1)+m_-(1)=\log\frac{c}{\sigma^2}. \tag{27}$$

To this end put

$$a(n)=\frac{1}{\pi}\int_{-\pi}^{\pi}F(t)\cos nt\,dt = -\frac{1}{\pi}\int_{-\pi}^{\pi}\log\left|\frac{\rho(t)/\sigma^2}{2\sin(t/2)}\right|\cos nt\,dt$$

and

$$b(n)=\frac{1}{\pi}\int_{-\pi}^{\pi}F(t)\sin nt\,dt = -\frac{i}{\pi}\int_{-\pi}^{\pi}\theta(t)\sin nt\,dt$$

(so that $F^\vee(n)=\frac{1}{2}(a(n)-ib(n))$) and consider the Fourier series

$$F(t)=\frac{a(0)}{2}+\sum_{n=1}^{\infty}(a(n)\cos nt+b(n)\sin nt) \tag{28}$$

and its conjugate series

$$\overline{F}(t) := -\frac{1}{\pi}\int_0^{\pi}\frac{F(t+y)-F(t-y)}{2\tan(y/2)}\,dy = \sum_{n=1}^{\infty}(a(n)\sin nt - b(n)\cos nt). \tag{29}$$

These trigonometric series converge to the functions on the left for every $t\neq 0$ at which F is continuously differentiable. We shall see shortly that this holds true also for $t=0$.

Lemma 2.1. $\int_0^\pi |\theta(t)/t|\,dt<\infty$.

Proof. Since

$$\rho(t)\rho(-t)=|\rho(t)|^2 \quad\text{and}\quad \rho(t)-\rho(-t)=i2|\rho(t)|\sin\theta(t),$$

it suffices to prove that

$$\int_0^\pi\left|\frac{1}{\rho(t)}-\frac{1}{\rho(-t)}\right|dt<\infty.$$

We write

$$\frac{i}{\rho(t)}-\frac{i}{\rho(-t)}=-\operatorname{Im}\left(\frac{2}{\rho(t)}\right)=-\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{E_0[\sin(tX+lY)]}{|1-\psi(t,l)|^2}\,dl,$$

and, using the decomposition

$$E_0[\cos lY\sin tX]=E_0[(\cos lY-1)\sin tX]+E_0[\sin tX-tX]$$

and a similar one for $E_0[\sin lY\cos tX]$, obtain

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{|E_0[\sin(tX+lY)]|}{|1-\psi(t,l)|^2}\,dt\,dl<\infty$$

(see Lemma 3.1 of [11]). Thus the lemma is verified. □

Lemma 2.2. (i) *If* $E_0|S_1|^{2+\delta}<\infty$ *for some* $0\le\delta<1$, *then as* $n\to\infty$, $a(n)=o(n^{-\delta-1})$ *and* $b(n)=o(n^{-\delta-1})$.

(ii) *Both* $\sum_{n=0}^{\infty}a(n)$ *and* $\sum_{n=0}^{\infty}b(n)$ *are convergent and* $\sum_{n=1}^{\infty}a(n)=-a(0)/2$.

Proof. Split the integral

$$\int_{-\pi}^{\pi}F(t)\begin{pmatrix}\cos nt\\ \sin nt\end{pmatrix}dt$$

by dividing the range $|t|<\pi$ at $\pm1/n$. With the help of (24) the contribution of the interval $|t|<1/n$ is immediately disposed of; and for the other intervals we perform integration by parts twice to conclude that (i) is true. Similarly, from $F'=o(1/t)$ and $F(\pm0)=0$ we infer that

$$\lim_{n\to\infty}\int_{-\pi}^{\pi}F(t)\frac{\sin nt}{t}\,dt=0,$$

which is equivalent to the validity of (28) at 0 (see e.g., [15], II.7.1). This gives (26), hence the identity of (ii). It remains to verify the convergence of $\sum_{n=0}^{\infty}b(n)$. A sufficient condition for it is that $|F(t)-F(-t)|/|t|$ is integrable about the origin, which indeed is true owing to Lemma 2.1. □

Now we compute $m_+(1)$. Recalling the defining expression (19) of $m_+(z)$, we find (from the Fourier expansions (28) and (29)) that

$$2m_+(e^{it})=-\frac{a(0)}{2}+F(t)+i\overline{F}(t). \tag{30}$$

By (25)

$$\frac{1}{2}a(0)=-\log\frac{c}{\sigma^2}.$$

On the other hand $F(0)=0$ and, since $\log[\rho(t)/\rho(-t)]=2i\theta(t)$,

$$i\overline{F}(0) = -\frac{1}{\pi}\int_0^\pi \frac{\theta(t)}{\tan(t/2)}\,dt = -2\theta_+,$$

where the integral is absolutely convergent owing to Lemma 2.1. Hence

$$m_+(1) = \log\sqrt{c/\sigma^2} - \theta_+. \tag{31}$$

We shall need further regularity properties of $\overline{F}$.

