

The topological center of the spectrum of some distal algebras

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Abstract. The topological center of the spectrum of the Weyl algebra W , i.e. the norm closure of the algebra generated by the set of functions $\{n \mapsto \lambda^{n^i}; \lambda \in \mathbb{T} \text{ and } i \in \mathbb{N}\}$, is characterized in a recent paper by JABBARI and NAMIOKA (Ellis group and the topological center of the flow generated by the map $n \mapsto \lambda^{n^k}$, to appear in *Milan J. Math.*). By the techniques essentially used in the cited paper, the topological center of the spectrum of the subalgebra W_k , the norm closure of the algebra generated by the set of functions $\{n \mapsto \lambda^{n^i}; \lambda \in \mathbb{T} \text{ and } i = 0, 1, 2, \dots, k\}$, will be characterized, for all $k \in \mathbb{N}$. Also an example of a non-minimal dynamical system, with the enveloping semigroup Σ , for which the set of all continuous elements of Σ is not equal to the topological center of Σ , is given.

1. Introduction

The history of characterizing the topological center of the spectrum of a left-invariant non-distal algebra on a group G goes back to a paper by Lau, Milnes and Pym [18] (see also [19]). They showed that the topological center of the spectrum of the largest compactification of any locally compact group G equals G itself. But, similar results for distal algebras are rather meager (see [11]). A distal algebra is a left-invariant conjugate-closed Banach algebra of distal functions. The spectrum of any distal subalgebra A of $\ell^\infty(\mathbb{Z})$, the Banach algebra of all bounded complex-valued functions on \mathbb{Z} , is an example of a compact admissible right-topological group. Namioka [22], and also Milnes and Pym in [20] and [21], have studied the structure of compact admissible right-topological groups, and Lau and Loy [17] have investigated harmonic analysis on these groups.

Distal functions on groups were first introduced by Auslander and Hahn in [1]. By definition, a bounded function on a group G is *distal* if whenever $\{g_n\}_{n=1}^\infty$,

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$\{h_i\}_{i=1}^\infty$ and $\{k_j\}_{j=1}^\infty$ are nets in G such that for all $g \in G$,

$$\lim_n \lim_i f(gg_n h_i) = \lim_n \lim_j f(gg_n k_j)$$

then $\lim_i f(gh_i) = \lim_j f(gk_j)$, for all g in G . Later, Knapp [14] gave an analysis and synthesis of distal functions on groups. He showed that the set of all distal functions on a group G is itself a distal algebra. For a general reference on distal functions on semigroups see [2].

Let \mathbb{T} denote the unit circle in the complex plane. Then \mathbb{T} is a compact topological group under the complex multiplication. Let W denote the norm closure of the algebra generated by the set of functions $\{n \mapsto \lambda^{n^i}; \lambda \in \mathbb{T} \text{ and } i \in \mathbb{N}\}$, where, as usual, \mathbb{N} is the set of all positive integers. The algebra W has been studied from different aspects: Knapp [14] showed that all of the elements of W are distal. Later, Namioka [22, Theorem 3.6] gave a simpler proof of this fact. By using a result of Furstenberg [6], Salehi [24] showed that all of the elements of W , called the Weyl algebra, are uniquely ergodic, and he derived that W does not exhaust all distal functions on $(\mathbb{Z}, +)$ [24, Theorem 2.14]. In [12] (see also [10]), by giving a characterization of the Weyl algebra in a more general setting of semitopological semigroups, the authors showed that the algebra W is actually a distal algebra. In a recent work of Isaac Namioka joint with the author [11], they proved that, for each irrational member λ of the unit circle, the shift-orbit closure X_f of the function $f(n) = \lambda^{n^k}$ is homeomorphic to a k -torus. Using this homeomorphism, by generalizing an interesting result of Namioka [23], they also characterized the topological center of the dynamical system X_f , as well as the topological center of the spectrum of the Weyl algebra W .

Throughout this paper we fix $k \in \mathbb{N}$, $k > 1$. Let W_k be the norm closure of the algebra generated by the set of functions $\{n \mapsto \lambda^{n^i}; \lambda \in \mathbb{T} \text{ and } i = 0, 1, 2, \dots, k\}$. In [12] (see also [10]), this algebra is generalized to arbitrary semitopological semigroups. It is proved that W_k is left invariant, and hence is a distal algebra [12, Theorem 3.5].

The aim of the present paper is to determine the topological center of the spectrum $M(W_k)$ of the distal algebra W_k (Theorem 1.1). To this end, we characterize $M(W_k)$ with the Ellis group $\Sigma(W_k, U)$ of the distal flow (W_k, U) [11, Proposition 5.3], where $U: W_k \rightarrow W_k$ is the shift operator defined by $U(g)(n) = g(n+1)$, for all $g \in W_k$ and all $n \in \mathbb{Z}$. Then we show that $\Sigma(W_k, U)$ is homeomorphically isomorphic to a subgroup of $E(\mathbb{T})^k$ (Theorem 3.1), where $E(\mathbb{T})$ is the set of all endomorphisms of the group \mathbb{T} .

Note that $M(W_k)$ is the set of all multiplicative means μ on W_k , that is, $\mu(1) = 1$, $\mu(f) \geq 0$ whenever $f \in W_k$ and $f \geq 0$, and $\mu(fg) = \mu(f)\mu(g)$ for all $f, g \in W_k$. $M(W_k)$ is a weak* compact subsemigroup of W_k^* , with the product $\langle \mu * \nu, f \rangle = \langle \mu, T_\nu f \rangle$, where $T_\nu f(n) = \nu(U^n f)$ for all $\mu, \nu \in M(W_k)$, $f \in W_k$ and $n \in \mathbb{Z}$.

Let H be the torsion subgroup of \mathbb{T} , and let V denote the quotient space \mathbb{T}/H . Then V is a vector space over \mathbb{Q} . Let $\text{Hom}(V, \mathbb{T})$ denote the set of all \mathbb{Q} -vector space homomorphisms from V into \mathbb{T} . The following is the main result of this paper.

Theorem 1.1. *Let $k \in \mathbb{N}$, with $k > 1$. The topological center of $M(W_k)$ is isomorphic to the group $\mathbb{Z} \times \text{Hom}(V, \mathbb{T})$ with the group structure*

$$(p, \theta)(q, \theta') = (p+q, \theta\theta'(\cdot)^{(p+q)^k - p^k - q^k}).$$

We remark that the topological center of $M(W)$ is essentially characterized in [11].

