Ark. Mat., **50** (2012), 331–357
DOI: 10.1007/s11512-011-0149-1

On the scaling limit of loop-erased random walk excursion

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Abstract. We use the known convergence of loop-erased random walk to radial SLE(2) to give a new proof that the scaling limit of loop-erased random walk excursion in the upper half-plane is chordal SLE(2). Our proof relies on a version of Wilson's algorithm for weighted graphs which is used together with a Beurling-type estimate for random walk excursion. We also establish and use the convergence of the radial SLE path to the chordal SLE path as the bulk point tends to a boundary point. In the final section we sketch how to extend our results to more general simply connected domains.

1. Introduction and main results

1.1. Introduction

Let D be an approximation, using the square lattice with small mesh size, of a simply connected planar domain. Loop-erased random walk (LERW) [4] in D is a self-avoiding random walk which is constructed by chronologically erasing the loops from a simple random walk on the scaled lattice that is started from an interior point and stopped when the boundary of D is hit.

The Schramm–Loewner evolution, SLE(κ) for short, is a one-parameter family of random Loewner chains that was introduced by Schramm [13] as a candidate for the lattice-size scaling limit of planar loop-erased random walk (when $\kappa=2$). Subsequently, Lawler–Schramm–Werner proved in their paper [10] that radial SLE(2) is indeed the scaling limit of loop-erased random walk. This implies in particular that loop-erased random walk has a conformally invariant scaling limit. Other discrete models with appropriate boundary conditions have also been shown to contain random curves which converge to SLE(κ) for different values of κ. We mention the critical percolation exploration path [18] ($\kappa=6$), cluster interfaces in the spin Ising and FK-Ising models at criticality ($\kappa=3$ and $\kappa=\frac{16}{3}$), see [19] and the references therein, a perimeter curve for the uniform spanning tree [10] ($\kappa=8$), and contour lines in the discrete Gaussian free field [15] ($\kappa=4$).

In this paper we consider a version of loop-erased random walk, namely loop-erased random walk *excursion*. This is the self-avoiding random walk defined as the loop-erasure of a random walk excursion, that is, simple random walk started from a given boundary point, conditioned on taking the first step into the domain and then exiting at a prescribed boundary point. We prove that loop-erased random walk excursion in the upper half-plane converges to chordal SLE(2) as the lattice size tends to zero. The proof relies on the known convergence of loop-erased random walk to radial SLE(2) but also on the convergence of radial to chordal SLE and a version of Wilson's algorithm.

We remark that the convergence of loop-erased random walk excursion to chordal SLE(2) in the upper half-plane has previously been considered by Beneš [2] who obtained several partial results. Zhan proved convergence to chordal SLE(2) using the "direct" method of proving convergence of the chordal Loewner driving function to Brownian motion, see [22]. Our method of proof is different from the approaches taken by these authors and we believe that some of the results that we establish along the way may be of independent interest.

1.2. Main results

To state our main results, let us set some notation. Let $\mathbb{H}=\{z:\operatorname{Im} z>0\}$ denote the complex upper half-plane. For $x\in\mathbb{R}$ and $\delta>0$ define $\lfloor x\rfloor_\delta:=\delta\lfloor x/\delta\rfloor$ and similarly for $z=x+iy\in\mathbb{C}$ we write $\lfloor z\rfloor_\delta=\lfloor x\rfloor_\delta+i\lfloor y\rfloor_\delta$ for the $\delta\mathbb{Z}^2$ lattice approximation of z. Set $D(z,R)=\{w\in\mathbb{C}:|w-z|<R\}$. We let $\gamma_\delta(t)$, $t\geqslant 0$, denote loop-erased random walk excursion on $\delta\mathbb{Z}^2\cap\mathbb{H}$ started from 0 parameterized by capacity (we add the edges to the discrete walk to get a curve) and we let $\gamma(t)$, $t\geqslant 0$, denote the chordal SLE(2) path also parameterized by capacity. Let $T<\infty$ be fixed and let μ and μ_δ denote the laws of the curves $\gamma(t)$ and $\gamma_\delta(t)$, $t\in[0,T]$, respectively, as elements in the space $\mathcal{K}$ of unparameterized curves in $\mathbb{C}$. We use the metric

$$\rho(\alpha,\beta)=\inf_{\varphi}\sup\{|\alpha-\beta\circ\varphi|:t\in[0,t_\alpha]\}$$

for elements in $\mathcal{K}$, where the infimum is taken over strictly increasing reparameterizations, see Section 2.4.

Theorem 1.1. *As $\delta\to 0$, the measures μ_δ converge weakly to μ with respect to the metric ρ.*

The proof, which is given in Section 5, is by a "three epsilon argument". We want to produce a coupling where loop-erased random walk excursion on $\delta\mathbb{Z}^2$, with δ small, and chordal SLE(2) are close with high probability in the sense of the metric ρ. The idea is to first compare loop-erased random walk excursion from a boundary point with loop-erased random walk excursion from an interior point using a version of Wilson's algorithm together with a Beurling-type estimate for random walk excursion started from the boundary.

Proposition 1.2. *Let $R<\infty$ be fixed. For each $\varepsilon>0$ there exists $d_0>0$ such that the following holds. Suppose $d\leqslant d_0$ and set $z=\lfloor di\rfloor_\delta$, $\delta>0$. Let γ_δ and γ_δ^z denote the loop-erased random walk excursion paths on $\delta\mathbb{Z}^2$ from 0 and from z, respectively, both stopped when first hitting $\{z:|z|=R\}$. There exists $\delta_0=\delta_0(d)>0$ and for $\delta<\delta_0$ a coupling of γ_δ with γ_δ^z such that*

$$\mathbb{P}\Big(\inf_{\varphi}\sup_{0\leqslant t\leqslant t_{\gamma_\delta^z}}|\gamma_\delta^z(t\wedge\eta)-\gamma_\delta(\varphi(t\wedge\eta))|>\varepsilon\Big)<\varepsilon,$$

where the infimum is taken over strictly increasing reparameterizations and

$$\eta=\inf\{t\geqslant 0:\max\{|\gamma_\delta^z(t)|,|\gamma_\delta(\varphi(t))|\}=R\}.$$

The proof of Proposition 1.2 is given in Section 4. The Beurling-type estimate that is used in the proof is stated and proved in Proposition 4.2.

The second step is to compare loop-erased random walk excursion from an interior point with radial SLE(2) using the convergence of loop-erased random walk from an interior point to radial SLE(2) as established by Lawler–Schramm–Werner, see Corollary 2.5.

In the last step we will use the fact that the radial SLE path converges to the chordal SLE path, as the bulk point tends to the boundary. Consider a sequence $\{z^{(n)}\}_{n=1}^\infty$ in the upper half-plane that tends to ∞ with n. We fix $\kappa\in[0,4]$ and let $\mu^{(n)}$ denote the law of the radial SLE(κ) path in $\mathbb{H}$ between 0 and $z^{(n)}$, stopped when hitting $\{z:|z|=R\}$. Similarly, we let μ' denote the law of the chordal SLE(κ) path stopped when hitting $\{z:|z|=R\}$.