Lemma 2.3. *The function $\overline{F}(t)$ is continuous on $[-\pi,\pi]$ and continuously differentiable except at* 0*, about which $\overline{F}'(t)=o(1/t)$.*

Proof. It suffices to show that the function

$$g(t) = \int_0^1 \frac{F(t+y)-F(t-y)}{y}\,dy$$

satisfies the property asserted for $\overline{F}$ in the lemma. The continuity of g is obvious for $0<|t|<2\pi$, in which F is continuously differentiable with

$$\int_0^\varepsilon y^{-1}(F(t+y)-F(t-y))\,dy$$

approaching zero as $\varepsilon\downarrow 0$ locally uniformly. The continuity at $t=0$ is proved by using that $F'(t)=o(1/t)$ as follows. By symmetry we have only to consider the case $t>0$. Decompose the integral $\int_0^1$ into the three parts $g_1=\int_0^{t/2}$, $g_2=\int_{t/2}^{2t}$ and $g_3=\int_{2t}^1$. By the mean-value theorem

$$g_1 = 2\int_0^{t/2} F'(t+\eta_t(y))\,dy$$

with $|\eta_t(y)|\le t/2$, so that $F'(t+\eta_t(y))=o(1/t)$; hence $g_1=o(1)$. The integrand of g_2 is $o(1/t)$ uniformly on the range of integration; hence $g_2=o(1)$. As for g_3 we decompose it as

$$g_3 = \int_{2t}^1 \frac{F(t+y)-F(y)}{y}\,dy + \int_{2t}^1 \frac{F(y)-F(-y)}{y}\,dy + \int_{2t}^1 \frac{F(-y)-F(t-y)}{y}\,dy. \tag{32}$$

Since $F(t+y)-F(y)=to(1/y)$, the first integral is $t\int_{2t}^1 o(1/y^2)\,dy=o(1)$. In a similar way we see that the third integral vanishes in the limit. By Lemma 2.1(ii) the second integral converges to $g(0)$. Thus $g(t)\to g(0)$ as required.

Employing the fact that F is twice continuously differentiable on $t>0$, the same reasoning advanced for the continuity of g at $t>0$ is adapted to verify that $g(t)$ is continuously differentiable there and its derivative is given by

$$g'(t) = \int_0^1 \frac{F'(t+y)-F'(t-y)}{y}\,dy.$$

Making decomposition parallel to (32) and using $F''(t)=o(1/t^2)$ we can readily show that $g'(t)=o(1/t)$. □

What is proved for F and $m_\pm$ is summarized in the following corollary.

Corollary 2.4. (i) *Both* $F(t)$ *and* $\overline{F}(t)$ *are continuous,* $F(0)=0$ *and* $i\overline{F}(0)=-2\theta_+$.

(ii) *The functions* $m_\pm(z)$ *are continuous on the closed disc* $|z|\le 1$ *with the boundary functions* $m_\pm(e^{it})=\frac{1}{2}(\log(c/\sigma^2)+F(\pm t)\pm i\overline{F}(\pm t))$.

Proof. We have only to examine the continuity of $m_\pm$. From the definition it is clear that $m_\pm$ is in the Hardy class H^2 and therefore represented as the Poisson integral of its boundary function determined by nontangential limits (see e.g. [9], Theorem 17.11). Thus the continuity follows from that of F and $\overline{F}$. □

3. Estimates of v_n

Let $\{c_n\}_{n=1}^\infty$ be a bounded sequence of complex numbers, put $f(z)=\sum_{n=1}^\infty c_n z^n$ and let $a_0=1$ and

$$e^{f(z)} = 1+a_1 z+a_2 z^2+\dots.$$

From $(e^f)'=f'e^f$ it follows that

$$(n+1)a_{n+1} = a_n b_0 + a_{n-1}b_1+\dots+a_0 b_n, \tag{33}$$

where we set $b_n=(n+1)c_{n+1}$ so that $f'(z)=\sum_{n=0}^\infty b_n z^n$.

Lemma 3.1. *Let* δ *be a positive constant. If* $c_n=O(1/n^{1+\delta})$, $n\ge 1$, *then* $a_n=O(1/n^{1+\delta})$.

Proof. We have $a_0=1$ and, by the assumption on f, $|b_k|\le A/(k+1)^\delta$ for some A. First consider the case $\delta=1$. Set $a'_n=n^2|a_n|$ and $M_n=\max\{a'_1,\dots,a'_n\}$. Then, taking

N so large that $A\sum_{k>N}k^{-2}+A\sup_{n>N}\sum_{k=1}^{n}[k(n+1-k)]^{-1}<\frac{1}{2}$, we obtain from the recursion relation (33) that for $n\geq N$,

$$\begin{aligned}a'_{n+1} &\leq A\sum_{k=1}^{n}\frac{(n+1)a'_k}{k^2(n+1-k)}+A\\ &= A\sum_{k=1}^{n}a'_k\left[\frac{1}{k^2}+\frac{1}{k(n+1-k)}\right]+A\\ &\leq A+AM_N\sum_{k=1}^{\infty}\frac{1}{k^2}+\frac{1}{2}M_n,\end{aligned}$$

which shows by induction that $M_n\leq 2A+2AM_N\sum_{k=1}^{\infty}k^{-2}$. Thus a'_n is bounded as desired. For $\delta\neq 1$ taking $a'_n=n^{1+\delta}|a_n|$ we can proceed in a similar way but using the inequalities

$$\sum_{k=N+1}^{n}\frac{(n+1)^\delta}{k^{1+\delta}(n+1-k)^\delta}\leq\frac{1}{(n+1)^\delta}\int_{N/(n+1)}^{1-1/(1+n)}\frac{dt}{t^{1+\delta}(1-t)^\delta}+\frac{(1+n)^\delta}{n^{1+\delta}}\leq\frac{C}{N^\delta}. \quad\square$$

The argument given above can be readily adapted to verify that if $c_n=o(1/n^{1+\delta})$, $n\geq 1$, then $a_n=o(1/n^{1+\delta})$.