Finally, we prove that the set of all continuous elements of $\Sigma(W_k, U)$ is isomorphic to \mathbb{Z} (Theorem 3.10), which along with Theorem 1.1 shows that there exists a non-minimal dynamical system with the Ellis group Σ such that the topological center of Σ is different from Σ_c , the set of all continuous elements of Σ (see [11, Lemma 4.3]).

Remark. The notion of topological center began with the earlier works of Isik–Pym–Ülger [9], Lau [15] and Lau–Losert [16]. In recent years there has been significant interest in the subject. Interested readers are referred to the recent papers and references contained in [3] and [4].

2. Preliminaries

A *dynamical system* is a pair (X, T) , where X is a Hausdorff space and T is a homeomorphism from X onto X . The dynamical system (X, T) is said to be *compact* if X is a compact Hausdorff space. We always assume that the space X^X is provided with the product topology. The closure $\Sigma(X, T)$ of the set $\{T^n; n \in \mathbb{Z}\}$ is a sub-semigroup of X^X called the *enveloping semigroup* of the dynamical system (X, T) . With the relativization of the product topology from X^X , the mapping $\sigma \mapsto \sigma \circ \tau: \Sigma \rightarrow \Sigma$ is continuous for all $\tau \in \Sigma$, in other words, $\Sigma = \Sigma(X, T)$ is a Hausdorff right-topological semigroup with the topology of pointwise convergence; and it is not left-topological, in general. The *topological center* $\Lambda(\Sigma)$ of Σ is defined as follows,

$$\Lambda(\Sigma) = \{\tau \in \Sigma; \sigma \mapsto \tau \circ \sigma: \Sigma \rightarrow \Sigma \text{ is continuous}\}.$$

$\Lambda(\Sigma)$ is a sub-semigroup of Σ containing $\{T^n; n \in \mathbb{Z}\}$. Also, for each x in X , the orbit closure of x is $\Sigma(x)$ and the map $\sigma \mapsto \sigma(x): \Sigma \rightarrow X$ is continuous. A compact dynamical system (X, T) is called *distal* if $\lim_{\alpha} T^{n_{\alpha}}x = \lim_{\alpha} T^{n_{\alpha}}y$ for some net $\{n_{\alpha}\}_{\alpha}$ in \mathbb{Z} and $x, y \in X$ implies that $x = y$. It was Ellis [5, Proposition 5.3] who showed

that a compact dynamical system is distal if and only if its enveloping semigroup is a group (whose identity is the identity mapping of X). The enveloping semigroup of a distal dynamical system is called the *Ellis group* of the dynamical system. A closed non-empty subset M of a dynamical system (X, T) is called *minimal* if M is invariant (i.e. $T^n M \subseteq M$ for all $n \in \mathbb{Z}$) and no proper closed subset of M is invariant. It is readily seen that M is minimal if and only if $\Sigma(x) = M$ for each x in M . A *minimal dynamical system* is a dynamical system (X, T) for which the phase space X is minimal. Two dynamical systems (X, T) and (X', T') , or briefly X and X' , are *isomorphic* if there exists a homeomorphism $\Gamma: X \rightarrow X'$ such that $\Gamma \circ T = T' \circ \Gamma$.

By $\ell^\infty(\mathbb{Z})$ (or ℓ^∞), we mean the Banach space of all bounded complex-valued functions on \mathbb{Z} , with the supremum norm. The topology of ℓ^∞ is the weak* topology, where ℓ^∞ is regarded as the dual space of $\ell^1(\mathbb{Z})$. Recall that the weak* topology of ℓ^∞ coincides with the topology of pointwise convergence on norm-bounded subsets. Define the shift operator $U: \ell^\infty \rightarrow \ell^\infty$ by $U(g)(n) = g(n+1)$ for all $g \in \ell^\infty$ and all $n \in \mathbb{Z}$. It is clear that the shift operator U is a continuous map of ℓ^∞ into itself. Therefore the pair $(\ell^\infty(\mathbb{Z}), U)$ is a dynamical system. Namioka [22, Lemma 3.1] showed that the enveloping semigroup $\Sigma(\ell^\infty, U)$ of the flow (ℓ^∞, U) is compact. We observe here that each element of $\Sigma(\ell^\infty, U)$ is a multiplicative bounded linear transformation of norm 1 of the Banach space ℓ^∞ into itself. The object of the present paper is the dynamical system (W_k, U) , where W_k is given the topology induced by that of ℓ^∞ . With a proof similar to the proof of [11, Theorem 5.1], one can readily verify that $\Sigma(W_k, U)$ is a compact right-topological group. Also, it is easily seen that the topological center of $\Sigma(W_k, U)$ coincides with its “center” in the group theoretic sense (just because of the commutativity of the acting group \mathbb{Z}).

By giving a new characterization of Weyl algebras in a more general setting of semitopological semigroups S , it is shown in [12] that, for each $\sigma \in \Sigma(\ell^\infty, U)$, $\sigma(W_k) \subset W_k$, that is W_k is left-invariant and therefore a distal algebra. Hence the enveloping semigroup $\Sigma(W_k, U)$ of the dynamical system (W_k, U) is $\{\sigma|_{W_k}; \sigma \in \Sigma(\ell^\infty, U)\}$, where $\sigma|_{W_k}$ denotes the restriction of σ to W_k .

For each $f \in \ell^\infty$, let X_f denote the orbit closure of f with respect to the shift mapping U . Then $U(X_f) \subseteq X_f$. Therefore (X_f, U) is a dynamical system as well. In fact $X_f = \{\sigma f; \sigma \in \Sigma(\ell^\infty, U)\}$. It follows that $X_f \subset W_k$ for each $f \in W_k$. Thus $\Sigma(X_f, U) = \{\sigma|_{X_f}; \sigma \in \Sigma(W_k, U)\}$. Now, because of the continuity of the restriction mapping, the enveloping semigroups $\Sigma(W_k, U)$ and $\Sigma(X_f, U)$, for $f \in W_k$, are also compact. Finally, we remark that a function $f \in \ell^\infty$ is distal if and only if the dynamical system (X_f, U) is distal. The structure of distal flows is essentially studied in [7].