Proposition 1.3. *As $n\to\infty$, the measures $\mu^{(n)}$ converge weakly to μ' with respect to the metric ρ.*

The proof of Proposition 1.3 is given in Section 3. The statement is a direct consequence of Lemma 3.2 which states that the laws of the driving processes corresponding to the paths converge in total variation. (See also [9] where a similar result is obtained.) Corollary 3.3 is the form of Proposition 1.3 that we use for the proof of Theorem 1.1. In Section 5 we combine the results from previous sections to prove Theorem 1.1. Although we work in the upper half-plane in the bulk of the paper, we outline in the final section how similar arguments may be used together with some additional results from the literature to get an analogous convergence result in a certain class of simply connected domains.

Acknowledgements. This paper was written while the author was a graduate student at KTH Royal Institute of Technology and supported by the Knut and Alice Wallenberg Foundation. I thank Wendelin Werner for suggesting the problem addressed in this paper and for preliminary discussions. I am grateful to Michael Benedicks, Ilia Binder, Michael Kozdron, and Alan Sola for helpful comments on earlier versions of the manuscript and to the anonymous referee for his/her suggestions that helped me improve the quality of the paper.

2. Preliminaries

2.1. Random walk excursion

Let S' be simple random walk on $\mathbb{Z}^2$ started from $z\in\mathbb{Z}^2\cap\mathbb{H}$. For $m\in\mathbb{Z}$ let $\tau_m=\min\{j:\operatorname{Im} S'(j)=m\}$. It follows from a gambler's ruin estimate that $\mathbb{P}^z(\tau_m<\tau_0)=\operatorname{Im} z/m$. If $\operatorname{Im} z<m$ and $\operatorname{Im} w\leqslant m$ the Markov property of simple random walk implies that

$$\mathbb{P}^z(S'(1)=w \mid \tau_m<\tau_0)=p(z,w)\,\frac{\operatorname{Im} w}{\operatorname{Im} z}, \tag{2.1}$$

where $p(z,w)$ are the transition probabilities for simple random walk. Since the right-hand side is independent of m we may let $m\to\infty$, and in this way we obtain the transition probabilities for *random walk excursion* started from z. We note the useful fact that random walk excursion stopped at τ_m has the same distribution as simple random walk conditioned on $\{\tau_m<\tau_0\}$, stopped at τ_m.

Let S be random walk excursion. We allow S to be started from $x\in\mathbb{Z}$ and then we set $\mathbb{P}^x(S(1)=x+i)=1$. By its construction, random walk excursion has the strong Markov property, and can be thought of as simple random walk conditioned to exit $\mathbb{H}$ at ∞ or as simple random walk on $\mathbb{Z}^2$ with edge weights defined using (2.1). Let us note also that the formula for transition probabilities for random walk excursion on the scaled lattice $\delta\mathbb{Z}^2$ are the same as for $\mathbb{Z}^2$. That is, if S_δ is random walk excursion on $\delta\mathbb{Z}^2$ then $\mathbb{P}^z(S_\delta(1)=w)=p_\delta(z,w)(\operatorname{Im} w/\operatorname{Im} z)$ (if $\operatorname{Im} z>0$), where $p_\delta(z,w)$ are the transition probabilities of simple random walk on $\delta\mathbb{Z}^2$.

The Markov property and (2.1) imply that probabilities for random walk excursion can be related to corresponding probabilities for simple random walk, S'. We use the notation $S[i,j]=\{S(i),...,S(j)\}$. Suppose $z_0\in\mathbb{Z}^2\cap\mathbb{H}$. We have

$$\mathbb{P}^{z_0}(S[1,k]=\{z_1,...,z_k\})=\mathbb{P}^{z_0}(S'[1,k]=\{z_1,...,z_k\})\frac{\operatorname{Im} z_k}{\operatorname{Im} z_0}, \tag{2.2}$$

provided $\operatorname{Im} z_j>0$, $j=1,...,k$; otherwise the probability on the left-hand side is zero. In particular, (2.2) implies the following formula for hitting probabilities. Let $E\subset D\subset\mathbb{Z}^2\cap\mathbb{H}$. Set $\tau_D=\min\{j\geqslant 0:S(j)\in D\}$ and let τ'_D be the corresponding time for S'. Then

$$\mathbb{P}^z(S(\tau_D)\in E)=\sum_{w\in E}\mathbb{P}^z(S'(\tau'_D)=w,\ \operatorname{Im} S'(j)>0 \text{ for } j\leqslant\tau'_D)\frac{\operatorname{Im} w}{\operatorname{Im} z}.$$

In contrast with simple random walk, random walk excursion is not recurrent. Indeed, let $\sigma_z=\min\{j\geqslant 1:S(j)=z\}$. By the strong Markov property, for $z\in\mathbb{H}$, it is enough to check that

$$\mathbb{P}^z(\sigma_z<\infty)<1.$$

By (2.2), $\mathbb{P}^z(\sigma_z<\infty)$ is equal to the probability that simple random walk started from z returns to z before leaving the upper half-plane, and this probability is clearly strictly smaller than 1.

Let us now briefly sketch how to define random walk excursion in other domains. For simplicity we assume that $D\subsetneq\mathbb{Z}^2$ is simply connected, that is, $\mathbb{Z}^2\setminus D$ is connected. Let ζ be a vertex on the boundary of D. Let S' be simple random walk and define $\tau=\min\{j\geqslant 0:S'(j)\in\partial D\}$. For $z\in D$ define

$$H(z,\zeta)=\mathbb{P}^z(S'(\tau)=\zeta).$$

Then $z\mapsto H(z,\zeta)$ is discrete harmonic, that is, H has the discrete mean-value property in D. If $H(z,\zeta)>0$, then we can condition S' on $\{S'(\tau)=\zeta\}$, and the thus constructed random walk S will have transition probabilities

$$\mathbb{P}^z(S(1)=w)=p(z,w)\,\frac{H(w,\zeta)}{H(z,\zeta)},$$

and a formula analogous to (2.2) holds in this case too. We allow S to be started from $z\in\partial D$ such that $\mathbb{P}^z(S'(1)\in D)>0$ and set

$$\mathbb{P}^z(S(1)=w):=\mathbb{P}^z(S'(1)=w\mid S'(1)\in D,\ S'(\tau)=\zeta).$$

2.2. Loop-erased walks

Let $S=\{S(0), S(1), ...\}$ be a nearest-neighbor path in $\mathbb{Z}^2$ visiting no vertex infinitely many times. The loop-erasure of S, $L\{S\}$, is the non-self-crossing nearest-neighbor path defined as follows. Set

$$t_0 = \max\{j \geqslant 0 : S(j) = S(0)\},$$

and inductively for $k=1, 2, ...,$

$$t_k = \max\{j > t_{k-1} : S(j) = S(t_{k-1}+1)\}.$$

Then we define $L\{S\}(j):=S(t_j)$ so that

$$L\{S\} = \{S(t_0), S(t_1), ...\}.$$

Note that $L\{S\}(0)=S(0)$. One can equivalently start from $S(0)$ and chronologically erase loops as they form. Another interpretation is the cycle-popping procedure used in [20].

If S' is (stopped) simple random walk, then we call $L\{S'\}$ *loop-erased random walk*, and if S is random walk excursion, then we call $L\{S\}$ *loop-erased random walk excursion*. Note that the infinite loop-erased random walk in $\mathbb{Z}^2$ has to be defined by taking a limit due to the recurrence of simple random walk, see [5].