Lemma 3.2. *If $c_n=O(1/n^2)$, as $n\to\infty$, and the infinite series $\sum_{n=1}^{\infty}nc_n$ is convergent, then $\{n^2a_n\}_{n=1}^{\infty}$ is bounded and the series $\sum_{n=1}^{\infty}na_n$ converges to the sum $f'(1)\,e^{f(1)}$.*

Proof. That $\{n^2a_n\}_{n=1}^{\infty}$ is bounded follows from the preceding lemma. The second half then follows from the Littlewood version of Tauberian theorems ([14], p. 360) applied to the power series $f'(z)\,e^{f(z)}=a_1+2a_2z+3a_3z^2+\dots$. $\square$

Recall the first identity of (18):

$$\sum_{k=0}^{\infty}v_kz^k=\frac{1}{\sqrt{c}\sqrt{1-z}}\,e^{m_+(z)},\qquad |z|<1;$$

and also that $F^\vee(n)$ is the Fourier coefficient of F defined in (16).

Lemma 3.3. *Let δ be a positive constant. If $F^\vee(n)=O(1/n^{1+\delta})$, then as $n\to\infty$,*

$$v_n-\frac{e^{-\theta_+}}{\sqrt{\pi\sigma^2n}}=\begin{cases}O(n^{-3/2}), & \text{if } \delta>1,\\ O(n^{-3/2}\log n), & \text{if } \delta=1,\\ O(n^{-\delta-1/2}), & \text{if } \delta<1.\end{cases}$$

Proof. Let $1/\sqrt{1-z}=\alpha_0+\alpha_1 z+\alpha_2 z^2+\ldots$, namely $\alpha_0=1$ and

$$\alpha_n = \frac{(2n-1)!!}{(2n)!!} = \frac{1}{\sqrt{\pi n}}+O(n^{-3/2}) \quad \text{for } n>0. \tag{34}$$

If $e^{m_+(z)}=1+a_1 z+a_2 z^2+\ldots$, then Lemma 3.1 together with the assumption of the lemma (see (19)) yields $a_n=O(1/n^{1+\delta})$, so that

$$\sqrt{c}v_n = \alpha_0 a_n+\ldots+\alpha_n a_0 = \sum_{k=0}^{n-1}\frac{a_k}{\sqrt{\pi}\sqrt{n-k}}+O(n^{-1/2-(1\wedge\delta)}) = \frac{1}{\sqrt{\pi n}}\sum_{0\le k<n/2} a_k+R,$$

with

$$R\le \frac{C}{2n\sqrt{n}}\sum_{0\le k<n/2} k|a_k|+O(n^{-1/2-(1\wedge\delta)}).$$

Since $\sum_{k=0}^{\infty} a_k=e^{m_+(1)}=e^{-\theta_+}\sqrt{c/\sigma^2}$, the relations above show those of the lemma. □

Lemma 3.3 provides reasonable estimates for v_n under the $(2+\delta)$-moment condition with $\delta>0$, whereas without assuming it we have the following result.

Proposition 3.4.

$$v_n = \frac{e^{-\theta_+}}{\sigma\sqrt{\pi n}}(1+o(1)).$$

Proof. The proof depends on the representation

$$v_n = \frac{1}{2\pi}\int_{-\pi}^{\pi}\hat{v}(t)\, e^{-int}\, dt, \tag{35}$$

where $\hat{v}(t)$, $t\neq 0$, is the boundary value at $z=e^{it}$ of the analytic function $\sum_{n=0}^{\infty} v_n z^n$ in $|z|<1$, and, according to (30), given by

$$\hat{v}(t) = \frac{e^{m_+(e^{it})}}{\sqrt{c}\sqrt{1-e^{it}}} = \frac{\exp\left(\frac{1}{2}F(t)+i\frac{1}{2}\overline{F}(t)\right)}{\sigma\sqrt{1-e^{it}}}.$$

Owing to Lemma 2.3 and (24) it holds that $\hat{v}(t)=O(1/\sqrt{|t|})$ and as $t\to 0$,

$$\hat{v}'(t) = \left(\frac{d}{dt}\frac{1}{\sqrt{1-e^{it}}}\right)\left(\frac{e^{-\theta_+}}{\sigma}+o(1)\right). \tag{36}$$

We are going to compare v_n with $e^{-\theta_+}\alpha_n/\sigma$, where α_n is the same as in the preceding proof so that $\widehat{\alpha}(t)=\sum_{n=0}^{\infty}\alpha_n e^{itn}=1/\sqrt{1-e^{it}}$, $0<|t|<\pi$. Now employing (36) and setting $A=e^{-\theta_+}/\sigma$ we have $\hat{v}(t)-A\widehat{\alpha}(t)=o(|t|^{-1/2})$ and $\hat{v}'(t)-A\widehat{\alpha}'(t)= o(|t|^{-3/2})$, and then

$$
\begin{aligned}
2\pi(v_n - A\alpha_n) &= \int_{-\pi}^{\pi} (\hat{v}(t) - A\widehat{\alpha}(t))\, e^{-int}\, dt \\
&= \int_{|t|<1/n} (\hat{v}(t) - A\widehat{\alpha}(t))\, e^{-int}\, dt + o\left(\frac{1}{\sqrt{n}}\right) \\
&\quad + \frac{1}{in} \int_{1/n<|t|<\pi} (\hat{v}'(t) - A\widehat{\alpha}'(t))\, e^{-int}\, dt \\
&= o(1/\sqrt{n}).
\end{aligned}
$$

