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On the holomorphic extension of CR functions from non-generic CR submanifolds of $\mathbb{C}^n$

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Abstract. We give a holomorphic extension result for continuous CR functions on a non-generic CR submanifold N of $\mathbb{C}^n$ to complex transversal wedges with edges containing N. We show that given any $v\in\mathbb{C}^n\setminus(T_pN+iT_pN)$, there exists a wedge of direction v whose edge contains a neighborhood of p in N, such that any continuous CR function defined locally near p extends holomorphically to that wedge.

1. Introduction

1.1. Statement of results

Let N be a CR submanifold of $\mathbb{C}^n$ ($\dim(T_pN\cap iT_pN)$ is independent of p); we say that N is *generic* if $T_pN+iT_pN=\mathbb{C}^n$. The main question we address in this paper is the possible holomorphic extension of CR functions from a non-generic CR submanifold N to some wedge $\mathcal{W}$ in a complex transversal direction. We say that a vector v in $\mathbb{C}^n$ is *complex transversal* to N at $p\in N$, if $v\notin\operatorname{span}_{\mathbb{C}}T_pN$. For totally real submanifolds of $\mathbb{C}^n$, we have the following well-known result (due to Nagel [13] and Rudin [15]), but not stated as such in any of the papers. If N is a non-generic totally real submanifold of $\mathbb{C}^n$ and $v\in\mathbb{C}^n$ is complex transversal to N at p, then for any continuous function f on a neighborhood of p in N there exists $\mathcal{W}_v$, a wedge of direction v whose edge contains N, such that f has a holomorphic extension to $\mathcal{W}_v$. (See the remarks following for different proofs of this result.) Our main result is the following generalization of this result for non-generic CR submanifolds.

Theorem 1. *Let N be a smooth ($\mathcal{C}^\infty$) non-generic* CR *submanifold of $\mathbb{C}^n$. Let $p\in N$ and let U be a neighborhood of p in N. For any v complex transversal to N at p, there exists a wedge $\mathcal{W}_v$ of direction v whose edge contains a neigh-*

borhood V of p in N, $V \subset U$, such that any continuous CR *function on U extends holomorphically to $\mathcal{W}_v$. The holomorphic extension F is continuous on $N \cup \mathcal{W}_v$.*

1.2. Remarks

In [8] we proved the above result for CR distributions that are decomposable and in [9] we proved it in the case of positive defect. In this paper we will study the last remaining case in which the defect is null. By [16], the union of the CR orbits through a point p forms a CR submanifold of N of same CR dimension. Denote this manifold by $\mathcal{O}_p^{\mathrm{CR}}$. By [8] and [9] in order to prove Theorem 1, it suffices to prove it in the case where $\mathcal{O}_p^{\mathrm{CR}}$ is a complex manifold.

If the CR function we seek to extend is of class $\mathcal{C}^\ell$, the extension obtained in Theorem 1 is only continuous up to the boundary of the wedge, this differs from the generic case in which the holomorphic extension to a wedge (if it exists) has the same smoothness as the CR function it extends.

The result corresponding to Theorem 1 for totally real manifolds (mentioned in the introduction) can be proved in several ways. One way is to follow the theory of analytic vectors of an elliptic operator due to Baouendi and Trèves (see [4], Section 3.2). Another way is to use the following result: Let N be a smooth submanifold of the boundary of Ω, a strictly pseudoconvex domain in $\mathbb{C}^n$. If N is complex tangential ($TN \subset (T(\partial\Omega) \cap iT(\partial\Omega))$), then N is a Pic interpolating set. See, for example, [13] or [15]. Given N, a totally real non-generic submanifold of $\mathbb{C}^n$, one can easily construct Ω as above and deduce the theorem. Finally, the simplest proof is to use the fact that any CR function on a totally real submanifold is decomposable, and hence by a projection argument (see [8]) any CR function extends holomorphically in a complex transversal wedge.

The main difference between this result and the "classical" holomorphic extension to wedges is that in the latter case, the holomorphic extension is forced and unique, since the CR functions are locally uniform limits of holomorphic polynomials. In our setting, Corollary 19 implies that the extension obtained is not unique. One should note that the question of CR extensions can be viewed as a Cauchy problem with Cauchy data on a characteristic set N. In [8] we constructed an example of an abstract CR structure, where there is no such CR extension property. It is from the perspective of trying to understand CR extensions from manifolds of the same CR dimension that we approached the problem treated in this paper (Corollary 20). Also note that in our setting, we can choose the direction of extension (provided it is complex transversal). This differs from the theory of holomorphic extensions in generic submanifolds, where not only one cannot choose the direction of extension, but in general, one does not know that direction.

1.3. Background

Most of the work dealing with holomorphic extensions of CR functions takes place on a generic CR submanifold of $\mathbb{C}^n$. We only give a short and incomplete survey of the work on this subject. The first result is due to H. Lewy [12]. He proved that if a hypersurface was Levi non-degenerate at p_0, then CR functions extend holomorphically to one side of the hypersurface. This result was generalized by Boggess and Polking [6] for arbitrary dimensions. In the case of Levi flatness, Trépreau [17] proved that if a hypersurface in $\mathbb{C}^n$ is minimal at p_0 (it contains no $(n-1)$-dimensional complex manifold through p_0), then CR functions extend holomorphically to one side of the hypersurface. The generalization of Trépreau's result to arbitrary codimension is due to Tumanov [21], in which he states that if the manifold M is minimal at p_0 (it contains no proper submanifolds of the same CR dimension through p_0), then CR functions extend to a wedge in $\mathbb{C}^n$ with edge M. It turns out that minimality is a necessary and sufficient condition for extension of CR functions. The converse of Tumanov's result is due to Baouendi and Rothschild [3]. If the CR manifold is not minimal, then one cannot obtain holomorphic wedge extensions. However, one should note that there is a variation of Tumanov's theorem, due essentially to Tumanov [22] and Trépreau [18], which can be seen as a refinement of the Aĭrapetyan edge of the wedge theorem [1], which describes a possible CR extension to a CR submanifold of higher CR dimension (see [18] and [22]). For those interested in the subject, we recommend the survey paper on holomorphic extension by Trépreau [19]. For general background on CR geometry, see the books by Baouendi, Ebenfelt and Rothschild [2], Boggess [5] and Jacobowitz [11].