Let S be random walk excursion and set $S^m:=S[0, \tau_m]$, that is, S^m is defined by S stopped when first reaching $\{z:\operatorname{Im} z \geqslant m\}$. Suppose $m>m_0>\operatorname{Im} z$. By considering the event that S never revisits $\{z:\operatorname{Im} z \leqslant m_0\}$ after first visiting $\{z:\operatorname{Im} z \geqslant m\}$, using (2.2) and recurrence of simple random walk, we see that

$$\mathbb{P}^z(L\{S^m\}[1, \tau'_{m_0}] = L\{S\}[1, \tau''_{m_0}]) \geqslant 1 - \frac{m_0}{m} - \frac{1}{m}, \tag{2.3}$$

where τ'_{m_0} and τ''_{m_0} denote the first times the loop-erased walks reach $\{z:\operatorname{Im} z \geqslant m_0\}$. The corresponding inequality for random walk excursion on $\delta\mathbb{Z}^2$ reads

$$\mathbb{P}^z(L\{S_\delta^m\}[1, \tau'_{m_0}] = L\{S_\delta\}[1, \tau''_{m_0}]) \geqslant 1 - \frac{m_0}{m} - \frac{\delta}{m}. \tag{2.4}$$

2.3. Wilson's algorithm

A uniform spanning tree on a finite graph is a random spanning tree chosen from the uniform distribution over all spanning trees. (There are only finitely many such trees for a finite graph.) It was shown by Pemantle [11] that the branches in a uniform spanning tree of an unweighted, undirected, finite graph have the distribution of loop-erased random walks. Wilson, see [20] and references therein, later showed that one can go in the opposite direction by giving an elegant and efficient algorithm for sampling uniform spanning trees using loop-erased random walk.

We will use Wilson's algorithm to couple loop-erased random walk excursions started from different points. On one hand our situation is different from the usual setting for Wilson's algorithm, since we are dealing with random walks on an infinite graph. On the other hand we are not interested in sampling uniform spanning trees but rather in coupling loop-erased random walks. In fact, one may argue that using Wilson's algorithm for random walk excursion is a natural way to *define* a random weighted spanning tree (rooted at ∞) on the infinite graph $\mathbb{Z}^2 \cap \overline{\mathbb{H}}$.

We now give a proof of a version of Wilson's algorithm which is suited for our needs. We follow closely Lawler's discussion in [6], where an elegant proof of Wilson's algorithm for a finite graph is given. The identity (2.5) below is key to the approach. Suppose X is a Markov chain on $\mathbb{Z}^2$ with transition probabilities p, and for $z \notin A \subset \mathbb{Z}^2$ define

$$G(z, A) = \sum_{j=0}^{\infty} \mathbb{P}^z(X(j) = z,\ j < \tau_A),$$

that is, G is the expected number of visits to z before hitting A of the chain started from z. For a transient chain, G may be interpreted as the Green function for A^c evaluated at the "pole". For $x, y \notin A$ (we allow A to be the empty set)

$$G(x, A)G(y, A \cup \{x\}) = G(y, A)G(x, A \cup \{y\}). \tag{2.5}$$

Indeed, by decomposing the sum and using the Markov property we find that

$$\begin{aligned} G(x, A) &= G(x, A \cup \{y\}) + \mathbb{P}^x(\tau_y < \tau_A) \sum_{j=0}^{\infty} \mathbb{P}^y(X(j) = x,\ j < \tau_A) \\ &= G(x, A \cup \{y\}) + \mathbb{P}^x(\tau_y < \tau_A)\mathbb{P}^y(\tau_x < \tau_A)G(x, A), \end{aligned}$$

so that

$$(1 - \mathbb{P}^x(\tau_y < \tau_A)\mathbb{P}^y(\tau_x < \tau_A))G(x, A) = G(x, A \cup \{y\}).$$

Note that the factor in front of $G(x, A)$ is symmetric in x and y. Since the analogous identity holds for $G(y, A)$, (2.5) follows directly. Suppose X is started from z_0 and stopped when hitting A. Let $\{z_j\}_{j=0}^k$ be a self-avoiding nearest-neighbor path connecting z_0 with A such that $z_0, ..., z_{k-1} \in A^c$ and $z_k \in A$. It follows directly from the definition of the loop-erasing procedure that

$$\mathbb{P}(L\{X\} = \{z_0, ..., z_k\}) = \prod_{j=0}^{k-1} p(z_j, z_{j+1}) \prod_{j=0}^{k-1} G\Big(z_j, \bigcup_{n=0}^{j-1} \{z_n\} \cup A\Big),$$

and by (2.5), since any permutation can be written as a product of transpositions, the second product on the right is symmetric as a function of z_j, $j=0,...,k$.

Lemma 2.1. (Wilson's algorithm for LERW excursion) *Let* $z_1,z_2\in\delta\mathbb{Z}^2\cap\overline{\mathbb{H}}$, $\delta>0$. *Let* S_1 *be random walk excursion from* z_1, *and set* $L_1=L\{S_1\}$. *Let further* S_2 *be independent random walk excursion from* z_2, *stopped when hitting* L_1. *Set* $L_2=L\{S_2\}$ *and suppose* $k\in\mathbb{N}\cup\{\infty\}$ *is such that* $L_1(k)=L_2\cap L_1$. *Then* $L_2\cup L_1[k,\infty)$ *has the distribution of loop-erased random walk excursion from* z_2.

Proof. Without loss of generality we assume that $\delta=1$. The conclusion is trivial if $k=\infty$, that is, if S_2 never hits L_1, so we may assume that k is finite. Fix $\mathbb{Z}\ni m>\max\{\operatorname{Im} z_1,\operatorname{Im} z_2\}$, put $I_m=\{z\in\mathbb{Z}^2:0\leqslant\operatorname{Im} z\leqslant m\}$, and write $J_m\subset I_m$ for the vertices with imaginary part equal to m. We shall sample a random subset of I_m. Let $\tau_{1,m}=\min\{j\geqslant 0:S_1(j)\in J_m\}$ and $L_1^m=L\{S_1[0,\tau_{1,m}]\}$. Define $\tau_{2,m}$ in the same way with S_1 replaced by S_2 and let $\tau_{L_1,m}$ denote the first time S_2 hits L_1^m (set it to ∞ if this never happens). Let $L_2^m=L\{S_2[0,\tau_{L_1,m}\wedge\tau_{2,m}]\}$, set $T^m(z_1,z_2):=L_1^m\cup L_2^m$, and let $T^m(z_2,z_1)$ denote the random set obtained by repeating the same procedure starting with random walk excursion from z_2 instead. We will show that $T^m(z_1,z_2)\overset{d}{=}T^m(z_2,z_1)$. The lemma then follows from (2.3) by letting $m\to\infty$. Indeed, let $X=\{x_0,...,x_n\}\ni z_1,z_2$ be a rooted tree in I_m (with J_m collapsed to one single vertex x_0 that we take to be the root) with at most two branches connecting z_1 and z_2 with x_0 such that at least one of z_1 and z_2 is a leaf. For each $x_j\in X\setminus\{x_0\}$, denote by x_j^* the unique vertex following x_j in the path from x_j to the root in X. Define