Thus we conclude that $v_n = e^{-\theta_+}/\sqrt{\pi\sigma^2 n} + o(1/\sqrt{n})$ as desired. □

4. Estimation of H_x^-

The Green function $g(x,y)$ for the one-dimensional random walk X_n killed on $\{x \le -1\}$ is given by

$$
g(x,y) = \sum_{0\le n\le x\wedge y} u_{x-n} v_{y-n}, \quad x, y \ge 0,
$$

(cf. Spitzer [10], Section 19, Proposition 3). Our estimation of H_x^- will be based on the formula

$$
\begin{aligned}
H_x^-(s) &= \sum_{y=0}^{\infty} g(x,y) H_0(s-y) \\
&= \sum_{n=0}^{x} u_{x-n} \sum_{y=n}^{\infty} v_{y-n} H_0(s-y) \\
&= \sum_{n=0}^{x} u_{x-n} \sum_{y=0}^{\infty} v_y H_0(s-n-y)
\end{aligned}
$$

as well as the estimates of v_n and u_n and of H_0 in Lemma 3.3 and Proposition 3.4 and in [12], respectively.

Remark. Using the estimate in Lemma 3.3 as well as the identity $m_+(1) + m_-(1) = \log(c/\sigma^2)$,

$$
g(x,y) = \frac{2+o(1)}{\pi\sigma^2} \log \frac{\sqrt{y}+\sqrt{x}}{\sqrt{|y-x|\vee 1}},
$$

as $x \vee y \to \infty$, $x, y \ge 0$. This asymptotic form of g however is not useful for our present purpose (and is thus not applied below).

The case $E_0|S_1|^{2+\delta}<\infty$, $\delta>0$. The form of dependence on δ of the estimates changes at $\delta=\frac{1}{2}$ and in order to express them concisely we introduce the function $\ell_\delta(y)$ defined for $y\geq 0$ by

$$\ell_\delta(y)=\begin{cases}(y\vee 1)^{1/2-\delta}\log(y\vee e), & \text{if } \delta\leq\frac{1}{2},\\ 1, & \text{if } \delta>\frac{1}{2}.\end{cases} \tag{37}$$

The next theorem reduces to Theorem 1.3 in the case $\delta>\frac{1}{2}$.

Theorem 4.1. *Suppose* $E_0|S_1|^{2+\delta}<\infty$, $\delta>0$. *Then uniformly for* $x\geq 0$ *and* $s<0$,

$$H_x^-(s)=\frac{1}{\pi}\frac{1}{x-s}\sqrt{\frac{x\vee 1}{-s}}\left[1+\frac{O(\ell_\delta(-s))}{\sqrt{-s}}+\frac{O(\ell_\delta(x))}{\sqrt{x\vee 1}}\right]; \tag{38}$$

if $-s>x\geq 0$, *then*

$$H_x^-(s)=\frac{\sigma^2\nu(x)}{2\pi\,(x-s)^{3/2}}\left[1+\frac{O(\ell_\delta(-s))}{\sqrt{-s}}+O\left(\frac{x\vee 1}{x-s}\right)\right]; \tag{39}$$

if $x\geq -s$, *then*

$$H_x^-(s)=\frac{\mu(s)}{\pi\sqrt{x-s}}\left[1+o\left(\frac{1}{x^\delta}\right)+O\left(\sqrt{\frac{-s}{x}}\right)\right]. \tag{40}$$

Here (and in what follows as well) $O(\ell_\delta(\,\cdot\,))$ *can be replaced by* $o(\ell_\delta(\,\cdot\,))$ *if* $\delta\leq\frac{1}{2}$.

Proof. We may restrict the range of the parameter δ to the open unit interval $(0,1)$; in the case $\delta\geq 1$ some of the estimates given below have to be altered but without making any effect to the net result. Thus let $0<\delta<1$ and suppose $E_0|S_1|^{2+\delta}<\infty$. Under this condition it is shown in [12] that

$$H_0(s)=\frac{\sigma^2}{\pi}\frac{1}{s^2}\left[1+o\left(\frac{\log|s|}{|s|^\delta}\right)\right],\quad \text{as } s\to\infty.$$

Observing that $\int_0^\infty[y^{-1/2}(y+k)^{-2-\delta}]\log(k+y)\,dy\leq\text{const}\cdot k^{-(3/2+\delta)}\log k$ and as $k\to\infty$,

$$\int_0^\infty\frac{1}{(y+1)^{1/2+\delta}(y+k)^2}\,dy=O\left(\frac{\ell_\delta(k)}{k^2}\right) \tag{41}$$

we see that for $k>0$,

$$\sum_{y=0}^{\infty} v_y H_0(-k-y) = \frac{\sigma^2}{\pi} \sum_{y=0}^{\infty} v_y \frac{1}{(y+k)^2} + o\left(\frac{\log k}{k^{3/2+\delta}}\right)$$
$$= \frac{\sigma^2\, e^{m_+(1)}}{\pi\sqrt{\pi c}} \int_0^\infty \frac{dy}{\sqrt{y}\,(y+k)^2} + O\left(\frac{\ell_\delta(k)}{k^2}\right).$$