3. The Ellis group of the dynamical system (W_k, U)

Let us recall some results from [11]. The binomial coefficients are extended as follows: Let $j \in \mathbb{N}$ and $n \in \mathbb{Z}$, then

$$\binom{n}{0} = 1 \quad \text{and} \quad \binom{n}{j} = \frac{1}{j!} n(n-1)\dots(n-j+1).$$

Fix an irrational (i.e. a non-root of unity) element $\lambda \in \mathbb{T}$. Define $f \in \mathbb{T}^{\mathbb{Z}}$ by $f(n) = \lambda^{n^k}$ for all $n \in \mathbb{Z}$. It is a result of [11] that there exists a continuous map $T: \mathbb{T}^k \rightarrow \mathbb{T}^k$ such that the mapping $\Gamma: \mathbb{T}^k \rightarrow X_f$ defined by

$$(1) \quad \Gamma(x_1, x_2, \dots, x_k)(n) = \lambda^{n^k} x_1^{Q_{k-1}(n)} x_2^{Q_{k-2}(n)} \dots x_{k-1}^{Q_1(n)} x_k$$

(for $(x_1, x_2, \dots, x_k) \in \mathbb{T}^k$ and $n \in \mathbb{Z}$) is an isomorphism between the dynamical systems (\mathbb{T}^k, T) and (X_f, U) [11, Theorem A], where for each $n \in \mathbb{Z}$ and $j \in \{0\} \cup \mathbb{N}$,

$$(2) \quad Q_j(n) = \binom{n + [j/2]}{j}.$$

By using this isomorphism, it is also proved that the mapping $\Theta: \Sigma(X_f, U) \rightarrow E(\mathbb{T})^{k-1} \times \mathbb{T}$ defined by $\Theta(\sigma) = (\theta_1, \theta_2, \dots, \theta_{k-1}, u)$ is a homeomorphic embedding into $E(\mathbb{T})^{k-1} \times \mathbb{T}$ [11, Theorem B], where $(\theta_1, \theta_2, \dots, \theta_{k-1}, u)$ is associated with $\sigma = \lim_{\alpha} U^{m_{\alpha}} \in \Sigma$, that is for each $j \in \{1, 2, \dots, k-1\}$, $\theta_j \in \mathbb{T}^{\mathbb{T}}$, is given by

$$(3) \quad \theta_j(x) = \lim_{\alpha} x^{Q_j(m_{\alpha})},$$

and

$$(4) \quad u = \lim_{\alpha} \lambda^{m_{\alpha}^k} \in \mathbb{T}.$$

Finally, if $\sigma \in \Lambda(\Sigma(X_f, U))$, then there exists an integer p such that $\theta_j = (\)^{Q_j(p)}$, for each $j \in \{1, 2, \dots, k-1\}$ [11, Lemma 4.5].

To characterize the topological center of $\Sigma(W_k, U)$, we need some preliminaries. Let $\sigma \in \Sigma(W_k, U)$, and let $\{m_{\alpha}\}_{\alpha}$ be a net in \mathbb{Z} such that $\sigma = \lim_{\alpha} U^{m_{\alpha}}$. By taking a subnet of m_{α} if necessary, we may assume that for each $j \in \{0, 1, \dots, k-1\}$, $\lim_{\alpha} x^{Q_j(m_{\alpha})}$ exists for each $x \in \mathbb{T}$ and $\lim_{\alpha} x^{m_{\alpha}^k}$ exists for each $x \in \mathbb{T}$. For each $j \in \{0, 1, \dots, k-1\}$ define $\theta_j \in \mathbb{T}^{\mathbb{T}}$ by

$$(5) \quad \theta_j(x) = \lim_{\alpha} x^{Q_j(m_{\alpha})}$$

for all $x \in \mathbb{T}$. Also define $\theta: \mathbb{T} \rightarrow \mathbb{T}$ by

$$(6) \quad \theta(x) = \lim_{\alpha} x^{m_{\alpha}^k}.$$

With this introduction, we can prove the next theorem.

Theorem 3.1. *Let $\Sigma = \Sigma(W_k, U)$, and define the mapping $\Theta_k: \Sigma \rightarrow E(\mathbb{T})^k$ by $\Theta_k(\sigma) = (\theta_1, \dots, \theta_{k-1}, \theta)$, where $\sigma, \theta_1, \dots, \theta_{k-1}$ and θ are defined as in the paragraph preceding the theorem. Then Θ_k is a homeomorphic embedding into $E(\mathbb{T})^k$.*

Proof. Let $\sigma = \lim_{\alpha} U^{m_{\alpha}} = \lim_{\beta} U^{n_{\beta}} \in \Sigma$. Let $y \in \mathbb{T}$, and let $j \in \{1, 2, \dots, k-1\}$. Let $f \in \mathbb{T}^{\mathbb{Z}}$ be the function defined by $f(n) = y^{j!Q_j(n)}$. Since $j!Q_j(n)$ is a polynomial of degree j with integral coefficients, one has $f \in W_k$. Hence $\lim_{\alpha} U^{m_{\alpha}}(f)(0) = \lim_{\beta} U^{n_{\beta}}(f)(0)$. Thus $\lim_{\alpha} y^{j!Q_j(m_{\alpha})} = \lim_{\beta} y^{j!Q_j(n_{\beta})}$. Since each $x \in \mathbb{T}$ can be written of the form $y^{j!}$ for some $y \in \mathbb{T}$, it follows that

$$(7) \quad \lim_{\alpha} x^{Q_j(m_{\alpha})} = \lim_{\beta} x^{Q_j(n_{\beta})}.$$