$$f(x_1...,x_n;\{x_0\})=\prod_{j=1}^{n}G(x_j,\{x_0\}\cup A_{j-1}), \tag{2.6}$$

where $A_j=\bigcup_{k=1}^{j}\{x_k\}$ and $A_0=\varnothing$. Then in view of the discussion preceding the statement of the lemma we have

$$\mathbb{P}(T^m(z_1,z_2)=X)=f(Z_1,Z_2;\{x_0\})\prod_{j=1}^{n}p(x_j,x_j^*), \tag{2.7}$$

where $Z_1=(z_1,...,z_m)$ are the vertices in the unique path in X from z_1 to x_0 (ordered according to the path), and Z_2 are the vertices in the path from z_2 to x_0 (in order) not already listed in Z_1. We showed that f is a symmetric function and so is the product in (2.7). Hence we can write $\mathbb{P}\,(T^m(z_2,z_1)=X)$ in the same manner starting instead with all the vertices in the path from z_2 to x_0, and rearrange the right-hand side using the symmetry of f, to find that

$$\mathbb{P}(T^m(z_1,z_2)=X)=\mathbb{P}(T^m(z_2,z_1)=X),$$

which completes the proof. $\square$

2.4. A metric space of curves

Let $\mathcal{K}$ denote the space of unparameterized paths in $\mathbb{C}$, that is, the space of equivalence classes of continuous functions (defined on intervals, taking values in $\mathbb{C}$) modulo increasing reparameterizations. This means that $\alpha(t)$, $t\in[0,t_\alpha]$, and $\beta(t)$, $t\in[0,t_\beta]$, are in the same equivalence class if and only if there exists a continuous strictly increasing function $\varphi\colon [0,t_\alpha]\to[0,t_\beta]$ such that $\alpha=\beta\circ\varphi$ pointwise. (Of course, that 0 is on the boundary of both intervals is not important.) We endow $\mathcal{K}$ with the metric ρ [1] defined by

$$\rho(\alpha,\beta)=\inf_{\varphi}\sup\{|\alpha(t)-\beta\circ\varphi(t)|:t\in[0,t_\alpha]\},$$

where the infimum is over strictly increasing reparameterizations as above. The metric space $(\mathcal{K},\rho)$ is complete and separable, but it is not necessarily compact, see [1]. We will usually not distinguish between a representative of a given element in $\mathcal{K}$ and the element itself.

We state two easily verified lemmas. Let $\gamma_1,\gamma_2\in\mathcal{K}$. Pick parameterizations for both curves and assume without loss of generality that they are defined on $[0,1]$. For a curve γ we define γ^{rev} to be the time-reversed curve, that is, the equivalence class determined by tracing γ backwards.

Lemma 2.2. *Fix $z\in\mathbb{C}$ and $\varepsilon>0$. For $j=1,2$, suppose $\gamma_j(0)\in D(z,\varepsilon)$, and let τ_j^ε be such that $\gamma_j[0,\tau_j^\varepsilon)\subset D(z,\varepsilon)$. Let γ_j^ε be the curves determined by $\gamma_j(t)$, $t\in[\tau_j^\varepsilon,1]$. Then*

$$\rho(\gamma_1,\gamma_2)\leqslant\rho(\gamma_1^\varepsilon,\gamma_2^\varepsilon)+2\varepsilon. \tag{2.8}$$

Lemma 2.3. *It is true that*

$$\rho(\gamma_1,\gamma_2)=\rho(\gamma_1^{\mathrm{rev}},\gamma_2^{\mathrm{rev}}).$$

Suppose now that γ_j, $j=1,2,3$, are such that $\gamma_j(0)\in D(0,R)$. Suppose further that the curves intersect $\partial D(0,R)$ and for each $j=1,2,3$, let γ_j^R be the curve determined by γ_j stopped at the first hitting time of $\partial D(0,R)$. Note that an upper bound on $\rho(\gamma_1,\gamma_2)$ gives no non-trivial information on $\rho(\gamma_1^R,\gamma_2^R)$. Since we need to consider stopped paths, we can instead use

$$\rho_R(\gamma_1,\gamma_2):=\inf_{\varphi}\sup_{t\in[0,1]}|\gamma_1(\varphi(t\wedge\eta))-\gamma_2(t\wedge\eta)|,$$

where the infimum is over increasing reparameterizations and

$$\eta=\eta(\varphi,R)=\inf\{t\geqslant 0:\max\{|\gamma_1(\varphi(t))|,|\gamma_2(t)|\}=R\}.$$

Clearly $\rho(\gamma_1,\gamma_2)<\varepsilon$ implies $\rho_R(\gamma_1,\gamma_2)<\varepsilon$. The function ρ_R does not satisfy the triangle inequality but when ε is small enough compared to R it has the property that $\rho_R(\gamma_1,\gamma_2)<\varepsilon$ and $\rho_R(\gamma_2,\gamma_3)<\varepsilon$ implies that $\rho_{R/2}(\gamma_1,\gamma_3)<2\varepsilon$ (if $\gamma_j(0)\in D(0,R/2)$) and this is sufficient for our purposes.

2.5. Loewner differential equation and versions of SLE

Given K, a compact subset of $\overline{\mathbb{H}}$ such that $\mathbb{H}\setminus K$ is simply connected, we define the *half-plane capacity* of K by

$$\mathrm{hcap}(K)=\lim_{z\to\infty} z(g_K(z)-z),$$

where $g_K\colon \mathbb{H}\setminus K\to\mathbb{H}$ is the unique Riemann map normalized so that

$$\lim_{z\to\infty}(g_K(z)-z)=0.$$

Let B_t be standard Brownian motion on $\mathbb{R}$. Let $\kappa>0$ and consider the solution to the chordal Loewner equation for $\mathbb{H}$,

$$\partial_t g_t(z)=\frac{2}{g_t(z)-\sqrt{\kappa}B_t} \text{ and } g_0(z)=z, \quad z\in\overline{\mathbb{H}}. \tag{2.9}$$

For each $t>0$, $g_t\colon \mathbb{H}\setminus K_t\to\mathbb{H}$ is a conformal map, where K_t is a compact subset of $\overline{\mathbb{H}}$ such that $\mathbb{H}\setminus K_t$ is simply connected. It follows from (2.9) that $\mathrm{hcap}(K_t)=2t$, and we say that K_t is parameterized by half-plane capacity. We call the family of solutions $(g_t)_t$ the chordal SLE(κ) Loewner chain. It describes a continuous path growing in $\mathbb{H}$ from 0 to ∞: the function $t\mapsto\gamma(t):=\lim_{y\to 0+} g_t^{-1}(\sqrt{\kappa}B_t+iy)$ can be proven to be a.s. continuous and is called the chordal SLE(κ) path. It is known that γ is a.s. simple if and only if $\kappa\in[0,4]$ and that $\lim_{t\to\infty}|\gamma(t)|=\infty$ a.s., see [12] or [7].

Radial SLE(κ) is defined similarly but using the radial Loewner equation

$$\partial_t g_t(z)=g_t(z)\frac{U_t+g_t(z)}{U_t-g_t(z)} \text{ and } g_0(z)=z,\ z\in\overline{\mathbb{D}}, \tag{2.10}$$

where $U_t=\exp(i\sqrt{\kappa}B_t)$. The conformal maps $g_t\colon \mathbb{D}\setminus K_t\to\mathbb{D}$ now describe a continuous path growing from 1 to 0 in $\mathbb{D}$ (see [8]); the radial SLE(κ) path. It is usually parameterized by logarithmic capacity.