(Here O appearing above may be replaced by o if $\delta \le \frac{1}{2}$ but not if $\delta > \frac{1}{2}$.) Hence, by the formula

$$\frac{1}{\pi}\int_0^\infty \frac{1}{(a+y)^2\sqrt{y}}\, dy = \frac{1}{2a^{3/2}}, \tag{42}$$

$$\sum_{y=0}^{\infty} v_y H_0(-k-y) = \frac{\sigma^2\, e^{m_+(1)}}{2\sqrt{\pi c}} \frac{1}{k^{3/2}} + O\left(\frac{\ell_\delta(k)}{k^2}\right), \quad \text{as } k\to\infty. \tag{43}$$

Applying the last formula with $k=n-s$ and the following formulas:

$$u_n = \frac{e^{m_-(1)}}{\sqrt{\pi c n}} + o\left(\frac{1}{n^{1/2+\delta}}\right),$$

$$\int_0^x \frac{1}{(a-t)^{3/2}\sqrt{t}}\, dt = \frac{2}{a}\sqrt{\frac{x}{a-x}}, \qquad 0<x<a \; (=x-s),$$

$$\sum_{n=1}^{x-1} \frac{1}{(x-n)^{1/2+\delta}(n-s)^{3/2}} = \sum_{n\le x/2} + \sum_{n>x/2} \le \frac{C'\sqrt{x}}{x^\delta (x-s)\sqrt{-s}} + \frac{C\ell_\delta(x)}{(x-s)^{3/2}}, \tag{44}$$

$$\sum_{n=1}^{x-1} \frac{\ell_\delta(n-s)}{\sqrt{x-n}\,(n-s)^2} \le C\frac{\ell_\delta(-s)\sqrt{x}}{(x-s)(-s)}, \tag{45}$$

we obtain (38).

Further applying the simple estimates

$$(n-s)^{-3/2} = (x-s)^{-3/2} + O((x-n)(x-s)^{-5/2})$$

and $\sum_{n=1}^{x-1}(x-n)/\sqrt{x-n} \le Cx^{3/2}$ we also obtain (39).

Similarly, using that $1/\sqrt{x-n} = (x-s)^{-1/2} + O((n-s)(x-s)^{-3/2})$ as well as (42), (43) and (44) and observing that

$$\sum_{n=0}^{x-1} \frac{n-s}{(n-s)^{3/2}} \le \sqrt{x-s} - \sqrt{-s} \le \frac{x}{\sqrt{x-s}},$$

we find that

$$H_x^-(s) = \sum_{n=0}^{x} \frac{e^{m_-(1)}}{\sqrt{\pi c(x-n)}} \sum_{y=0}^{\infty} v_y H_0(s-n-y) + o\bigg(\frac{\sqrt{x}}{x^\delta (x-s)\sqrt{-s}}\bigg) + \frac{O(\ell_\delta(x))}{(x-s)^{3/2}},$$

$$= \frac{1}{\pi\sqrt{x-s}}\mu_x(s) + O\bigg(\frac{x}{(x-s)^2}\bigg) + o\bigg(\frac{\sqrt{x}}{x^\delta (x-s)\sqrt{-s}}\bigg),$$

where

$$\mu_x(s) = \frac{e^{m_-(1)}}{\sqrt{c/\pi}} \sum_{n=0}^{x}\sum_{y=0}^{\infty} v_y H_0(s-n-y).$$

But by (43),

$$\mu(s) - \mu_x(s) = \frac{\sqrt{\pi}\, e^{\theta_+}}{\sigma} \sum_{n=x+1}^{\infty}\sum_{y=0}^{\infty} v_y H_0(s-n-y) \le \frac{C}{\sqrt{x-s}}.$$

Now we have only to take into account the constraint $x \ge -s$ to modify the error terms to obtain (40). □

Remark on the proof of Theorem 4.1. As mentioned in Section 1 it seems hard to improve the estimates of H^- in Theorem 4.1 for $\delta > \frac{1}{2}$ by the present method. The bottlenecks are the term in (41) and the last term in (44), which arise from the discrepancies of $e^{-\theta_\pm}/\sqrt{\pi\sigma^2 n}$ from v_n and u_n, respectively (for small values of n).

It follows that $\mu(s) = (\sqrt{\pi}\, e^{\theta_+}/\sigma) \sum_{y=0}^{\infty} v_y \sum_{j=y}^{\infty} H(s-j)$, or what is the same thing,

$$\mu(s) - \mu(s-1) = \frac{\sqrt{\pi}\, e^{\theta_+}}{\sigma} \sum_{y=0}^{\infty} v_y H_0(s-y). \tag{46}$$

Hence (43) and (27) yield the following proposition in the case $\delta > 0$.

Proposition 4.2. *Suppose $E_0|S_1|^{2+\delta} < \infty$, $\delta > 0$, and let ℓ_δ be defined by* (37). *Then*

$$\mu(s) - \mu(s-1) = \frac{1}{2(-s)^{3/2}} + \frac{O(\ell_\delta(-s))}{(-s)^2} \tag{47}$$

as $s \to -\infty$. Here $O(\ell_\delta(-s))$ may be replaced by $o(\ell_\delta(-s))$ if $\delta \le \frac{1}{2}$.