Again let $x \in \mathbb{T}$. Then the function $g \in \mathbb{T}^{\mathbb{Z}}$ defined by $g(n) = x^{n^k}$, $n \in \mathbb{Z}$, is an element of W_k . Hence $\lim_{\alpha} U^{m_{\alpha}}(g)(0) = \lim_{\beta} U^{n_{\beta}}(g)(0)$. Therefore $\lim_{\alpha} x^{m_{\alpha}^k} = \lim_{\beta} x^{n_{\beta}^k}$. From this and (7), it follows that $\Theta_k(\sigma)$ depends only on σ , not on the choice of the net representing σ . Hence Θ_k is well-defined. To show that Θ_k is one-to-one, assume that $\sigma, \tau \in \Sigma$ are such that $\Theta_k(\sigma) = \Theta_k(\tau) = (\theta_1, \dots, \theta_{k-1}, \theta)$. Then we must prove that $\sigma = \tau$. Since σ and τ are bounded linear transformations on W_k , and since W_k is the norm-closed subalgebra of ℓ^{∞} generated by the set $A_k = \{n \mapsto x^{n^i}; 0 \leq i \leq k \text{ and } x \in \mathbb{T}\}$, it is enough to show that $\sigma(f) = \tau(f)$ for all $f \in A_k$. To this end, fix $x \in \mathbb{T}$. First, let f be the function $n \mapsto x^{n^k}$. Then for each $m \in \mathbb{Z}$, $(U^m f)(n) = x^{(n+m)^k} = f(n)x^{S(m)}x^{m^k}$ with $S \in \mathcal{S}(k-1)$, where $\mathcal{S}(k-1)$ is the set of all integral linear combinations of the functions Q_1, Q_2, \dots, Q_{k-1} , as defined in Section 4 of [11]. Let $S(m) = \sum_{j=1}^{k-1} a_j Q_j(m)$ with $a_j \in \mathbb{Z}$. Let $\{m_{\alpha}\}_{\alpha}$ be a net in \mathbb{Z} such that $\lim_{\alpha} U^{m_{\alpha}}(g) = \sigma(g)$ for each $g \in W_k$, as in the beginning of this proof, then

$$\sigma(f)(n) = f(n) \lim_{\alpha} x^{S(m_{\alpha})} \lim_{\alpha} x^{m_{\alpha}^k} = f(n) \prod_{j=1}^{k-1} \theta_j(x^{a_j}) \theta(x).$$

Similarly $\tau(f)(n) = f(n) \prod_{j=1}^{k-1} \theta_j(x^{a_j}) \theta(x)$. Therefore by the hypothesis, $\sigma(f) = \tau(f)$. Now, for $i \in \{1, 2, \dots, k-1\}$, let $f_i \in A_k$ be defined by $f_i(n) = x^{n^i}$. A similar, but simpler, proof applies to show that $\sigma(f_i) = \tau(f_i)$, for all $i = 1, 2, \dots, k-1$. Hence $\sigma = \tau$. Finally, to show that Θ_k is continuous, we must prove that for each $x \in \mathbb{T}$ the map $\varphi: \sigma \mapsto \theta(x)$ and the maps $\varphi_j: \sigma \mapsto \theta_j(x)$, for $j = 1, 2, \dots, k-1$, are continuous on Σ . For the latter, fix $j = 1, 2, \dots, k-1$ and $x, y \in \mathbb{T}$ with $x = y^{j!}$. If $g \in W_k$ is defined by $g(n) = y^{j!Q_j(n)}$, then $\varphi_j(\sigma) = \theta_j(x) = \theta_j(y^{j!}) = \sigma(g)(0)$, hence the continuity of φ_j follows from the continuity of the map $\sigma \mapsto \sigma(g)(0)$ on Σ . It remains to prove that $\varphi: \sigma \mapsto \theta(x)$ is continuous for all $x \in \mathbb{T}$. To this end, fix $x \in \mathbb{T}$. Define $f \in W_k$ by $f(n) = x^{n^k}$. Then $\varphi(\sigma)(x) = \theta(x) = \sigma(f)(0)$. Hence the continuity of φ follows from

the continuity of the map $\sigma \mapsto \sigma(f)(0)$ on $\Sigma(W_k, U)$. That is, Θ_k is continuous on Σ . \square

In the next lemma, we show that for an element σ , with $\Theta_k(\sigma) = (\theta_1, \dots, \theta_{k-1}, \theta)$, to be in the topological center of Σ it is enough that the first $k-1$ components are continuous elements of $E(\mathbb{T})$.

Lemma 3.2. *Let Σ and $\Theta_k: \Sigma \rightarrow E(\mathbb{T})^k$ be as in Theorem 3.1. Let $\sigma \in \Sigma$ and let $\Theta_k(\sigma) = (\theta_1, \dots, \theta_{k-1}, \theta)$. Then $\sigma \in \Lambda(\Sigma)$ if and only if θ_i is a continuous endomorphism of \mathbb{T} for each $i \in \{1, 2, \dots, k-1\}$.*

Proof. Let $\sigma \in \Lambda(\Sigma)$. An observation similar to the one at the beginning of the proof of [11, Theorem E], with W replaced by W_k , shows that $\sigma|_{X_f} \in \Lambda(\Sigma(X_f, U))$ for each $f \in W_k$. Let λ be an irrational member of \mathbb{T} , and let $f(n) = \lambda^{n^k}$. Then $\sigma|_{X_f} \in \Lambda(\Sigma(X_f, U))$. Now we can apply Lemma 4.4 of [11]. (In that lemma, $\sigma \in \Sigma(\mathbb{T}^k, T)$ is expressed as $\sigma = \lim_{\alpha} T^{m_{\alpha}}$, but since, by [11, Theorem A], the dynamical systems (X_f, U) and (\mathbb{T}^k, T) are isomorphic, this is equivalent to writing $\sigma|_{X_f} = \lim_{\alpha} U^{m_{\alpha}}$ in this section. Recall that $U: \mathbb{T}^{\mathbb{Z}} \rightarrow \mathbb{T}^{\mathbb{Z}}$ is the shift map and $T: \mathbb{T}^k \rightarrow \mathbb{T}^k$ is the map satisfying $U \circ \Gamma = \Gamma \circ T$, where $\Gamma: \mathbb{T}^k \rightarrow X_f$ is the isomorphism of Theorem 1.1 in [11], as defined in (1) of the present paper.) Let $\Theta_k(\sigma|_{X_f}) = (\theta_1, \dots, \theta_{k-1}, u)$ as in Theorem B of [11] (or as illustrated at the beginning of this section). Then by [11, Lemma 4.4] $\theta_1, \theta_2, \dots, \theta_{k-1}$ are all continuous. Conversely, let $\sigma \in \Sigma$, let $\Theta_k(\sigma) = (\theta_1, \dots, \theta_{k-1}, \theta)$, and assume that $\theta_i \in E(\mathbb{T})$ is continuous for all $i = 1, 2, \dots, k-1$. Hence for each $i = 1, 2, \dots, k-1$, $\theta_i = (\cdot)^{n_i}$ for some $n_i \in \mathbb{Z}$. As we remarked in Section 2, $\Sigma = \Sigma(W_k, U)$ is a compact right-topological group, and the topological center of Σ coincides with its center in the group theoretic sense. Hence, to prove that $\sigma \in \Lambda(\Sigma)$ is to prove that $\sigma \circ \tau = \tau \circ \sigma$ for all $\tau \in \Sigma$. So fix $\tau = \lim_{\beta} U^{n_{\beta}} \in \Sigma$ and let $\Theta_k(\tau) = (\theta'_1, \dots, \theta'_{k-1}, \theta')$. Since σ and τ are bounded linear transformations on W_k and since W_k is generated by $A_k = \{n \mapsto x^{n^i}; x \in \mathbb{T} \text{ and } i = 0, 1, 2, \dots, k\}$, it is enough to show that $\sigma \circ \tau(f) = \tau \circ \sigma(f)$ for all $f \in A_k$. First note that for each $i \in \{1, 2, \dots, k\}$ the polynomial m^i is an element of $\mathcal{S}(i)$, the set of all integral linear combinations of $Q_1(m), \dots, Q_i(m)$ [11]. (For instance, $m = Q_1(m) \in \mathcal{S}(1)$, $m^2 = 2Q_2(m) - Q_1(m) \in \mathcal{S}(2), \dots$) For simplicity, we shall confine the proof for the special case $k=3$, since it contains all the necessary ideas. So let $k=3$, fix $x \in \mathbb{T}$ and let $f \in W_k$ be defined by $f(n) = x^{n^3}$. Let σ and τ be as above. Then