1.4. Outline of the paper

A non-generic CR submanifold of $\mathbb{C}^n$ is given locally as a CR graph of a generic submanifold of $\mathbb{C}^{n-m}$ (see [5]). So $N=\{(Z,h(Z)):Z\in\mathfrak{N}\}$, where $\mathfrak{N}$ is a generic CR submanifold of $\mathbb{C}^{n-m}$ and h is a CR map from $\mathfrak{N}\to\mathbb{C}^m$. Choose coordinates on $\mathfrak{N}(z,w)$, where z is the complex tangential coordinate. We first establish holomorphic extension in the variable w using the theory of analytic vectors due to Baouendi and Trèves. We construct an elliptic operator of degree two with no constant term for which we solve a Dirichlet problem. We then show that the solution to this Dirichlet problem (with the CR function we are extending as boundary values) is holomorphic (in w) on an open set whose boundary contains N. Next, we solve a $\bar\partial$ problem in z with a vanishing condition on N. This is the main difficulty in the proof of Theorem 1, since there is no convexity assumption in the variable z. Since the w variable on this open set depends on the variable z, one cannot take a global convolution with a Bochner–Martinelli kernel.

2. Local coordinates

Let $p\in N$, our setting is the following:

$$N=\{(z,w',h(z,w')):(z,w')\in\mathfrak{N}\}\subset\mathbb{C}^{n-m}\times\mathbb{C}^m,$$

where $\mathfrak{N}$ is generic in $\mathbb{C}^{n-m}$ and h is a CR map into $\mathbb{C}^m$ defined in a neighborhood of p. Define $\mathbb{C}T_p\mathfrak{N}=T_p\mathfrak{N}\otimes\mathbb{C}$ and $T_p^{0,1}\mathfrak{N}=T_p^{0,1}\mathbb{C}^m\cap\mathbb{C}T_p\mathfrak{N}$. The CR vector fields of $\mathfrak{N}$ and hence on N, are vector fields L on $\mathfrak{N}$ (or N) such that for any $p\in\mathfrak{N}$ we have $L_p\in T_p^{0,1}\mathfrak{N}$. We introduce local coordinates near p so that p is the origin in $\mathbb{C}^n$ and $\mathfrak{N}$ is parameterized in $\mathbb{C}^{n-m}=\mathbb{C}_z^d\times\mathbb{C}_{w'}^k$ by

$$\mathfrak{N}=\{(z,w')\in\mathbb{C}^d\times\mathbb{C}^k:\operatorname{Im}(w')=a(z,\operatorname{Re}(w'))\},\quad T_0\mathfrak{N}=\mathbb{C}^d\times\mathbb{R}^k. \tag{2.1}$$

We let $s=\operatorname{Re}(w')\in\mathbb{R}^k$, and set $w'(z,s)=s+ia(z,s)$. We thus have

$$\mathfrak{N}=\{(z,w'(z,s):z\in\mathbb{C}^d \text{ and } s\in\mathbb{R}^k\}\subset\mathbb{C}^d\times\mathbb{C}^k,\quad T_0\mathfrak{N}=\mathbb{C}^d\times\mathbb{R}^k. \tag{2.2}$$

By [8] and [9], we may suppose that $\mathcal{O}_0^{\mathrm{CR}}$ is a complex submanifold. Indeed, in [8] we show that in the case where N is minimal at 0 then all CR distributions holomorphically extend to any complex transversal wedge (it is a simple projection argument). In the case where $\mathcal{O}_0^{\mathrm{CR}}$ is not a complex manifold (that is when the defect in the sense of Trépreau–Tumanov is positive) we showed again in [9] that any continuous CR function extends holomorphically to any complex transversal wedge. Therefore, to prove Theorem 1, it suffices to consider the case where $\mathcal{O}_0^{\mathrm{CR}}$ is a complex submanifold of $\mathfrak{N}$. Hence, after a holomorphic change of coordinates, we may assume that for some $\varepsilon>0$ we have

$$a(z,0)\equiv 0\quad\text{for } \|z\|<\varepsilon. \tag{2.3}$$

We are henceforth working in a fixed neighborhood $\mathcal{U}$ of the origin in N which we shall assume is parameterized by

$$\mathcal{U}=\{(z,w'(z,s)):\|z\|<\varepsilon \text{ and } s\in U\}, \tag{2.4}$$

where U is a neighborhood of the origin in $\mathbb{R}^k$.

We will prove Theorem 1 in the case where $m=1$. That is, h is a CR function defined in a neighborhood of the origin in $\mathfrak{N}$. The general case follows in precisely the same manner, see the remark at the end of the proof of Theorem 1. Note that if

v is complex transversal to N at the origin, then $v=(v',v'')\in\mathbb{C}^{n-1}\times\mathbb{C}$ with $v''\neq 0$. After a linear change of variables not affecting (2.2), we may assume that

$$v=(0,1)\in\mathbb{C}^{n-1}\times\mathbb{C}. \tag{2.5}$$

A basis $\mathcal{L}$ of $T^{0,1}\mathfrak{N}$ near the origin consists of vector fields L_j, $j=1,...,d$, of the form

$$L_j=\frac{\partial}{\partial\bar{z}_j}+\sum_{l=1}^{k}F_{jl}\frac{\partial}{\partial s_l}. \tag{2.6}$$

We are now going to consider the "simplest" generic submanifold of $\mathbb{C}^n$ which has the same CR structure as N, namely $N\times\mathbb{R}$. Define $M\subset\mathbb{C}^n$ by

$$M=\{(z,w'(z,s),t+h(z,w'(z,s))):w'(z,s)=s+ia(z,s),(z,w'(z,s))\in\mathfrak{N}\}. \tag{2.7}$$

We will denote the coordinate functions on M as $(z,w)=(z,w',w'')\in\mathbb{C}^d_z\times\mathbb{C}^k_{w'}\times\mathbb{C}_{w''}$ with

$$\begin{cases} w'(z,s,t)=s+ia(z,s)\subset\mathbb{C}^k,\\ w''(z,s,t)=t+h(z,s)\subset\mathbb{C}. \end{cases} \tag{2.8}$$

We are going to show that any CR function on N extends holomorphically to a wedge with edge N in the direction of $M^+=\{(z,w(z,s,t))\in M:t>0\}$.