Both the chordal and radial versions of SLE can be defined in arbitrary simply connected domains using a suitably normalized Riemann map from $\mathbb{H}$ and $\mathbb{D}$, respectively. In particular we will need to consider radial SLE(κ) in $\mathbb{H}$ between 0 and z, defined as the image of the radial SLE path in $\mathbb{D}$ under the Möbius transformation $\varphi_z\colon \mathbb{D}\to\mathbb{H}$ such that $\varphi_z(0)=z$ and $\varphi_z(1)=0$. Note that the radial path only intersects $\partial\mathbb{D}$ at 1 when $\kappa\leqslant 4$, so the image path under φ_z does not escape to infinity in this case.

We can think of the SLE paths as random elements of $\mathcal{K}$. We use the notation γ to refer to the chordal SLE path in $\mathbb{H}$, and $\gamma^{0,z}$ to refer to the radial SLE path in $\mathbb{H}$ from 0 to $z\in\mathbb{H}$. We will also need to consider the (properly stopped) time-reversal of $\gamma^{0,z}$ under the mapping $\iota(z):=-1/z$, and we denote the thus obtained curve by $\gamma^{w,\infty}$, where $w=\iota(z)$. We use this particular notation for convenience and $\gamma^{z,\infty}$ should be only interpreted as shorthand for $\iota(\gamma^{0,\iota(z)})^{\mathrm{rev}}$.

We now review the definition of chordal SLE(κ,ρ) with one force point in $\mathbb{H}$. See [16] for more details. Let $\kappa>0$ and $\rho\in\mathbb{R}$ and consider the solution to the following system of stochastic differential equations

$$\begin{cases} dW_t=\sqrt{\kappa}\,dB_t+\mathrm{Re}\left(\dfrac{\rho}{W_t-V_t}\right)dt, \\ dV_t=\dfrac{2}{V_t-W_t}\,dt, \end{cases} \tag{2.11}$$

with initial value $(W_0,V_0)=(0,z)$. We define SLE(κ,ρ) with force point z as the chordal Loewner chain $(g_t)_t$ driven by W_t for $0\leqslant t<\tau$, where $\tau=\tau(z)$ is the infimum of τ such that 0 is in the set of limit points of $|V_t-W_t|$, when $t\to\tau-$. If this never happens we put $\tau=\infty$. The corresponding trace is the SLE(κ,ρ) path. (It exists almost surely up to a stopping-time by absolute continuity.) Notice that $t\mapsto V_t$ describes the flow of z under g_t. Of course, $\rho=0$ yields ordinary SLE(κ). Intuitively, the effect of the term involving ρ in (2.11) is to introduce attraction of the path towards z if ρ is negative and repulsion if ρ is positive.

2.6. Convergence of LERW to radial SLE(2)

Our proof of Theorem 1.1 uses the convergence of (unweighted) loop-erased random walk started from a bulk point to radial SLE(2) [10, Theorem 1.1].

Theorem 2.4. (Lawler–Schramm–Werner) *Let $D\subsetneq\mathbb{C}$ be a simply connected domain with $0\in D$. For $\delta>0$, let μ_δ be the law of the time-reversal of loop-erased random walk on $\delta\mathbb{Z}^2$ started at 0 and stopped when hitting ∂D. Let ν_D be the law of radial* SLE(2) *in D, with bulk point 0, started according to harmonic measure. Then as $\delta\to 0$, the measures μ_δ converge weakly with respect to ρ to ν_D.*

Recall, from Section 2.5, that $\gamma^{z,\infty}$ is the curve determined by $\iota(\gamma^{0,\iota(z)})^{\text{rev}}$, where $\gamma^{0,\iota(z)}$ is the radial SLE(2) path in $\mathbb{H}$. We will now derive the following corollary from Theorem 2.4.

Corollary 2.5. *Fix $z\in\mathbb{H}$ and $R>|z|$. Let γ_δ^z denote the loop-erased random walk excursion path from $\lfloor z\rfloor_\delta$, and let $\gamma^{z,\infty}$ be the time-reversal of radial* SLE(2) *in $\mathbb{H}$. For any $\varepsilon>0$ there exists $\delta_0>0$ so that if $\delta<\delta_0$ there is a coupling of γ_δ^z and $\gamma^{z,\infty}$ such that*

$$\mathbb{P}\left(\rho_R(\gamma_\delta^z,\gamma^{z,\infty})>\varepsilon\right)<\varepsilon.$$

Proof. Let $m>\operatorname{Im} z$ and, as before, set

$$I_m=\{z:0<\operatorname{Im} z<m\} \quad\text{and}\quad J_m=\{z:\operatorname{Im} z=m\}.$$

Let $\psi_m\colon I_m\to\mathbb{D}$ denote the conformal mapping normalized by the conditions $\psi_m(z)=0$ and $\psi_m(\operatorname{Re} z+im)=1$. Let S_δ be random walk excursion from $\lfloor z\rfloor_\delta$, and set $\tau_m=\min\{j\geqslant0:\operatorname{Im} S_\delta(j)\geqslant m\}$. When $\delta\to0$ (the proof of) Theorem 2.4 implies that the law of the time-reversal of $L\{S_\delta[0,\tau_m-1]\}$ converges weakly with respect to ρ to the law of $\gamma^{\zeta,z}$, radial SLE(2) in I_m with bulk point z, where the point $\zeta\in J_m$ is chosen according to the density

$$\zeta\longmapsto\frac{\sin(\pi\operatorname{Im} z/m)/\operatorname{Im} z}{2\cosh(\pi(\zeta-\operatorname{Re} z)/m)+2\cos(\pi\operatorname{Im} z/m)}. \tag{2.12}$$

(See the remark immediately following this proof.)

Since the harmonic measure of J_m from z in I_m equals $\operatorname{Im} z/m$, conformal invariance implies that

$$\psi_m(J_m)=\left\{e^{i\theta}:\theta\in\left[-\pi\operatorname{Im}\frac{z}{m},\pi\operatorname{Im}\frac{z}{m}\right]\right\}.$$

It follows that $\psi_m(\gamma^{\zeta,z})$ for large m is a small random rotation of the standard radial SLE(2) path in $\mathbb{D}$ and as $m\to\infty$ these paths clearly converge weakly with respect to ρ. By mapping back to the half-plane we see that we can choose $m_1<\infty$ so that whenever $m>m_1$ we may couple $\gamma^{z,\infty}$ with $\gamma^{z,\zeta}=(\gamma^{\zeta,z})^{\text{rev}}$ and

$$\mathbb{P}(\rho_{3R}(\gamma^{z,\infty},\gamma^{z,\zeta})>\varepsilon)<\varepsilon.$$