$$\begin{aligned}
 \sigma \circ \tau(f)(n) &= \lim_{\alpha} \lim_{\beta} U^{(m_{\alpha} + n_{\beta})} f(n) = \lim_{\alpha} \lim_{\beta} f(m_{\alpha} + n_{\beta} + n) \\
 (8) \qquad &= \lim_{\alpha} \lim_{\beta} x^{(m_{\alpha} + n_{\beta} + n)^3}.
 \end{aligned}$$

But, for each $n \in \mathbb{Z}$ and for all α, β one has

$$\begin{aligned} (m_\alpha + n_\beta + n)^3 &= n^3 + 3n^2(Q_1(m_\alpha) + Q_1(n_\beta)) \\ &\quad + 3n(2Q_2(m_\alpha) - Q_1(m_\alpha) + 2Q_1(m_\alpha)Q_1(n_\beta) + 2Q_2(n_\beta) - Q_1(n_\beta)) \\ &\quad + 3(2Q_2(m_\alpha) - Q_1(m_\alpha))Q_1(n_\beta) + 3Q_1(m_\alpha)(2Q_2(n_\beta) - Q_1(n_\beta)) \\ &\quad + m_\alpha^3 + n_\beta^3. \end{aligned}$$

Hence

$$(9) \quad \sigma \circ \tau(f)(n) = f(n)[(\theta_1 \theta'_1)^{3n(n-1)}(\theta_2 \theta'_2)^{6n}(\theta_1 \circ \theta'_1)^{6(n-1)}(\theta_2 \circ \theta'_2)^6(\theta_1 \circ \theta'_2)^6 \theta \theta'](x).$$

Similarly

$$(10) \quad \tau \circ \sigma(f)(n) = f(n)[(\theta_1 \theta'_1)^{3n(n-1)}(\theta_2 \theta'_2)^{6n}(\theta'_1 \circ \theta_1)^{6(n-1)}(\theta'_2 \circ \theta_2)^6(\theta'_1 \circ \theta_2)^6 \theta \theta'](x).$$

Now, since $\theta_i = (\cdot)^{n_i}$ one has $\theta_i \circ \theta'_j(x) = \theta'_j(x)^{n_i} = \theta'_j(x^{n_i}) = \theta'_j \circ \theta_i(x)$. Therefore, it follows from (9) and (10) that $\sigma \circ \tau(f) = \tau \circ \sigma(f)$. A similar argument shows that $\sigma \circ \tau(g) = \tau \circ \sigma(g)$ for all $g \in A_k$ defined by $g(n) = x^{n^i}$, with $i = 1, 2, \dots, k-1$. That is $\sigma \circ \tau = \tau \circ \sigma$. Thus $\sigma \in \Lambda(\Sigma(W_k, U))$. \square

The proof of the next lemma is similar to [11, Lemma 4.5], but we shall give it here for the sake of completeness.

Lemma 3.3. *Using the notation of Lemma 3.2, let σ be an element of Σ and let $\Theta_k(\sigma) = (\theta_1, \dots, \theta_{k-1}, \theta)$. Then $\sigma \in \Lambda(\Sigma)$ if and only if there exists an $n \in \mathbb{Z}$ such that $\theta_j(x) = x^{Q_j(n)}$ for each $j \in \{1, 2, \dots, k-1\}$ and each $x \in \mathbb{T}$.*

Proof. Assume that $\sigma \in \Lambda(\Sigma)$. Then by the previous lemma, each θ_j is continuous. This means that for some $n_j \in \mathbb{Z}$; $\theta_j(x) = x^{n_j}$ for each $x \in \mathbb{T}$. Hence it is enough to show that for each $j = 1, 2, \dots, k-1$, $(\cdot)^{n_j} = (\cdot)^{Q_j(n_1)}$. Let $\{m_\alpha\}_\alpha$ be a net in \mathbb{Z} such that $\sigma = \lim_\alpha U^{m_\alpha}$. Fix a prime number $p > (k-1)!$. Then for $\eta = e(1/p)$, $\theta_1(\eta) = \lim_\alpha \eta^{m_\alpha} = \eta^{n_1}$, hence $m_\alpha = n_1 \pmod p$ eventually. Here recall that for each $t \in \mathbb{R}$, $e(t) = e^{2\pi it}$. Note that $\mathbb{Z} \pmod p$ is a field. Since $p > j!$, the division by $j!$ is well defined in $\mathbb{Z} \pmod p$, and we see that $Q_j(m_\alpha) = Q_j(n_1) \pmod p$ eventually, for $j = 2, \dots, k-1$. Therefore $\lim_\alpha e((1/p)Q_j(m_\alpha)) = e((1/p)Q_j(n_1))$. It follows that for each integer q with $0 < q < p$, one has $\lim_\alpha e((q/p)Q_j(m_\alpha)) = e((q/p)Q_j(n_1))$. On the other hand $\lim_\alpha e((q/p)Q_j(m_\alpha)) = \theta_j(e(q/p)) = e(q/p)^{n_j}$. Thus $e(q/p)^{Q_j(n_1)} = e(q/p)^{n_j}$. Now since $x \mapsto x^{Q_j(n_1)}$ and $x \mapsto x^{n_j}$ are both continuous functions on \mathbb{T} and the set $\{e(q/p); 0 < q < p, p > (k-1)! \text{ and } p \text{ is prime}\}$ is dense in \mathbb{T} , we have $x^{Q_j(n_1)} = x^{n_j}$ for each $x \in \mathbb{T}$. The converse is clear from the previous lemma and the fact that for each j the map $x \mapsto x^{Q_j(n_1)}$ is continuous on \mathbb{T} . \square