3. Elliptic theory

3.1. Construction of Δ

Lemma 2. *On the manifold M, there exist $k+1$ vector fields R_j of the form*

$$R_j=\sum_{l=1}^{k}a_{jl}(z,s)\frac{\partial}{\partial s_l},\ j=1,...,k,\quad\text{and}\quad R_{k+1}=\frac{\partial}{\partial t},$$

where the a_{jl} are smooth functions, such that for $j,l\in\{1,...,k+1\}$,

(i) $R_j(w_l)=\delta_{jl}$;

(ii) $[R_j,R_l]=0$;

(iii) $[L_j,R_l]=0$;

(iv) *the set $\{L_1,...,L_d,\bar{L}_1,...,\bar{L}_d,R_1,...,R_{k+1}\}$ spans the complex tangent plane to M near the origin.*

Proof. The proof of the lemma is classic. Cases (ii), (iii) and (iv) follow from (i). We thus determine the R_j by solving for their coefficients in (i) (see for example [2], Lemma 8.7.13, p. 234). □

On M, we construct an elliptic differential operator Δ (following Baouendi and Trèves [4]) for which we shall solve a Dirichlet problem.

Remark 3. Since $T_0\mathfrak{N}=\mathbb{C}^d\times\mathbb{R}^k$, we note that the R_j's satisfy

$$R_j(0)=\frac{\partial}{\partial s_j}.$$

Define Δ by

$$\Delta=\sum_{j=1}^{k} R_j^2+\frac{\partial^2}{\partial t^2}. \tag{3.1}$$

From Remark 3 we immediately deduce the following result.

Lemma 4. *The operator $-\Delta$ is strongly elliptic of degree two on M with smooth ($\mathcal{C}^\infty$) coefficients and no constant term.*

3.2. Resolution of a Dirichlet problem

Δ is a differential operator in the variables s and t, whose coefficients depend smoothly on the variable z. Hence we view Δ as a differential operator on $\mathbb{R}^{k+1}=\mathbb{R}^k_s\times\mathbb{R}_t$ with z acting as a parameter. Let Ω be an open set in $\{(z,w(z,s,t))\in M: t>0\}$ such that if Π is the projection from $\mathbb{C}^d\times\mathbb{C}^k\times\mathbb{R}$ onto $\mathbb{C}^d\times\mathbb{C}^k\times\{0\}$, then Ω satisfies

$$\Pi(\overline{\Omega})\subset\mathcal{U}. \tag{3.2}$$

We shall now define a boundary of Ω on which we shall impose a boundary condition of a Dirichlet problem. Since Δ is a differential operator in the variables s and t, we shall "close" Ω in the variables s and t. That is, we can assume that Ω is parameterized by $\Omega=\{(z,w(z,s,t)):z\in V$ and $(s,t)\in\omega\}$, where ω is an open set in $\mathbb{R}^k\times\mathbb{R}$. We then define the boundary $\partial\Omega$ of Ω on which we shall impose the Dirichlet condition by

$$\partial\Omega=\{(z,w(z,s,t)):(s,t)\in\partial\omega\}. \tag{3.3}$$

Denote by (D) the Dirichlet problem on Ω, i.e.

$$\text{(D)} \qquad \begin{cases} \Delta(u) = 0 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases}$$

Then we have the fundamental result.

Theorem 5. *For $g \in \mathcal{D}'(\partial\Omega)$, (D) has a unique solution. If the boundary data g is continuous, then $S(g)$ is continuous up to $\partial\Omega$.*

For a reference, we refer the reader to [7], Theorem 5.2 and Remark 5.3, pp. 263–265, as well as to [10], Chapter 10.

Let S be the solution operator for (D). That is, for $g \in \mathcal{D}'(\partial\Omega)$, we have

$$\begin{cases} \Delta(S(g)) = 0 & \text{in } \Omega, \\ S(g) = g & \text{on } \partial\Omega. \end{cases} \tag{3.4}$$

3.3. Analytic vector theory

In this section we present the results developed by Baouendi and Trèves [4] and later by Trèves in [20]. We have included this section for the sake of completeness and claim no originality whatsoever.

Definition 6. Using the notation $R_{k+1} = \partial/\partial t$ and $R^\alpha = R_1^{\alpha_1} \dots R_{k+1}^{\alpha_{k+1}}$, $\alpha \in \mathbb{N}^{k+1}$, we shall say that a continuous function f in ω is an analytic vector of the system of vector fields $\{R_1, \dots, R_{k+1}\}$ if $R^\alpha f \in \mathcal{C}^0$ for any $\alpha \in \mathbb{N}^{k+1}$, and if to every compact set K of ω there is a constant $\rho > 0$ such that in K,

$$\sup_{\alpha \in \mathbb{N}^{k+1}} \left(\rho^{|\alpha|} \frac{|R^\alpha f|}{|\alpha|!} \right) < \infty. \tag{3.5}$$

Set $V = \{z \in \mathbb{C}^d : \|z\| < \varepsilon\}$. We have

$$(z, w(z, s, t)) \in \Omega \quad \text{if and only if} \quad (z, (s, t)) \in V \times \omega.$$

The next proposition is a simplified version of Lemma 4.1 in [4].

Let B be the ball with center x and radius ρ in $\mathbb{R}^{k+1}$ and for $s \in (0, 1]$, let B_s be the ball with center x and radius ρs.

Proposition 7. *Let B be as above and such that $B \Subset \omega$. Then there exists C_1 and C_2 depending only on ω and Δ such that for any $\mathcal{C}^\infty$ function f in an open neighborhood of the closure of B with*

$$\Delta f = 0 \quad \text{in } B,$$

for any z in V and every $\alpha \in \mathbb{N}^{k+1}$,

$$\|R^\alpha f\|_{L^2(B_s)} \leq C_1 \left(\frac{C_2}{(1-s)} \right)^{|\alpha|} |\alpha|!. \tag{3.6}$$

The L^2 norm can be replaced by the L^∞ norm.

For the sake of completeness, we include the proof of the proposition, which is found in [4], p. 403.