The case $\delta=0$. To complete the proof of Proposition 4.2 we must prove the following lemma, which in particular implies (5).

Lemma 4.3.

$$\sum_{y=0}^{\infty} v_y H_0(-k-y) = \frac{\sigma\, e^{-\theta_+}}{2\sqrt{\pi} k^{3/2}}(1+o(1)), \quad \text{as } k\to\infty.$$

Proof. Under only the existence of variance the asymptotic form of $H_0(s)$ as $|s|\to\infty$ is not always given by $\sigma^2/\pi s^2$; we resort to the Fourier analytic method.

Owing to Proposition 3.4, v_y may be replaced by any sequence whose asymptotic form, as $y\to\infty$, is $e^{-\theta_+}/\sqrt{\sigma^2\pi y}$. For our present purpose it is convenient to take $e^{-\theta_+}\alpha_y/\sigma$, where $\{\alpha_n\}_{n=0}^{\infty}$ is the sequence introduced in the proof of Lemma 3.3. Let $\widehat{\alpha}(t)=1/\sqrt{1-e^{it}}$ as before so that

$$\sum_{y=0}^{\infty} \alpha_y H_0(-k-y) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \widehat{\alpha}(t)\phi(t)\, e^{ikt}\, dt, \tag{48}$$

which is valid since $H_0(n)$ and $\widehat{\alpha}(t)$ (with Fourier coefficients $\widehat{\alpha}_y=0$ for $y<0$) are summable over $n\in\mathbf{Z}$ and $|t|<\pi$, respectively. By the same formula (applied in the reverse way),

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \widehat{\alpha}(t)\, e^{-\sigma^2|t|}\, e^{ikt}\, dt = \sum_{y=0}^{\infty} \alpha_y \frac{\sigma^2}{\pi[\sigma^4+(k-y)^2]} = \frac{\sigma^2}{2\sqrt{\pi}}\frac{1}{k^{3/2}}(1+o(1))$$

as $k\to\infty$. It therefore suffices to prove that if $f(t)=\widehat{\alpha}(t)(\phi(t)-e^{-\sigma^2|t|})$, then

$$\int_{-\pi}^{\pi} f(t)\, e^{ikt}\, dt = o(k^{-3/2}), \quad \text{as } k\to\infty.$$

We know that $\rho(t)=\sigma^2|t|(1+o(t))$, $\rho'(t)=\sigma^2 t/|t|+o(t)$ and $\rho''(t)=o(1/t)$ as $t\to 0$, which show that $f(t)=o(|t|^{1/2})$, $f'(t)=o(|t|^{-1/2})$ and $f''(t)=o(|t|^{-3/2})$. By using these bounds we observe, as in the last step of the proof of Proposition 3.4, that $\int_{-\pi}^{\pi} f(t)\, e^{ikt}\, dt=-(ik)^{-1}\int_{-\pi}^{\pi} f'(t)\, e^{ikt}\, dt=o(k^{-3/2})$. □

Proof of Theorem 1.1. If x, the initial site, is nonnegative, the formula (3) is proved as in the proof of Theorem 4.1 (especially its last step) owing to Lemma 4.3. If $x<0$ and $|x-s|\to\infty$, then (3) is a special case of Theorem 1.4, but in the latter theorem the additional condition $E_0[|S_1^{(1)}|^2\log|S_1^{(1)}|]<\infty$ is supposed. In order to dispense with it we proceed as in the proof of the preceding lemma. In [12] it is shown that $H_0(s)=\sigma^2/\pi s^2+o(|s|^{-2}\log|s|)$ as $|s|\to\infty$ and owing to it our task is readily reduced to showing that under $(\log|s|)\sqrt{s/x}\le C$,

$$\sum_{\xi=1}^{\infty}\frac{\sqrt{\xi}}{\xi-s}\bigg[H_0(\xi-x)-\frac{\sigma^2}{\pi[\sigma^4+(x-\xi)^2]}\bigg]=o\bigg(\frac{1}{(-x)^{3/2}}\bigg) \tag{49}$$

as $x\to-\infty$. On the left the numerator $\sqrt{\xi}$ may be replaced by $\sqrt{\pi}\xi\alpha_\xi$. Taking this into account define

$$g(t)=\sum_{\xi=0}^{\infty}\frac{\xi\alpha_\xi}{\xi-s}\,e^{i\xi t}.$$

Then, on writing $\widehat{\alpha}(t)=1/\sqrt{1-e^{it}}$ as before,

$$g(t)=\widehat{\alpha}(t)+isG(t),\quad \text{where } G(t)=\frac{1}{i}\sum_{\xi=0}^{\infty}\frac{\alpha_\xi}{\xi-s}\,e^{i\xi t}.$$

Put $\eta(t)=\phi(t)-e^{-\sigma^2|t|}$ and $f(t)=g(t)\eta(t)=\widehat{\alpha}(t)\eta(t)+isG(t)\eta(t)$ and observe that $G'=g$, $|sG(t)|\le C\sqrt{-s}$ and that for $k=0,1,2$, $|(d/dt)^k\widehat{\alpha}(t)|=O(|t|^{-k-1/2})$ and $|(d/dt)^k\eta(t)|=o(|t|^{1-k})$ as $t\to0$. Then by integration by parts

$$\begin{aligned}\Big(1-\frac{s}{x}\Big)\int_{-\pi}^{\pi}f(t)\,e^{-ixt}\,dt&=\frac{1}{ix}\int_{-\pi}^{\pi}(\widehat{\alpha}\eta)'\,e^{-ixt}\,dt+\frac{1}{x}\int_{-\pi}^{\pi}sG\eta'\,e^{-ixt}\,dt\\&=o\bigg(\frac{1}{(-x)^{3/2}}\bigg)\end{aligned}$$

as $x\to-\infty$ under $(\log|x|)\sqrt{s/x}\le C$, where the computation for the last estimate is carried out as in the proof of Lemma 2.2(i).