Remark 3.4. Notice that the previous lemma implies that each element $\sigma \in \Lambda(\Sigma)$ determines a unique element (n, θ) of the space $\mathbb{Z} \times E(\mathbb{T})$, such that $\theta(x) = x^{n^k}$ for all $x \in H$, where H denotes the torsion subgroup of \mathbb{T} .

Let H be as above. Then $\mathbb{T} = H \times V$, where V is isomorphic to \mathbb{T}/H . Hence V is a divisible torsion-free subgroup. Consequently, it is a linear space over the rationals \mathbb{Q} [8, Appendix A]. As remarked in [11], for $v, w \in V$ and $q \in \mathbb{Q}$, the linear space “addition” $v + w$ is the complex multiplication vw and the multiplication by scalar qv is actually the power v^q . For instance, $\frac{1}{2}v$ is actually $v^{1/2}$ which is the unique member $u \in V$ such that $u^2 = v$.

Let $\text{Hom}(V, \mathbb{T})$ denote the set of all homomorphisms from the vector space V (over \mathbb{Q}) into \mathbb{T} . Then $\mathbb{Z} \times \text{Hom}(V, \mathbb{T})$ is a group with the group structure defined as follows: for $(p, \theta), (q, \theta') \in \mathbb{Z} \times \text{Hom}(V, \mathbb{T})$, $(p, \theta)(q, \theta') = (p + q, \theta\theta'(\cdot)^{(p+q)^k - (p^k + q^k)})$.

We are going to show that $\Lambda(\Sigma)$ is isomorphic to $\mathbb{Z} \times \text{Hom}(V, \mathbb{T})$. To this end, let $\Phi: \Lambda(\Sigma) \rightarrow \mathbb{Z} \times \text{Hom}(V, \mathbb{T})$ be defined by $\Phi(\sigma) = (n, \theta)$, where σ, n and θ are as in the above remark. To show that the map Φ is an isomorphism of groups, we need some preliminaries.

Note that we shall confine to the case $k = 3$, since this case exhibits all the necessary ideas (even the ideas for the case $k = 2$). The general case is then derived similarly.

Now, as in [11], let $\{v_\gamma; \gamma \in \Gamma\}$ be a \mathbb{Q} -basis of V . Similar to Lemmas (i) and (ii) of [11] we have the following lemmas.

Lemma 3.5. *For each $\varepsilon > 0$, each finite subset F of Γ , each $h \in \mathbb{N}$ and each $\theta \in \text{Hom}(V, \mathbb{T})$ there is an $m \in \mathbb{N}$ such that*

- (a) $m \equiv 0 \pmod{d}$ for each integer $d, 0 < d \leq h$;
- (b) $|(v_\gamma^{1/d})^m - 1| < \varepsilon$ for each $\gamma \in F$ and for each $d \in \{1, 2, \dots, h\}$;
- (c) $|(v_\gamma^{1/d})^{m^2} - 1| < \varepsilon$ for each $\gamma \in F$ and for each $d \in \{1, 2, \dots, h\}$;
- (d) $|(v_\gamma^{1/d})^{m^3} - \theta(v_\gamma^{1/d})| < \varepsilon$ for each $\gamma \in F$ and for each $d \in \{1, 2, \dots, h\}$.

Proof. Consider the following statements:

- (b') $|(v_\gamma^{1/h!})^m - 1| < \varepsilon/h!$ for each $\gamma \in F$;
- (c') $|(v_\gamma^{1/h!})^{m^2} - 1| < \varepsilon/h!$ for each $\gamma \in F$;
- (d') $|(v_\gamma^{1/h!})^{m^3} - \theta(v_\gamma^{1/h!})| < \varepsilon/h!$ for each $\gamma \in F$.

It is proved in [11, Lemma (i)] that the statements (b) and (c) are implied by (b') and (c'), respectively. We show that the statement (d) is also implied by (d'). First, recall that if $x, y \in \mathbb{T}$, then for each $n \in \mathbb{N}$, $|x^n - y^n| \leq n|x - y|$. Now assume (d'). Then

for each $d \in \{1, 2, \dots, h\}$,

$$|(v_\gamma^{1/d})^{m^3} - \theta(v_\gamma^{1/d})| = |((v_\gamma^{1/h!})^{m^3})^{h!/d} - (\theta(v_\gamma^{1/h!}))^{h!/d}| < \frac{h!}{d} \frac{\varepsilon}{h!} = \frac{\varepsilon}{d} \leq \varepsilon.$$

Let $F = \{\gamma_j; j \in \{1, 2, \dots, p\}\}$ for some $p \in \mathbb{N}$. For each $j \in \{1, 2, \dots, p\}$ choose $\xi_j \in [0, 1)$ such that $v_{\gamma_j}^{1/h!} = e(\xi_j)$. For $j \in \{1, 2, \dots, p\}$, let

$$\phi_j^1(t) = \xi_j(th!), \quad \phi_j^2(t) = \xi_j(th!)^2 \quad \text{and} \quad \phi_j^3(t) = \xi_j(th!)^3.$$

Then the polynomials $\phi_j^1(t), \phi_j^2(t)$ and $\phi_j^3(t), 1 \leq j \leq p$, satisfy the hypothesis of Satz 14 of H. Weyl [25] since $\xi_1, \xi_2, \dots, \xi_p$ are independent (mod 1) over \mathbb{Q} . It follows that the sequence

$$\{(\phi_1^1(n), \dots, \phi_p^1(n), \phi_1^2(n), \dots, \phi_p^2(n), \phi_1^3(n), \dots, \phi_p^3(n)); n \in \mathbb{N}\}$$

is dense in $(\mathbb{R}/\mathbb{Z})^{3p}$. Hence the image of this sequence under the map e is dense in \mathbb{T}^{3p} . Thus, for some $n, m = nh!$ satisfies conditions (a), (b'), (c'), and (d'). Hence the lemma is proved. \square

Lemma 3.6. *Given $\theta \in \text{Hom}(V, \mathbb{T})$, there is a net $\{n_\alpha\}_\alpha$ in \mathbb{N} such that*

- (a) $\lim_\alpha x^{n_\alpha} = 1$ for each $x \in \mathbb{T}$;
- (b) $\lim_\alpha x^{n_\alpha^2} = 1$ for each $x \in \mathbb{T}$;
- (c) $\lim_\alpha x^{n_\alpha^3} = \theta(x)$ for each $x \in V$.