Proof. Denote by $\|\cdot\|_s$ the L^2 norm on B_s. By Sobolev's embedding theorem and the linear independence of the R_j's, the L^2 result implies the L^∞ result. We will prove the following for all z in V,

$$(1-s)^{|\alpha|} \|R^\alpha \Delta^k f\|_s \leq D_1 D_2^{|\alpha|} D_3^k (2k + |\alpha|!). \tag{3.7}$$

Claim 8. *There exists a positive R such that for all $0 < s < s' < 1$, $\beta \in \mathbb{N}^{k+1}$, $|\beta| = 2$, $z \in V$ and $u \in \mathcal{C}^\infty(\overline{B})$,*

$$\|R^\alpha u\|_s \leq R \left(\|\Delta u\|_{s'} + \frac{1}{s'-s} \sum_{j=1}^{k+1} \|R_j(u)\|_{s'} + \frac{1}{(s'-s)^2} \|u\|_{s'} \right), \tag{3.8}$$

where the D_j depend only on ω.

Choose $\phi \in \mathcal{C}_0^\infty(B)$ such that $0 \leq \phi \leq 1$, $\phi = 1$ on B_s, $\operatorname{supp} \phi \subset B_{s'}$ and

$$|\partial^\alpha \phi| \leq C_\alpha \frac{1}{(s'-s)^{|\alpha|}}, \tag{3.9}$$

where C_α is independent of s and s'.

By the ellipticity of Δ, there exists R' depending only on ω such that

$$\|\phi u\|_{H^2(\omega)} \leq R' (\|\Delta \phi u\|_{L^2(\omega)} + \|\phi u\|_{L^2(\omega)}). \tag{3.10}$$

Hence (3.8) follows from (3.9) and (3.10) and the linear independence of the R_j's.

We now prove (3.7) by induction on $|\alpha|$. Using that

$$\|\phi u\|_{H^1(\omega)} \leq R'' (\|\Delta \phi u\|_{H^{-1}(\omega)} + \|\phi u\|_{L^2(\omega)})$$

and taking $u=\Delta^k f$, we can assume that (3.7) holds for $|\alpha|=1$ and all $k\in\mathbb{N}$, i.e.

$$\|R^\alpha \Delta^k f\|_{L^2(B)} \le C_1''(C_2'')^{k+2}(2k+1)!. \tag{3.11}$$

Suppose that (3.7) holds for $|\alpha|=l$. Let $\alpha\in\mathbb{N}^{k+1}$ and $|\alpha|=l+1$. We will now prove (3.7) for D_1, D_2 and large enough D_3. Write $\alpha=\beta+\alpha'$, where $|\beta|=2$. By (3.8) with $u=R^{\alpha'}\Delta^k f$ and $s'=s+1-s/|\alpha|$ we have

$$\|R^\alpha\Delta^k f\|_s \le R\Bigg(\|R^{\alpha'}\Delta^{k+1}f\|_{s+1-s/|\alpha|}+\frac{|\alpha|}{1-s}\sum_{j=1}^{k+1}\|R_jR^{\alpha'}\Delta^k f\|_{s+1-s/|\alpha|} + \frac{|\alpha|^2}{(1-s)^2}\|R^{\alpha'}\Delta^k f\|_{s+1-s/|\alpha|}\Bigg). \tag{3.12}$$

Using the induction hypothesis to estimate the right-hand side of (3.12) we obtain

$$\begin{aligned}\|R^\alpha\Delta^k f\|_s \le RD_1\Bigg(&(1-s)^{-|\alpha|+2}\left(1+\frac{1}{|\alpha|-1}\right)^{|\alpha|-1} D_2^{|\alpha|-2}D_3^{k+1}(|\alpha|+2k)!\\ &+(k+1)(1-s)^{-|\alpha|}\left(1+\frac{1}{|\alpha|-1}\right)^{|\alpha|-1}|\alpha|D_2^{|\alpha|-2}D_3^{k}(|\alpha|-1+2k)!\\ &+(1-s)^{-|\alpha|}|\alpha|(|\alpha|-1)\left(1+\frac{1}{|\alpha|-1}\right)^{|\alpha|-1}D_2^{|\alpha|-2}D_3^{k}(|\alpha|-2+2k)!\Bigg).\end{aligned}$$

Hence (3.7) follows for $|\alpha|=l+1$ if

$$\frac{D_3}{D_2^2}+\frac{k+1}{D_2}+\frac{1}{D_2^2}<1. \tag{3.13}$$

It suffices to choose $D_1>C_1''$, $D_3>C_2''$ and then D_2 large enough so that $D_2>C_2''$ and (3.13) holds. □

Proposition 9. (Proposition II.4.1 in [20]) *Let Ω be an open set in $M\times\mathbb{R}$ and let $p\in\Omega$. A continuous function $f=f(z,w(z,s,t))$ on Ω is an analytic vector of the system $\{R_1,...,R_{k+1}\}$ if and only if there exists an open neighborhood $\mathcal{V}$ of p in $\mathbb{C}^{m+1}$ with $\mathcal{V}\cap\Omega=\Omega$ and a continuous function $F=F(z,w)$ in $\mathcal{V}$, holomorphic with respect to the variable w, such that we have $f(z,w(z,s,t))=F(z,w(z,s,t))$ on Ω.*

The main difficulty in the proof of Proposition 9 is to show that the function defined by

$$F(z,w,s,t)=\sum_{\alpha\in\mathbb{N}^{k+1}}\frac{R^\alpha f(z,s,t)}{|\alpha|!}[w-w(z,s,t)]^\alpha$$

is equal to f for w near $w(z,s,t)$ in Ω, if f is an analytic vector of the vector fields $\{R_1, ..., R_{k+1}\}$.

We shall use Proposition 7 to construct analytic vectors of the vector fields $\{R_1, ..., R_{k+1}\}$ and then apply Proposition 9 to these vectors to obtain a holomorphic extension in the variables w.

4. Proof of Theorem 1

Let f be a continuous CR function defined in $\mathcal{U}$. Then, trivially, f extends to a CR function on M (since the CR structure of M is the same as $\mathfrak{N}$). By (3.2), f is CR on Ω. Choose $T>0$ small enough so that

$$\text{for any } z\in V,\ s\in U \text{ and } t\in(0,T],\quad \operatorname{dist}((z,w(z,s,t)),\partial\Omega)=t. \tag{4.1}$$

Consider then the continuous function $S(f)$ given by (3.4). By construction, we have $\Delta S(f)=0$ on Ω.

Define Ω_T by

$$\Omega_T=\{(z,w(z,s,t)): z\in V,\ s\in U \text{ and } t\in(0,T]\}. \tag{4.2}$$

We then obtain as a consequence of Propositions 7 and 9 the following result.