For the case when $s\to-\infty$ under $0<-x\le C|s|/(\log|s|)^2$ one has only to consider the time reversed walk.

The proof of Theorem 1.1 is finished. □

Appendix A

For simple random walk (for which $\psi(t,l)=\frac{1}{2}(\cos t+\cos l)$ and $\sigma^2=\frac{1}{2}$) we have

$$\frac{\rho(t)}{\sigma^2}=\sqrt{(3-\cos t)(1-\cos t)}\quad\text{and}\quad F(t)=-\log\sqrt{(3-\cos t)/2};$$

if $a=3-2\sqrt{2}$, then $F(t)=-\log\sqrt{(1-2a\cos t+a^2)/(1-a)^2}$ and, by simple computations,

$$F(t) = \log(1-a) + \sum_{n=1}^{\infty} \frac{a^n}{n} \cos nt, \quad c = \frac{1}{2(1-a)},$$

$$m_+(z) = m_-(z) = -\log\sqrt{(1-az)} \quad \text{and} \quad u_n = v_n = \sqrt{2/\pi n}\left(1 - \frac{1+a}{16n} + O(n^{-2})\right).$$

Appendix B

Let $h_z^-(s)$, $s<0$, be the density of the distribution of the position s of the first visit to the negative real axis $L_- = (-\infty, 0)$ of the standard two-dimensional Brownian motion B_t starting at $z \in \mathbf{C} \setminus L_-$. We must distinguish whether the hitting of L_- takes place through the upper or lower half plane. To this end we assign to every number $s<0$ two points $s+i0$ and $s-i0$ representing the points of the 'upper edge' and the 'lower edge', respectively, of the region $\mathbf{C} \setminus L_-$ along the slit L_-. Thus, e.g., for $s<0$,

$$h_z^-(s+i0) = \frac{d}{ds} P_z[B_{\tau(L_-)} \le s;\ \text{there is } \varepsilon > 0 \text{ such that } \operatorname{Im} B_{\tau(L_-)-t} > 0 \text{ for } 0 < t < \varepsilon]$$

and $h_z^-(s) = h_z^-(s+i0) + h_z^-(s-i0)$. Since the function $f(z) = \sqrt{z} = \sqrt{r}\, e^{i\theta/2}$, $r>0$, $-\pi < \theta < \pi$, conformally and univalently maps the region $\mathbf{C} \setminus L_-$ onto the right half plane $\{z : \operatorname{Re} z > 0\}$ and the density at $\sqrt{s}$, $s<0$, of hitting distribution of the imaginary axis for the process B_t starting at $w = \sqrt{z} = u + iv$ equals

$$\frac{\pi^{-1} u}{u^2 + (\sqrt{-s} - v)^2},$$

we obtain that

$$h_z^-(s+i0) = \frac{1}{\pi} \frac{\sqrt{r}\cos(\theta/2)}{r - s - 2\sqrt{r(-s)}\sin(\theta/2)} \frac{1}{2\sqrt{-s}}$$

and, by simple computation, that for $z = re^{i\theta} = x + iy \notin L_-$, $s<0$,

$$h_z^-(s) = \frac{1}{\pi} \frac{(r-s)\sqrt{r}\ \cos(\theta/2)}{r^2 + s^2 - 2rs\cos\theta} \frac{1}{\sqrt{-s}} = \frac{1}{\pi} \frac{(r-s)\sqrt{r+x}}{[(x-s)^2 + y^2]\sqrt{-2s}};$$

in particular

$$h_x^-(s) = \frac{1}{\pi} \frac{\sqrt{x}}{(x-s)\sqrt{-s}}, \quad x > 0, s < 0.$$

Let $h_{x+iy}(s)$ denote the Poisson kernel: $h_{x+iy}(s) = y/\pi(y^2 + (s-x)^2)$. For the Brownian motion $\widetilde{B}_t = Q^{1/2} B_t$ the densities of the first hitting distribution of the real line and the negative real line are given by $h_{x-\omega y+i\lambda y}(s)$ and $h^-_{x-\omega y+i\lambda y}(s)$, respectively, so that

$$h^-_{x-\omega y+i\lambda y}(s)=h_{x-\omega y+i\lambda y}(s)+\int_0^\infty h_{x-\omega y+i\lambda y}(u)h^-_u(s)\,du \tag{50}$$

(λ and ω are defined in (12)). It is remarked that $h_{x-\omega y+i\lambda y}(s)$ and $h_{x+iy}(s)$ are uniformly comparable (since $\lambda^2>\omega^2$), hence so are $h^-_{x-\omega y+i\lambda y}(s)$ and $h^-_{x+iy}(s)$.