Proof. As in the proof of [11, Lemma (ii)], let \mathcal{F} be the family of all finite subsets F of Γ and let $D = \mathcal{F} \times (0, \infty) \times \mathbb{N}$. Partially order D as follows: for $\alpha = (F, \varepsilon, h)$ and $\alpha' = (F', \varepsilon', h') \in D$, $\alpha \leq \alpha'$ if and only if $F \subset F', \varepsilon \geq \varepsilon'$ and $h \leq h'$. Then clearly D is a directed set. For each $\alpha = (F, \varepsilon, h)$, let n_α be an integer $m \in \mathbb{N}$ satisfying the conditions (a)–(d) of Lemma 3.5.

The fact that (a) and (b) are satisfied follows from a proof similar to the proof of Lemma (ii) of [11].

For each $\gamma \in \Gamma$ let V_γ be $\mathbb{Q}v_\gamma = \{v_\gamma^r; r \in \mathbb{Q}\}$. Suppose $x \in V$. Then there is an $F \in \mathcal{F}$ such that $x \in \prod\{V_\gamma; \gamma \in F\}$. It remains to prove (c) for the case $x \in V_\gamma$ for each $\gamma \in \Gamma$, so let $x \in V_\gamma$. Then $x = (v_\gamma^{1/d})^c$ for some $c \in \mathbb{Z}$ and $d \in \mathbb{N}$. Hence in order to prove (c) for $x \in V_\gamma$, it is sufficient to show (c) for $x = v_\gamma^{1/d}$. Let $\varepsilon > 0$ and let $\alpha = (\{\gamma\}, \varepsilon, d) \in D$. If $\beta = (G, \delta, h) \geq \alpha$ in D , then $\gamma \in G, \delta \leq \varepsilon$ and $d \leq h$. Hence by Lemma 3.5(d), $|(v_\gamma^{1/d})^{n_\beta^3} - \theta(v_\gamma^{1/d})| < \delta \leq \varepsilon$ whenever $\beta \geq \alpha$. This completes the proof of (c). \square

Corollary 3.7. *Let $(q, \theta) \in \mathbb{Z} \times \text{Hom}(V, \mathbb{T})$. Then there is a net $\{m_\alpha\}_\alpha$ in \mathbb{N} such that for all $x \in \mathbb{T}$,*

$$\lim_\alpha x^{m_\alpha} = x^q, \quad \lim_\alpha x^{m_\alpha^2} = x^{q^2} \quad \text{and} \quad \lim_\alpha v^{m_\alpha^3} = \theta(v) \text{ for each } v \in V.$$

Proof. Let $\{n_\alpha\}_\alpha$ be the net given by Lemma 3.6 for $\theta' \in \text{Hom}(V, \mathbb{T})$ instead of θ , in which $\theta'(v) = \theta(v)v^{-q^3}$ for all $v \in V$. For each α , let $m_\alpha = n_\alpha + q$. Then the first two equations follow from (a) and (b) of Lemma 3.6. Similarly

$$\lim_\alpha v^{m_\alpha^3} = \lim_\alpha v^{(n_\alpha^3 + 3n_\alpha^2q + 3n_\alpha q^2 + q^3)} = \theta'(v)v^{q^3} = \theta(v)v^{-q^3}v^{q^3} = \theta(v). \quad \square$$

Corollary 3.8. *For each $(q, \theta) \in \mathbb{Z} \times \text{Hom}(V, \mathbb{T})$, there is a net $\{m_\alpha\}_\alpha$ in \mathbb{Z} such that the following conditions are satisfied:*

- (a) $\lim_\alpha x^{Q_1(m_\alpha)} = x^{Q_1(q)}$ for each $x \in \mathbb{T}$;
- (b) $\lim_\alpha x^{Q_2(m_\alpha)} = x^{Q_2(q)}$ for each $x \in \mathbb{T}$;
- (c) $\lim_\alpha x^{m_\alpha^3} = \theta(x)$ for each $x \in V$.

Proof. Let $\{m_\alpha\}_\alpha$ be the net given in the Corollary 3.7. Since $Q_1(n) = n$ for each $n \in \mathbb{Z}$, (a) is obvious. Now $2Q_2(n) = n^2 + n$. Again by Corollary 3.7, we have $\lim_\alpha y^{2Q_2(m_\alpha)} = y^{2Q_2(q)}$ for each $y \in \mathbb{T}$. For each $x \in \mathbb{T}$, let $y \in \mathbb{T}$ be chosen so that $x = y^2$. Then (b) follows easily. The statement (c) is trivially part (c) of Corollary 3.7. \square

Proof of Theorem 1.1. It is enough to show that the map Φ is an isomorphism of groups. From Lemma 3.3 and the results preceding Lemma 3.5, Φ is well defined and is clearly one-to-one. In order to show that Φ is onto, it is clear from (5) and (6) that for each $(n, \theta) \in \mathbb{Z} \times \text{Hom}(V, \mathbb{T})$ we must find a net $\{m_\alpha\}_\alpha$ such that for each $j \in \{1, \dots, k-1\}$ and for each $x \in \mathbb{T}$,

$$x^{Q_j(n)} = \lim_\alpha x^{Q_j(m_\alpha)} \quad \text{and} \quad \lim_\alpha v^{m_\alpha^k} = \theta(v) \text{ for all } v \in V.$$

This follows from Corollary 3.8 above for the case $k=3$, and as we remarked, the general case is completely analogous. It remains to show the group structure on $\mathbb{Z} \times \text{Hom}(V, \mathbb{T})$ induced by the map Φ .