Corollary 10. *There exists $C=C_\omega$ such that for any $(z,w(z,s,t))\in\Omega_T$, $S(f)$ extends holomorphically as a function of w for $\{w\in\mathbb{C}^{k+1}:\|w-w(z,s,t)\|<Ct\}$.*

Proof. Let $B_t(p)$ be a ball with radius t in $\mathbb{R}^{k+1}$. Thus, if $(z,w(z,s,t))\in\Omega_T$, then $B_t(s,t)\subset\omega$, by (4.1). Since $\Delta S(f)=0$ in ω, the corollary follows by Propositions 7 and 9. □

By (4.1) and Corollary 10 for $(z,w(z,\sigma,\tau))\in\Omega_T$ the function

$$F(z,w(z,s,t))=\sum_{\alpha\in\mathbb{N}^{k+1}}\frac{R^\alpha Sf(z,w(z,\sigma,\tau))}{|\alpha|!}[w(z,s,t)-w(z,\sigma,\tau)]^\alpha \tag{4.3}$$

is equal to $S(f)(z,w(z,s,t))$ when $(z,s,t)\in V\times B_\tau(\sigma,\tau)$, where $B_\tau(\sigma,\tau)$ is the ball with radius τ around (σ,τ) in ω, and extends holomorphically in the variable w when $\|w-w(z,\sigma,\tau)\|<C\tau$.

We now claim that the various holomorphic extensions of $S(f)$ agree where they are defined. Indeed, for fixed z, Ω is a totally real generic submanifold of $\mathbb{C}^{k+1}$. Hence it is a defining set for holomorphic functions of w. So any two extensions of F taken at different points $(z,w(z,\sigma_1,\tau_1))$ and $(z,w(z,\sigma_2,\tau_2))$ that are simultaneously defined as holomorphic functions of w in a neighborhood of Ω must be equal, since by Proposition 7 their difference vanishes on an open subset of Ω. We will now restrict ourselves to points in Ω_T. We have established the following lemma.

Lemma 11. *$S(f)$ extends to a function F in $\mathcal{W}$, which is holomorphic with respect to w, where $\mathcal{W}$ is a neighborhood of Ω_T given by*

$$\mathcal{W}=\{(z,w):\|w-w(z,\sigma,\tau)\|<C\tau,\ z\in V \text{ and } (z,w(z,\sigma,\tau))\in\Omega_T\}.$$

We are now going to define a product domain $\mathfrak{w}\subset\mathbb{C}^{n+1}$ to which F extends holomorphically as a function of w. For $(\xi,s)\in B\times U$, define $\mathfrak{w}(\xi,s,t)\subset\mathbb{C}^{k+1}$ by

$$\mathfrak{w}(\xi,s,t)=\left\{\omega\in\mathbb{C}^{k+1}:\|\omega-w(\xi,s,t)\|<\tfrac{1}{2}Ct\right\}, \tag{4.4}$$

and $\mathfrak{M}\subset\mathbb{C}^{n+1}$ by

$$\mathfrak{M}=B\times\left(\bigcup_{\substack{\xi\in B\\ s\in U\\ 0<t<T}}\mathfrak{w}(\xi,s,t)\right). \tag{4.5}$$

Proposition 12. *Let $\mathcal{W}$ and F be as in Lemma 11, then F extends holomorphically as a function of w to $\mathfrak{M}$.*

This is the main part of the proof of Theorem 1. We have no assumption of convexity in z, so there is no obvious way to use L^2 methods. Since the w coordinate in $\mathcal{W}$ depends on z, we cannot use a convolution kernel either. This is where we heavily depend on the fact that the union of the CR orbits through the origin is a complex manifold. It allows us to extend the function F written in a "good" way to $\mathfrak{M}$, where we can solve a $\bar\partial$ problem via convolution kernels.

Proof. Let $F\in\mathcal{O}(\mathcal{W})$, and $(z,w(z,u,t))\in M$, then F is given by

$$F(z,w)=\sum_{\alpha\in\mathbb{N}^{k+1}}\frac{D_w^\alpha(F(z,w(z,s,t)))}{\alpha!}[w-w(z,s,t)]^\alpha, \tag{4.6}$$

which converges normally for $z\in B$ and w such that $\|w-w(z,s,t)\|<Ct$. We now seek to isolate the dependence in the variable z to the coefficients of the power series in (4.6).

For $\omega\in\mathbb{C}^{k+1}$, define $K(\omega,s,t)\subset B\subset\mathbb{C}^d$ as

$$K(\omega,s,t)=\left\{z\in B:\|w(z,s)-\omega\|<\tfrac{1}{2}Ct\right\}. \tag{4.7}$$

Given $(s,t)\in U\times\mathbb{R}^+$, let w_0 be such that $K(w_0,s,t)\neq\varnothing$. Define $B_r(w_0)$ by

$$B_r(w_0)=\{w\in\mathbb{C}^{k+1}:\|w-w_0\|<r\}.$$

If $(z,w)\in K(w_0,t)\times B_{Ct/2}(w_0)$ then it follows that $(z,w)\in\mathcal{W}$. Indeed, if $(z,w)\in K(w_0,t)\times B_{Ct/2}(w_0)$ then we have

$$\|w-w(z,s,t)\|\leq\|w-w_0\|+\|w_0-w(z,s,t)\|<Ct.$$

Hence if w_0 is such that $K(w_0,s,t)\neq\varnothing$, we then have

$$K(w_0,s,t)\times B_{Ct/2}(w_0)\subset\mathcal{W}. \tag{4.8}$$

If $(z,w(\xi,s,t))\in K(w_0,s,t)\times B_{Ct/2}(w_0)$, then (4.8) and (4.6) yield

$$F(z,w,t)=\sum_{\alpha\in\mathbb{N}^{k+1}}c_\alpha(\xi,s,t,z)[w-w(\xi,s,t)]^\alpha. \tag{4.9}$$

Let $\Gamma(F)\subset\mathbb{C}^d\times\mathbb{R}^{k+1}$ be defined as

$$\Gamma(F)=\{(\xi,s,t)\in B\times U\times\mathbb{R}^+:K(w(\xi,s,t),s,t)=B\}. \tag{4.10}$$

We claim that $\Gamma(F)$ is such that $\overline{\Gamma(F)}\cap(B\times U\times\{0\})\neq\varnothing$. Indeed by (2.5) and (4.7), for any $t>0$ there exists $\sigma(t)\in\mathbb{R}^+$ such that

$$(\xi,s,t)\in\Gamma(F)\quad\text{for }\|s\|<\sigma(t).$$

We will show that there is a way to extend F so that $\Gamma(F)=B\times U\times\mathbb{R}^+$. If $\Gamma(F)=B\times U\times\mathbb{R}^+$, then we are done. Hence, assume that $\Gamma(F)\neq B\times U\times\mathbb{R}^+$.