Similarly using (2.4) we can find $m_2<\infty$ such that if $m>m_2$ then $\gamma_\delta^{z,m}$ with high probability is close (in the sense that ρ_{3R} is small) to γ_δ^z and this is true uniformly in δ. Let $m>\max\{m_1,m_2\}$. Theorem 2.4 implies that we can find $\delta_0=\delta_0(m,\varepsilon)$ such that for $\delta<\delta_0$ there is a coupling of the LERW path $\gamma_\delta^{z,m}$ with the radial SLE(2) path $\gamma^{z,\zeta}$ and the probability that $\rho_{3R}(\gamma_\delta^{z,m},\gamma^{z,\zeta})>\varepsilon$ is at most ε. Putting things together we get, whenever $\delta<\delta_0$, that

$$\mathbb{P}(\rho_R(\gamma^z_\delta,\gamma^{z,\infty})>3\varepsilon)\leqslant\mathbb{P}(\rho_{3R}(\gamma^z_\delta,\gamma^{z,m}_\delta)>\varepsilon)\\+\mathbb{P}(\rho_{3R}(\gamma^{z,m}_\delta,\gamma^{z,\zeta})>\varepsilon)+\mathbb{P}(\rho_{3R}(\gamma^{z,\zeta},\gamma^{z,\infty})>\varepsilon)<3\varepsilon,$$

and we have proved the corollary. □

Remark. The formula (2.12) comes from the fact that simple random walk conditioned on exiting I_m through J_m converges, as $\delta\to 0$, to Brownian motion conditioned on the analogous event. (Note also that this event has a probability which is bounded away from zero by a constant depending only on $\operatorname{Im} z$ and m.) Hence the density can be calculated using the half-plane Poisson kernel after a change of coordinates. Note that we may let $\operatorname{Im} z\to 0$ to obtain the hitting distribution of a *Brownian excursion.*

3. Convergence of radial to chordal SLE

In this section we prove Proposition 1.3 and we will assume that $\kappa\leqslant 4$. Consider a sequence $\{z^{(n)}\}_{n=1}^{\infty}$ in $\mathbb{H}$ such that $z^{(n)}\to\infty$ as $n\to\infty$. We write $z^{(n)}=x^{(n)}+iy^{(n)}$. For each n write $\gamma^{(n)}(t):=\gamma^{0,z^{(n)}}(t)$, $t\in[0,T_n]$, for the radial SLE(κ) path in $\mathbb{H}$ from 0 to $z^{(n)}$ parameterized by half-plane capacity, that is, $\operatorname{hcap}(\gamma^{(n)}[0,t])=2t$ and $\lim_{t\to T_n-}\gamma^{(n)}(t)=z^{(n)}$. Note that T_n is random. As before let $\gamma(t)$, $t\in[0,\infty)$, denote the chordal SLE(κ) path parameterized by half-plane capacity. By considering a Loewner chain driven by a constant function we can see that

$$\operatorname{hcap}(\beta[0,t])\geqslant\sup_{0\leqslant s\leqslant t}\frac{(\operatorname{Im}\beta(s))^2}{2} \tag{3.1}$$

holds for a non-self-intersecting path $\beta(t)$ in $\mathbb{H}$ parameterized by half-plane capacity with $\beta(0)\in\mathbb{R}$. Since $\operatorname{Im}\gamma^{(n)}(T_n)=y^{(n)}$, it follows that $T_n\geqslant(y^{(n)})^2/4$. For $R<|z^{(n)}|$, we define the stopping times $T_R=\inf\{t:|\gamma(t)|\geqslant R\}$ and $T_R^{(n)}=\inf\{t:|\gamma^{(n)}(t)|\geqslant R\}$. Note that the stopped paths are contained in $D(0,R)$. Since the paths are parameterized by half-plane capacity this implies that T_R and $T_R^{(n)}$ are both bounded by $R^2/2$. Indeed, the half-plane capacity of a semi-disc of radius R equals R^2.

In [16] Schramm and Wilson calculated explicitly the chordal Loewner driving process for the radial SLE path in $\mathbb{H}$. It turns out that the driving process has the same distribution as that of the SLE($\kappa,\kappa-6$) path, up to a suitable stopping time. To be more precise, the following result is contained in [16, Theorem 3].

Lemma 3.1. (Schramm–Wilson) *Let ϕ be a Möbius transformation mapping $\mathbb{D}$ onto $\mathbb{H}$. Denote by $\widetilde{\gamma}$ the standard radial* SLE(κ) *path in $\mathbb{D}$. Then the path*

$\phi\circ\tilde{\gamma}$, parameterized by half-plane capacity, has the same law as the SLE($\kappa,\kappa-6$) *path started at $\phi(1)$ with force point $\phi(0)$, stopped at an a.s. positive stopping time.*

Recall that we assume that $\kappa\in(0,4]$. In this case, the radial path is a.s. simple and only touches $\partial\mathbb{D}$ at $t=0$, so the image path under ϕ does not disconnect $\phi(0)$ from infinity, nor does it escape to infinity. We can take the stopping time in Lemma 3.1 to be the hitting time of $\partial D(\phi(1),R)$ as long as $|\phi(1)-\phi(0)|>R$.

It is known that SLE(κ,ρ) can be obtained by weighting SLE(κ) by a martingale in the sense that the driving process can be obtained by an absolutely continuous change of measure via Girsanov's theorem, see [21] and [16]. We shall review the case of one force point $z\in\mathbb{H}$. By applying Itô's formula one can show that the process

$$D_t:=|g_t'(z)|^{(8-2\kappa+\rho)\rho/8\kappa}y_t^{\rho^2/8\kappa}|\sqrt{\kappa}B_t-z_t|^{\rho/\kappa} \tag{3.2}$$

is a local martingale on the interval $[0,\tau(z))$. (See Section 2.5 for the definition of $\tau(z)$ and [16] for details about the local martingale.) Here $x_t+iy_t=z_t:=g_t(z)$, where g_t is the chordal SLE(κ) Loewner chain. We can take $\eta_m=\inf\{t:y_t\leqslant 1/m\}$, $m=1,2,\ldots$, as a localizing sequence. Clearly, η_m are stopping times increasing to $\tau(z)$ and $D_{t\wedge\eta_m}$ is a true martingale for each m.

Notice that the local martingale D_t takes on a particularly nice form when $\kappa=2$ and $\rho=\kappa-6=-4$, namely $D_t=y_t/|\sqrt{\kappa}B_t-z_t|^2$ which we recognize as the half-plane Poisson kernel with pole at $\sqrt{\kappa}B_t$, evaluated at z_t. A discrete version of this invariant martingale is the observable that is used to prove convergence of the Loewner driving function in [10].

We see that D_t is strictly positive, so it is an exponential local martingale. If we use a stopped and normalized version of D_t in Girsanov's theorem, a calculation of the resulting drift shows that we obtain a measure Q under which $\sqrt{\kappa}B$ properly stopped has the distribution of the driving process (2.11) for SLE(κ,ρ) with force point z.

This leads to the following result.