Appendix C

Here we prove the identity (7), which may be written as

$$\theta_+=\frac{1}{2}\sum_{k=1}^\infty\frac{1}{k}\sum_{n=1}^\infty[H_0^k(-n)-H_0^k(n)], \tag{51}$$

and the absolute convergence $\sum_{k=1}^\infty k^{-1}|\sum_{n=1}^\infty(H_0^k(n)-H_0^k(-n))|<\infty$ as well.

Proof. The proof is similar to that found in [8]. Recall that $m_\pm(1)$ exists and $2\theta_+=-m_+(1)+m_-(1)=-\sum_{n=1}^\infty(F^\vee(n)-F^\vee(-n))$ (owing to Lemma 2.2(ii) and (31), respectively). Substitution from (17) then yields

$$\theta_+=-\frac{1}{2}\sum_{n=1}^\infty\sum_{k=1}^\infty\frac{1}{k}[H_0^k(n)-H_0^k(-n)].$$

This becomes (51) on interchanging the order of summation which it is our task to justify. By Abel's theorem

$$\theta_+=-\lim_{r\uparrow 1}\sum_{k=1}^\infty\frac{1}{2k}T^k(r),\quad \text{where } T^k(r)=\sum_{n=1}^\infty r^{n-1}[H_0^k(n)-H_0^k(-n)].$$

It follows that for $r<1$,

$$\begin{aligned}T^k(r)&=\sum_{n=1}^\infty r^{n-1}\frac{1}{2\pi}\int_{-\pi}^{\pi}[\phi^k(t)-\phi^k(-t)]\,e^{-int}\,dt\\&=\frac{1}{\pi}\int_0^\pi|\phi(t)|^k\sin(k\arg\phi(t))\frac{2\sin t}{1-2r\cos t+r^2}\,dt.\end{aligned}$$

One can choose $\varepsilon>0$ so small that

$$|\phi(t)|<1-\varepsilon\quad\text{for } \varepsilon\le|t|\le\pi,$$

and

$$\frac{|\rho(t)|}{2}\le 1-|\phi(t)| \text{ and } |\rho(t)|<|\phi(t)|\quad \text{for } |t|<\varepsilon.$$

Of these three inequalities the last one implies $|\phi(t)\arg\phi(t)|\le|\rho(t)\theta(t)|$, which together with the first one and the inequality $1-2r\cos t+r^2\ge 1-\cos^2 t$ yields that

$$\frac{1}{2k}|T^k(r)|\le C_\varepsilon(1-\varepsilon)^k+\frac{4}{\pi}\int_0^\varepsilon|\phi(t)|^{k-1}|\rho(t)\theta(t)|\frac{dt}{t}.$$

Now use the second one to see that the sum of the right-hand side over $k \geq 1$ is dominated by $C_\varepsilon \varepsilon^{-1} + (8/\pi) \int_0^\varepsilon |\theta(t)| t^{-1}\, dt$, which is finite owing to Lemma 2.1. Thus the dominated convergence theorem is applied to conclude the desired identity. □

References

1. Bousquet-Mélou, M. and Schaeffer, G., Walks on the slit plane, *Probab. Theory Related Fields* **124** (2002), 305–344.
2. Fukai, Y., Hitting distribution to a quadrant of two-dimensional random walk, *Kodai Math. J.* **23** (2000), 35–80.
3. Fukai, Y., Hitting time of a half-line by two-dimensional random walk, *Probab. Theory Related Fields* **128** (2004), 323–346.
4. Kesten, H., Hitting probabilities of random walks on $\mathbf{Z}^d$, *Stochastic Process. Appl.* **25** (1987), 165–184.
5. Kesten, H., How long are the arms of DLA, *J. Phys. A* **20** (1987), L29–L33.
6. Kesten, H., Some caricatures of the multiple contact diffusion-limitted aggregation and η-model, in *Stochastic Analysis* (*Durham*, *1991*), pp. 179–227, Cambridge Univ. Press, Cambridge, 1991.
7. Lawler, G. F. and Limic, V., The Beurling estimate for a class of random walks, *Electron. J. Probab.* **9** (2004), 846–861.
8. Rosén, B., On the asymptotic distribution of sums of independent identically distributed random variables, *Ark. Mat.* **4** (1962), 323–332.
9. Rudin, W., *Real and Complex Analysis*, 3rd ed., McGraw-Hill, New York, 1987.
10. Spitzer, F., *Principles of Random Walks*, Van Nostrand, Princeton, NJ, 1964.
11. Uchiyama, K., Green's functions for random walks on $\mathbf{Z}^d$, *Proc. London Math. Soc.* **77** (1998), 215–240.
12. Uchiyama, K., The hitting distributions of a line for two dimensional random walks, to appear in *Trans. Amer. Math. Soc.*
13. Uchiyama, K., The hitting distribution of line segments for two dimensional random walks, *Preprint*, 2008.
14. Yosida, K., *Functional Analysis*, 6th ed., Springer, Berlin–Heidelberg, 1980.
15. Zygmund, A., *Trigonometric Series*, vol. **2**, 2nd ed., Cambridge Univ. Press, Cambridge, 1959.

Kôhei Uchiyama
Department of Mathematics
Tokyo Institute of Technology
Oh-okayama, Meguro
Tokyo 152-8551
Japan
uchiyama@math.titech.ac.jp

Received June 24, 2008
published online June 3, 2009