We next prove an elementary lemma, which is at the root of the extension of the function F to a product domain.

Lemma 13. *Let $(\xi_0,s_0,t_0)\in\Gamma(F)$, then we may holomorphically extend F such that if $w(\xi_1,s_1,t_0)\in B_{Ct_0/2}(w(\xi_0,s_0,t_0))$ then $(\xi_1,s_1,t_0)\in\Gamma(F)$.*

Proof. For $w\in B_{Ct_0/2}(w(\xi_0,u_0,t_0))\cap B_{Ct_0/2}(w(\xi_1,u_1,t_0))$ we have

$$\sum_{\alpha\in\mathbb{N}^{k+1}}c_\alpha(\xi_0,s_0,t_0,z)[w-w(\xi_0,s_0,t_0)]^\alpha =\sum_{\alpha\in\mathbb{N}^{k+1}}c_\alpha(\xi_1,s_1,t_0,z)[w-w(\xi_1,s_1,t_0)]^\alpha. \tag{4.11}$$

Note that the left-hand side of (4.11) is defined for all z in B as the right-hand side is defined for z in $K(w(\xi_1, s_1, t_0), t_0)$. Hence (4.11) enables us to holomorphically extend (as a function of z) the function given by

$$\sum_{\alpha\in\mathbb{N}^{k+1}} c_\alpha(\xi_1, s_1, t_0, z)[w-w(\xi_1, s_1, t_0)]^\alpha$$

to $B\times B_{Ct_0/2}(w(\xi_0, s_0, t_0))\cap B_{Ct_0/2}(w(\xi_1, s_1, t_0))$ in the following way.

Let $w(\xi_2, s_2, t_0)\in B_{Ct_0/2}(w(\xi_0, s_0, t_0))\cap B_{Ct_0/2}(w(\xi_1, s_1, t_0))$, and write

$$(4.12)\quad \sum_{\alpha\in\mathbb{N}^{k+1}} c_\alpha(\xi_0, s_0, t_0, z)[w-w(\xi_0, s_0, t_0)]^\alpha$$
$$= \sum_{\alpha\in\mathbb{N}^{k+1}} c_\alpha(\xi_0, s_0, t_0, z)[w-w(\xi_2, s_2, t_0)+w(\xi_2, s_2, t_0)-w(\xi_0, s_0, t_0)]^\alpha.$$

By resumability of absolutely convergent power series, (4.12) becomes

(4.13)
$$\sum_{\alpha\in\mathbb{N}^{k+1}} c_\alpha(\xi_0, s_0, t_0, z)[w-w(\xi_0, s_0, t_0)]^\alpha = \sum_{\alpha\in\mathbb{N}^{k+1}} e_\alpha(\xi_0, s_0, t_0, z)[w-w(\xi_2, s_2, t_0)]^\alpha$$

for w in a neighborhood N_1 of $w(\xi_2, s_2, t_0)$. Note that the expression on the right of (4.13) is defined for *all* z in B. Proceed in the same manner for the power series on the right-hand side of (4.11) to obtain

(4.14)
$$\sum_{\alpha\in\mathbb{N}^{k+1}} c_\alpha(\xi_1, s_1, t_0, z)[w-w(\xi_1, s_1, t_0)]^\alpha = \sum_{\alpha\in\mathbb{N}^{k+1}} e_\alpha(\xi_1, s_1, t_0, z)[w-w(\xi_2, s_2, t_0)]^\alpha$$

for w in a neighborhood N_2 of $w(\xi_2, s_2, t_0)$. Set $\mathcal{N}=N_1\cap N_2$. We then have by (4.13), (4.14) and (4.11), for $w\in\mathcal{N}$,

(4.15)
$$\sum_{\alpha\in\mathbb{N}^{k+1}} e_\alpha(\xi_0, s_0, t_0, z)[w-w(\xi_2, s_2, t_0)]^\alpha = \sum_{\alpha\in\mathbb{N}^{k+1}} e_\alpha(\xi_1, s_1, t_0, z)[w-w(\xi_2, s_2, t_0)]^\alpha.$$

We are now able to extend the right-hand side of (4.15) to $z\in B$. We have thus obtained an extension of the power series $\sum_{\alpha\in\mathbb{N}^{k+1}} e_\alpha(\xi_1, s_1, t_0, z)[w-w(\xi_2, s_2, t_0)]^\alpha$ to $B\times\mathcal{N}$. Hence, in (4.14), we obtain for $w\in\mathcal{N}$, that

$$\sum_{\alpha\in\mathbb{N}^{k+1}} c_\alpha(\xi_1, s_1, t_0, z)[w-w(\xi_1, s_1, t_0)]^\alpha$$

converges absolutely for $z\in B$. Therefore since $w(\xi_2,s_2,t_0)$ can be chosen arbitrarily, by Abel's lemma, we see that the power series

$$\sum_{\alpha\in\mathbb{N}^{k+1}} c_\alpha(\xi_1,s_1,t_0,z)[w-w(\xi_1,s_1,t_0)]^\alpha$$

converges absolutely for $z\in B$ and $w\in B_{Ct_0/2}(w(\xi_1,s_1,t_0))$. We now show that this extension is independent of ξ, s and t. Note that

$$\text{(4.16)}\qquad \begin{cases} \dfrac{\partial}{\partial\overline{w}}\left[\dfrac{\partial}{\partial\xi}\displaystyle\sum_{\alpha\in\mathbb{N}^{k+1}} c_\alpha(\xi_1,s_1,t_0,z)[w-w(\xi_1,s_1,t_0)]^\alpha\right]=0,\\ \dfrac{\partial}{\partial\overline{w}}\left[\dfrac{\partial}{\partial\bar{\xi}}\displaystyle\sum_{\alpha\in\mathbb{N}^{k+1}} c_\alpha(\xi_1,s_1,t_0,z)[w-w(\xi_1,s_1,t_0)]^\alpha\right]=0,\\ \dfrac{\partial}{\partial\overline{w}}\left[\dfrac{\partial}{\partial s}\displaystyle\sum_{\alpha\in\mathbb{N}^{k+1}} c_\alpha(\xi_1,s_1,t_0,z)[w-w(\xi_1,s_1,t_0)]^\alpha\right]=0,\\ \dfrac{\partial}{\partial\overline{w}}\left[\dfrac{\partial}{\partial t}\displaystyle\sum_{\alpha\in\mathbb{N}^{k+1}} c_\alpha(\xi_1,s_1,t_0,z)[w-w(\xi_1,s_1,t_0)]^\alpha\right]=0. \end{cases}$$