Lemma 3.2. *Assume that $\kappa\in(0,4]$ and that $\{z^{(n)}\}_{n=1}^{\infty}$ is a sequence in $\mathbb{H}$ such that $z^{(n)}\to\infty$, as $n\to\infty$. For each n, let $(W_t^{(n)})_t$ denote the chordal driving process for radial* SLE(κ) *in $\mathbb{H}$ between 0 and $z^{(n)}$, parameterized by half-plane capacity. Then for every fixed $R>0$, as $n\to\infty$, the law of $(W^{(n)}_{t\wedge T_R^{(n)}})_t$ converges in total variation to the law of $(\sqrt{\kappa}B_{t\wedge T_R})_t$, where B is standard Brownian motion.*

Proof. We may assume that $|z^{(n)}|>2R$ for all n. Let B be standard Brownian motion on $\mathbb{R}$ under the measure P. By Lemma 3.1, under P, $W^{(n)}$ has the same distribution as the driving process for the SLE($\kappa,\kappa-6$) path with force point $z^{(n)}$ up to a stopping time that we can take to be $T_R^{(n)}$; clearly $T_R^{(n)}<\tau(z^{(n)})$ a.s. Let $D^{(n)}=D^{(n)}_{t\wedge T_R^{(n)}}$ be defined by (3.2) with z replaced by $z^{(n)}$ and ρ replaced by $\kappa-6$ and normalized so that $D_0^{(n)}=1$. Hence if $\sigma:=t\wedge T_R$ then

$$D^{(n)}=|g_\sigma'(z^{(n)})|^{(2-\kappa)(\kappa-6)/8\kappa}\left(\frac{y_\sigma^{(n)}}{y^{(n)}}\right)^{(\kappa-6)^2/8\kappa}\left|\frac{\sqrt{\kappa}B_\sigma-z_\sigma^{(n)}}{z^{(n)}}\right|^{(\kappa-6)/\kappa}. \tag{3.3}$$

We claim that $D^{(n)}$ is a bounded martingale. Indeed, Koebe's estimate implies that

$$|g_t'(z^{(n)})|\asymp\frac{\operatorname{Im} g_t(z^{(n)})}{\operatorname{dist}(z^{(n)},\partial H_t)}$$

for $t<\tau(z^{(n)})$, where $H_t=\mathbb{H}\setminus\gamma^{(n)}[0,t]$ and $\asymp$ means that both sides are bounded by a constant times the other. Since $\gamma^{(n)}[0,T_R^{(n)}]$ is contained in a closed half-disc centered at the origin with radius R, we have for $t\leqslant T_R^{(n)}$ that

$$\operatorname{Im} g_t(z^{(n)})\geqslant\operatorname{Im}\psi(z^{(n)})=\operatorname{Im} z^{(n)}\left(1-\frac{R^2}{|z^n|^2}\right),$$

where $\psi(z)=z+R^2/z$ maps the complement of the half-disc onto $\mathbb{H}$. Since also $\operatorname{Im} g_t(z^{(n)})\leqslant\operatorname{Im} z^{(n)}$ we get that

$$\frac{\operatorname{Im} g_t(z^{(n)})}{\operatorname{dist}(z^{(n)},\partial H_t)}\asymp 1,$$

when $t\leqslant T_R^{(n)}$ and consequently the same holds for $|g_t'(z^{(n)})|$. Hence, for each n, $D^{(n)}$ is bounded. By dominated convergence, we may pass to the limit to obtain

$$1=\lim_{t\to\infty}\mathbb{E}[D^{(n)}_{t\wedge T_R^{(n)}}]=\mathbb{E}[D^{(n)}_{T_R^{(n)}}].$$

Consider the measure $Q^{(n)}$ defined by

$$dQ^{(n)}=D^{(n)}\,dP.$$

By Girsanov's theorem, under $Q^{(n)}$, $(\sqrt{\kappa}B_{t\wedge T_R})_t$ has the same law as $(W^{(n)}_{t\wedge T_R^{(n)}})_t$ under P. This means that under $Q^{(n)}$, the stopped chordal SLE(κ) path has the same law as the stopped radial path in $\mathbb{H}$ under P. We show that $D^{(n)}\to1$, as $n\to\infty$, P-a.s. By the series expansion of $g_{T_R^{(n)}}$ at infinity, keeping in mind that $T_R^{(n)}$ is bounded P-a.s. we have that $z^{(n)}_{T_R^{(n)}}=z^{(n)}+O(1/|z^{(n)}|)$, $y^{(n)}_{T_R^{(n)}}=y^{(n)}+O(y^{(n)}/|z^{(n)}|^2)$, and $x^{(n)}_{T_R^{(n)}}=x^{(n)}+O(x^{(n)}/|z^{(n)}|^2)$ for large n. Also $|g'_{T_R^{(n)}}(z^{(n)})|=1+O(1/|z^{(n)}|^2)$ if n is large enough. Thus, plugging this into (3.3), we obtain that P-a.s.

$$\lim_{n\to\infty}\frac{dQ^{(n)}}{dP}=\lim_{n\to\infty}D^{(n)}=1.$$

This implies that $Q^{(n)}$ converges to P in total variation, which in particular implies that the law of $(W^{(n)}_{t\wedge T_R^{(n)}})_t$ converges in total variation to that of $(\sqrt{\kappa}B_{t\wedge T_R})_t$. □

Proof of Proposition 1.3. By Lemma 3.2 the driving processes converge in law. Hence, since the paths are simple, it follows that the laws of the paths in the capacity parameterization converge weakly with respect to local Hausdorff distance (at any fixed time), see Proposition 5.1 of [14]. To prove that the paths converge with respect to ρ we have to prove tightness. To this end, note that there is an $\alpha=\alpha(\kappa)>0$ such that γ is Hölder-α in the capacity parameterization almost surely; see, e.g., [7]. Consequently, by absolute continuity, the same holds for $\gamma^{(n)}$. Fix $\varepsilon>0$ and let $M<\infty$ be such that $P(C_\alpha\leqslant M)\geqslant 1-\varepsilon/2$, where C_α is the (random) Hölder-α norm of γ. Then if $C_\alpha^{(n)}$ is the Hölder-α norm of $\gamma^{(n)}$ we have

$$P(C_\alpha^{(n)}\leqslant M)=Q^{(n)}(C_\alpha\leqslant M).$$

Hence, using the fact that $Q^{(n)}$ converges to P in total variation, we get that $P(C_\alpha^{(n)}\leqslant M)\geqslant 1-\varepsilon$, whenever n is sufficiently large. By the Arzelà–Ascoli theorem this proves tightness. Finally, since the stopped driving processes converge weakly with respect to the supremum norm on, say, $[0,R^2/2]$, we can couple them in such a way that $T_R^{(n)}$ converges to T_R in probability. (Recall that T_R and $T_R^{(n)}$ are bounded by $R^2/2$.) Indeed, the probability that there is a subinterval of $[0,R^2/2]$ of any fixed positive length on which the supremum of the modulus of a Brownian motion is smaller than ε is $o(1)$ as $\varepsilon\to0$. Hence, using absolute continuity and a modulus of continuity estimate for chordal SLE, we conclude that for any $\varepsilon>0$, if n is sufficiently large, we can couple the stopped paths so that their ρ distance exceeds ε with probability at most ε. It follows that the stopped paths converge weakly with respect to ρ. □

Here we again use the notation of Section 2.4.

Corollary 3.3. *Let $R>1$ and $\kappa\in(0,4]$. For every $\varepsilon>0$ there exists $d_0>0$ so that if $|z|<d_0$ then there is a coupling of $\gamma^{z,\infty}$ and γ such that*

$$\mathbb{P}\left(\rho_R(\gamma^{z,\infty},\gamma)>\varepsilon\right)<\varepsilon.$$

The idea is to argue that the inverted and reversed paths are close after the first time they hit $\partial D(0,\varepsilon)$, if $|z|$ is small enough. By reversibility of chordal SLE, the distribution of the chordal path is invariant under inversion and time-reversal.