Let $\lambda \in \mathbb{T}$ be irrational, and let $f \in \mathbb{T}^{\mathbb{Z}}$ be defined by $f(n) = \lambda^{n^k}$. Recall that $U: \mathbb{T}^{\mathbb{Z}} \rightarrow \mathbb{T}^{\mathbb{Z}}$ is the shift map and $T: \mathbb{T}^k \rightarrow \mathbb{T}^k$ is the map satisfying $U \circ \Gamma = \Gamma \circ T$, where $\Gamma: \mathbb{T}^k \rightarrow X_f$ is the isomorphism of Theorem 1.1 in [11]. Let $g \in X_f$ be arbitrary. Then there is an $\mathbf{x} = (x_1, x_2, \dots, x_k) \in \mathbb{T}^k$ such that $\Gamma(\mathbf{x}) = g$. Hence for each $m \in \mathbb{Z}$,

$$(11) \quad U^m(g)(0) = g(m) = \Gamma(x)(m) = \lambda^{m^k} x_1^{Q_{k-1}(m)} x_2^{Q_{k-2}(m)} \dots x_{k-1}^{Q_1(m)} x_k.$$

Now suppose $\sigma \in \Lambda(\Sigma(W_k, U))$ and $\Phi(\sigma) = (p, \theta) \in \mathbb{Z} \times \text{Hom}(V, \mathbb{T})$. Then there is a net $\{m_\alpha\}_\alpha$ in \mathbb{Z} such that $\lim_\alpha U^{m_\alpha} = \sigma$, $\lim_\alpha x^{Q_j(m_\alpha)} = x^{Q_j(p)}$ for each $j \in \{1, 2, \dots, k-1\}$ and each $x \in \mathbb{T}$, and $\lim_\alpha \lambda^{m_\alpha} = \theta(\lambda)$. Suppose that $\sigma, \tau \in \Lambda(\Sigma)$ and that $\Phi(\sigma) = (p, \theta)$, $\Phi(\tau) = (q, \theta')$ and $\Phi(\sigma \circ \tau) = (r, \theta'')$. If $u = \theta(\lambda)$, $v = \theta'(\lambda)$ and $w = \theta''(\lambda)$, then it is not hard to verify that $w = uv\lambda^{(p+q)^k - (p^k + q^k)}$ and $r = p + q$. Therefore $\Phi(\sigma \circ \tau) = (p + q, \theta\theta'(\cdot)^{(p+q)^k - (p^k + q^k)})$. This completes the proof of the Theorem 1.1. \square

Remark 3.9. Let G_k denote the set $\mathbb{Z} \times \text{Hom}(V, \mathbb{T})$, given the group operation defined in the proof of Theorem 1.1. Then G_1 is $\mathbb{Z} \times \text{Hom}(V, \mathbb{T})$ with the group operation $(p, \theta)(q, \theta') = (p + q, \theta\theta')$. The groups G_1 and G_k seem to be different, but they are actually isomorphic by the map $\phi: G_k \rightarrow G_1$ defined by $\phi((p, \theta)) = (p, \theta(\cdot)^{-p^k})$.

Finally, we are going to prove the next result, which, comparing with Theorem 1.1, shows that Lemma 4.3 of [11] (or [13, Proposition 2.1]), on the equivalence of the topological center of the enveloping semigroup S , corresponding to any minimal flow, with the set of all continuous elements of S , is not generally valid for non-minimal flows. Let $\Sigma_c(W_k, U)$ denote the set of all continuous elements of $\Sigma(W_k, U)$.

Theorem 3.10. *For each $k > 1$, $\Sigma_c(W_k, U) = \{U^n; n \in \mathbb{Z}\}$.*

Proof. Assume that $\sigma = \lim_\alpha U^{m_\alpha} \in \Sigma_c(W_k, U)$. Clearly $\sigma \in \Lambda(\Sigma(W_k, U))$. Let $\Theta_k(\sigma) = (\theta_1, \dots, \theta_{k-1}, \theta)$, as in Theorem 3.1. Then by Lemma 3.3 there exist an integer n such that $\theta_j = (\cdot)^{Q_j(n)}$ for each j , $1 \leq j \leq k-1$. On the other hand, $\Theta_k(U^n) = ((\cdot)^{Q_1(n)}, \dots, (\cdot)^{Q_{k-1}(n)}, (\cdot)^{n^k})$ and Θ_k is one-to-one, hence it remains to show that $\theta = (\cdot)^{n^k}$. First, we show that θ is continuous. To this end, let $\{x_\beta\}_\beta$ be an arbitrary net in \mathbb{T} which converges to $x \in \mathbb{T}$. We show that $\theta(x_\beta) \rightarrow \theta(x)$. Let $f_\beta, f \in W_k$ be defined by $f_\beta(n) = x_\beta^{n^k}$ and $f(n) = x^{n^k}$ for all $n \in \mathbb{Z}$. Then $f_\beta \rightarrow f$ (pointwise) in W_k . Hence, by the continuity of σ ,

$$\theta(x_\beta) = \lim_\alpha (x_\beta)^{m_\alpha^k} = \sigma(f_\beta)(0) \rightarrow \sigma(f)(0) = \lim_\alpha x^{m_\alpha^k} = \theta(x).$$

That is, θ is continuous. Therefore, $\theta = (\cdot)^m$ for some $m \in \mathbb{Z}$. Let p be any prime number. Since $\lim_\alpha x^{m_\alpha} = x^n$, for all $x \in \mathbb{T}$, we have $m_\alpha = n \pmod{p}$ eventually. Therefore $m_\alpha^k = n^k \pmod{p}$ eventually. Thus $\lim_\alpha e((1/p)m_\alpha^k) = e((1/p)n^k)$. It follows that for each integer q with $0 < q < p$, one has $\lim_\alpha e((q/p)m_\alpha^k) = e((q/p)n^k)$. On the other hand, $\lim_\alpha e((q/p)m_\alpha^k) = \theta(e(q/p)) = e(q/p)^m$. Hence $e(q/p)^{n^k} = e(q/p)^m$. Now since $x \mapsto x^{n^k}$ and $x \mapsto x^m$ are both continuous functions on \mathbb{T} and the set $\{e(q/p); 0 < q < p, p > (k-1)! \text{ and } p \text{ is prime}\}$ is dense in \mathbb{T} , we have $x^{n^k} = x^m = \theta(x)$ for each $x \in \mathbb{T}$. The theorem is now proved. \square

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