By (4.15) *for any* $z\in B$ we have on $\mathcal{N}$,

$$\text{(4.17)}\qquad \begin{cases} \dfrac{\partial}{\partial\xi}\displaystyle\sum_{\alpha\in\mathbb{N}^{k+1}} c_\alpha(\xi_1,s_1,t_0,z)[w-w(\xi_1,s_1,t_0)]^\alpha=0,\\ \dfrac{\partial}{\partial\bar{\xi}}\displaystyle\sum_{\alpha\in\mathbb{N}^{k+1}} c_\alpha(\xi_1,s_1,t_0,z)[w-w(\xi_1,s_1,t_0)]^\alpha=0,\\ \dfrac{\partial}{\partial s}\displaystyle\sum_{\alpha\in\mathbb{N}^{k+1}} c_\alpha(\xi_1,s_1,t_0,z)[w-w(\xi_1,s_1,t_0)]^\alpha=0,\\ \dfrac{\partial}{\partial t}\displaystyle\sum_{\alpha\in\mathbb{N}^{k+1}} c_\alpha(\xi_1,s_1,t_0,z)[w-w(\xi_1,s_1,t_0)]^\alpha=0. \end{cases}$$

Hence by (4.16) and (4.17) we see that the extended power series is independent of the variables ξ, s and t. This completes the proof of Lemma 13. □

Since $\overline{\Gamma(F)}\cap(B\times U\times\{0\})\neq\varnothing$, for any t, pick $(\xi,s,t)\in\Gamma(F)$. By Lemma 13 we extend F as a holomorphic function of w to $\mathfrak{M}$. Note that any two extensions, where simultaneously defined, must agree, since they are equal to our original function F on $K(w(\xi,s),s,t)\times B_{Ct/2}(w(\xi,s,t))$. This completes the proof of Proposition 12. □

Proposition 14. *The extension* $\widetilde{F}$ *satisfies the property*

$$\lim_{\substack{w''\to 0\\ (z,w)\in\mathfrak{M}}} \frac{\partial \widetilde{F}}{\partial \bar{z}_j} = 0 \quad \text{for any } j. \tag{4.18}$$

Proof. $S(f)$ is continuous up to $\mathcal{U}\times\{0\}$ and has boundary values f. Hence F is continuous up to $\mathcal{U}\times\{0\}$ with the same boundary values. Note that

$$\text{if } (z,w',w'')\in\mathfrak{M}' \quad \text{then } (z,w',w'')\to\mathcal{U}\times\{0\} \text{ if and only if } w''\to 0. \tag{4.19}$$

Hence (4.19) combined with the boundary values of $S(f)$ on $\mathcal{U}\times\{0\}$ yields

$$\lim_{w''\to 0} F(z,w)=f. \tag{4.20}$$

The next lemma holds in far greater generality, but we just need it in this simple form.

Lemma 15. *We have for any generator* L_j *of the* CR *vector fields on* Ω_T, *and a function* G, *which is continuous up to* $\mathcal{U}\times\{0\}$, *that*

$$\lim_{t\to 0} L_j(G)(z,w(z,s,t)) = L_j|_{\mathcal{U}\times\{0\}}(G|_{\mathcal{U}\times\{0\}}).$$

Proof. Set $G=g^0+g^1$, where

$$\lim_{t\to 0} G(z,w(z,s,t))=g_0 \quad \text{and} \quad \lim_{t\to 0} g^1(z,w(z,s,t))=0.$$

By (2.6) we have L_j given by

$$L_j=\frac{\partial}{\partial \bar{z}_j}+\sum_{l=1}^{k} F_{jl}\frac{\partial}{\partial s_l},$$

with no differentiation in t (or dependence on t). Therefore, since G is assumed to be continuous up to $\mathcal{U}\times\{0\}$ we have

$$\begin{aligned}\lim_{t\to 0} L_j(G)(z,w(z,s,t)) &= \lim_{t\to 0} \frac{\partial}{\partial \bar{z}_j}+\sum_{l=1}^{k} F_{jl}\frac{\partial}{\partial s_l}(g^0+g^1)\\ &= L_j(g^0)+\lim_{t\to 0} L_j(g^1)=L_j(g^0).\end{aligned}$$

□

Since F is holomorphic as a function of w, we have

$$L_j(F)(z,w(z,s,t))=\frac{\partial}{\partial \bar{z}_j}F(z,w(z,s,t)). \tag{4.21}$$

F is continuous up to $\mathcal{U}\times\{0\}$ and its boundary values are f, a CR function. Hence by (4.21) and Lemma 15 we deduce that

$$\lim_{t\to 0}\frac{\partial}{\partial\bar z_j}F(z,w(z,s,t))=0. \tag{4.22}$$

Note that $\partial F/\partial\bar z_j$ is holomorphic as a function of w, so

$$\lim_{\substack{w''\to 0\\ (z,w)\in\mathfrak{M}'}}\frac{\partial}{\partial\bar z_j}F(z,w)=0. \tag{4.23}$$

Hence by (4.23), we have

$$\lim_{t\to 0}\frac{\partial}{\partial\bar z_j}c_\alpha(\xi,s,t,z)=0. \tag{4.24}$$

Therefore the proposition follows from (4.24) and the extension used in (4.15). □

We will drop the tilde in the extended version of F. We now finish the proof of Theorem 1. Fix a ball $B'\subset B$ around the origin in $\mathbb{C}^d$. Let $\chi\in\mathcal{C}_0^\infty(\mathbb{C}^d)$ be such that $\chi=1$ on B' and $\chi=0$ of B. Let $\mathcal{K}$ be the Bochner–Martinelli kernel in $\mathbb{C}_z^d$ (see for instance [14], Corollary 1.11), and define G by

$$G(z,w)=-\int_{\mathbb{C}^d}\chi(\zeta)\left(\frac{\partial}{\partial\bar z_1}F(\zeta,w),\dots,\frac{\partial}{\partial\bar z_d}F(\zeta,w)\right)\wedge\mathcal{K}(\zeta,z). \tag{4.25}$$

Then for $j=1,\dots,d$,

$$\frac{\partial}{\partial\bar z_j}F(z,w)=\frac{\partial}{\partial\bar z_j}G(z,w)\quad\text{for } (z,w)\in\mathfrak{M}'=B'\times\left(\bigcup_{\substack{\xi\in B'\\ s\in U\\ 0<t<T}}\mathfrak{w}(\xi,s,t)\right). \tag{4.26}$$

So by Proposition 14 we have

$$\lim_{\substack{w''\to 0\\ (z,w)\in\mathfrak{M}'}}G(z,w)=0.$$

Therefore the holomorphic extensions of f to $\mathfrak{M}'$ is given by $F-G$. This concludes the proof of Theorem 1.

5. Remarks and corollaries

Remark 16. The proof of Theorem 1 for $m>1$ is as follows: After a linear change of variables, there is no loss of generality assuming that $v=(0;0;(0,...,0,1))\in \mathbb{C}^d\times\mathbb{C}^k\times\mathbb{C}^m$. Construct the R_k as in Lemma 2 and define Δ by $\Delta=\sum_{j=1}^k R_j^2+\sum_{j=1}^m \partial^2/\partial t_j^2$. In $\mathbb{R}^k\times\mathbb{R}^m$, define an open set Ω and $\partial\Omega$ as previously. Let f be a CR function on $\mathfrak{N}$. Trivially, f is CR on $\mathfrak{N}\times\mathbb{R}^m$. Solve a Dirichlet problem on Ω with the restriction of f to $\partial\Omega$ as boundary data. We therefore obtain, by the above argument, a holomorphic extension of f to a wedge of direction v and of edge $\mathfrak{N}\times\mathbb{R}^{m-1}$. Hence the result in higher dimension.

Remark 17. The tools used in this argument imply that the boundary values of the holomorphic function H is the CR function we wish to extend only in a small subset of the part of $\partial\Omega$ which is a CR submanifold of $\overline{\Omega}$.

We now give a few corollaries, arguments for proofs may be found in [9].

Corollary 18. *Let $N=\{(Z,h(Z)):Z\in\mathfrak{N}\}$ be a smooth ($\mathcal{C}^\infty$) non-generic* CR *submanifold of $\mathbb{C}^n$. If f is a* CR *distribution on N, then for any $p\in N$ and any v complex transversal to N at p, there exists a wedge $\mathcal{W}$ of direction v whose edge contains a neighborhood of p in N and $F\in\mathcal{O}(\mathcal{W})$, such that the boundary values of F on N are f.*

Corollary 19. *Let N be non-generic* CR *submanifold of $\mathbb{C}^L$ and let v be a complex transversal vector to N at p. If m is the complex codimension of* $\operatorname{span}_{\mathbb{C}} T_pN$ *in $\mathbb{C}^n$, then there exists a wedge $\mathcal{W}$ of direction v, and $\{F_l\}_{l=1}^m$, F_l holomorphic in $\mathcal{W}$, such that $dF_1\wedge...\wedge dF_m\neq0$ on $\mathcal{W}$ and each F_l vanishes on N near p.*

Corollary 20. *Let M be a smooth ($\mathcal{C}^\infty$) generic submanifold of $\mathbb{C}^n$ containing, through some $p\in M$, a proper smooth ($\mathcal{C}^\infty$)* CR *submanifold N of the same* CR *dimension and of codimension m. Let $v\in T_pM\setminus T_pN$. Then there exists a wedge $\mathcal{W}$ in M of direction v such that:*

(a) *any continuous* CR *function of N near p admits a $\mathcal{C}^\infty$* CR *extension to $\mathcal{W}$ near p, the extension is continuous up to N;*

(b) *if m is the complex codimension of* $\operatorname{span}_{\mathbb{C}} T_pN$ *in $\mathbb{C}^L$, then there exists a collection $\{g_l\}_{l=1}^m$ of smooth* CR *functions on $\mathcal{W}$, vanishing on N near p such that $dg_1\wedge...\wedge dg_m\neq0$ on $\mathcal{W}$.*

Suppose now that N is of codimension one in M. If u is a continuous CR *function near p in N, then u admits a continuous* CR *extension to a neighborhood of p in M. Furthermore, N cuts M in two parts: M^+ and M^-. There exists a*

$\mathcal{C}^\infty$ smooth CR *function φ of M in a neighborhood of p such that $\varphi\equiv 0$ on M^- and $\varphi\not\equiv 0$ on M^+ locally near p. Such a function is an example of a smooth* CR *function that admits no holomorphic wedge extension near p.*

Remark 21. The function φ does not extend holomorphically to any neighborhood of N. This situation differs greatly from the holomorphic extension situation in the generic case. Consider Trépreau's example given by

$$\mathfrak{M}=\{(z, s_1+is_2|z|^2, s_2-is_1|z|^2):(z,s_1,s_2)\in\mathbb{C}\times\mathbb{R}\times\mathbb{R}\}.$$

Let $\mathcal{W}$ be a wedge attached to $\mathfrak{M}$ in a neighborhood of the origin. Then any function, which is holomorphic in $\mathcal{W}$, extends holomorphically to a full neighborhood of the origin (see [18] for details). However, if we now consider the CR graph of $\mathfrak{M}$, $M=\{(Z,h(Z)):Z\in\mathfrak{M}\}$, where h is any CR function on $\mathfrak{M}$. Then, for any v complex transversal to M at the origin, the graphing function h extends holomorphically to a wedge $\mathcal{W}_v$ of direction v. Without loss of generality we may assume that, if we have chosen for coordinates $(z,w',w'')\in\mathbb{C}\times\mathbb{C}^2\times\mathbb{C}$, $\operatorname{Re}(w''-h)>0$ in $\mathcal{W}_v$. Set $g=\exp(-1/w''-h)$. Then g vanishes to infinite order on M, and hence does not extend holomorphically to a full neighborhood of M.

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