Proof of Corollary 3.3. Let $\varepsilon>0$ be smaller than R. Consider the last time γ hits $\partial D(0,1/\varepsilon)$:

$$\sigma=\sup\{t\geqslant 0:|\gamma(t)|=1/\varepsilon\},$$

where the path is parameterized by half-plane capacity. The transience of chordal SLE implies that $\sigma<\infty$ a.s. Clearly $|\gamma(t)|>1/\varepsilon$ for $t>\sigma$. Consequently we can choose a (non-random) $R_0<\infty$ such that

$$\mathbb{P}\,(\sigma<T_{R_0})\geqslant 1-\varepsilon, \tag{3.4}$$

where $T_{R_0}=\inf\{t\geqslant 0: |\gamma(t)|=R_0\}$ is a stopping time. Proposition 1.3 implies that for all $\varepsilon_1>0$ there exists R_1 so that if $|z|>R_1$ then there is a coupling of $\gamma^{0,z}$ and γ such that

$$\mathbb{P}\,(\rho(\gamma^{0,z},\gamma)>\varepsilon_1)<\varepsilon_1 \tag{3.5}$$

for the paths stopped when hitting $\partial D(0,R_0)$. By taking R_1 larger if necessary, we can assume that the estimate analogous to (3.4) holds also for $\gamma^{0,z}$. We now apply the mapping $\iota(z)=-1/z$ to the paths. Note that ι is uniformly continuous on $\{z:|z|\geqslant 1/R\}$. Hence, by choosing $\varepsilon_1<\varepsilon$ small enough (and the corresponding $R_1<\infty$ sufficiently large), (3.4) and (3.5) together with Lemmas 2.2 and 2.3 imply that if $|z|>R_1$ then there is a coupling of $\gamma^{0,z}$ and γ such that

$$\mathbb{P}\,(\rho_R(\iota(\gamma^{0,z})^{\mathrm{rev}},\iota(\gamma)^{\mathrm{rev}})>3\varepsilon)<3\varepsilon. \tag{3.6}$$

By reversibility of chordal SLE, see [23], considered as paths stopped when exiting $D(0,R)$, $\iota(\gamma)^{\mathrm{rev}}$ has the same distribution as γ. The corollary then follows from (3.6) by taking $d_0=1/R_1$ and recalling that $\gamma^{w,\infty}$ is shorthand for $\iota(\gamma^{0,\iota(w)})^{\mathrm{rev}}$. □

4. Coupling of LERW excursions

The goal of this section is to prove Proposition 1.2. We shall use our version of Wilson's algorithm (Lemma 2.1) to couple LERW excursions from nearby points. For this, we need a uniform estimate on a hitting probability for random walk excursion. The difficulty is that we start from a boundary point which prohibits using (2.2) directly.

For $L>0$ set

$$R=R(L)=\{z:-1<\operatorname{Re} z<1 \text{ and } 0\leqslant \operatorname{Im} z<L\},$$
$$E=E(L)=\{z\in\partial R:\operatorname{Im} z=L\}.$$

Lemma 4.1. *Let S be random walk excursion on $\delta\mathbb{Z}^2$ started from δi, $0<\delta<L$. Let p_S be the probability that S exits R using an edge intersecting E. Then there is a constant $c>0$ depending only on L such that $p_S>c$.*

Proof. Let X be simple random walk on $\delta\mathbb{Z}^2$ started from δi. Define the reflected random walk X' by

$$X' = \begin{cases} X, & \text{if } \operatorname{Im} X \geqslant 0, \\ \overline{X}, & \text{otherwise.} \end{cases}$$

It follows from the definition that the probability $p_{X'}$ that X' exits R through E is the same as the probability that X exits $R\cup\overline{R}$ through $E\cup\overline{E}$. Hence, by the convergence of simple random walk to Brownian motion, $p_{X'}>c$ for some constant $c>0$ depending only on L.

Set $\tau_S=\min\{j\geqslant 1 : \operatorname{Im} S\geqslant L\}$ and let $\tau_{X'}$ be the corresponding stopping-time for X'. We see from the transition probabilities for S and X' that

$$\mathbb{P}(\tau_S<j)\geqslant \mathbb{P}(\tau_{X'}<j), \quad j\geqslant 1. \tag{4.1}$$

Let $\eta_S=\min\{j\geqslant 1 : |\operatorname{Re} S|\geqslant 1\}$ and let $\eta_{X'}$ be the corresponding stopping-time for X'. The random walks $\operatorname{Re} X'$ and $\operatorname{Re} S$ have the same distribution, so by using (4.1) we get that

$$p_S=\mathbb{P}(\tau_S<\eta_S)=\sum_{j=1}^{\infty}\mathbb{P}(\tau_S<j)\mathbb{P}(\eta_S=j)\geqslant \sum_{j=1}^{\infty}\mathbb{P}(\tau_{X'}<j)\mathbb{P}(\eta_{X'}=j)=p_{X'},$$

and since $p_{X'}>c$ the proof is complete. □

Let S_δ denote random walk excursion on $\delta\mathbb{Z}^2$ started from 0. Consider the half-square

$$Q=Q(r)=\{x+iy : -r<x<r \text{ and } 0<y<r\},$$

where $r<\frac{1}{2}$. Let $h_\beta(x)=x/|\log(x)|^\beta$ for $\beta\in[0,1)$ fixed and define the *weak cone*

$$\mathcal{C}_\beta=\mathcal{C}(h_\beta)=\{x+iy : -1<x<1 \text{ and } y>h_\beta(|x|)\}.$$

We set $T_r=\min\{j\geqslant 0 : S_\delta(j)\notin Q(r)\}$ and consider the stopped walk $S_\delta(j)$, $j=0,...,T_r$.

Proposition 4.2. *Fix $0\leqslant\beta<1$ and $r<\frac{1}{2}$. For each $\varepsilon>0$ there exists a $d_0>0$ such that the following holds uniformly in $\delta>0$ sufficiently small. Let $K_\delta\subset\delta\mathbb{Z}^2\cap\mathbb{H}$ be a connected set with* $\operatorname{dist}(0,K_\delta)\leqslant d_0$ *that separates the two components of $\partial\mathcal{C}_\beta\cap(Q(r)\setminus Q(d_0))$ in $Q(r)\setminus Q(d_0)$. Then*

$$\mathbb{P}(S_\delta\cap K_\delta=\varnothing)<\varepsilon. \tag{4.2}$$

Remark. Connectedness is meant in the sense that K_δ contains a nearest-neighbor path between any two of its vertices.

Remark. For $\beta=0$ we can see from the proof that the upper bound in (4.2) can be taken to be $c_1(\operatorname{dist}(0,K_\delta)/r)^{c_2}$ for uniform constants $0<c_1,c_2<\infty$, and in this case we may take r to be arbitrarily large.

Remark. It is necessary to assume that K_δ separates the boundary components in a weak cone such as $\mathcal{C}_\beta$. Indeed, if $K(d)=\{x+iy: x\leqslant 0 \text{ and } y=d\}$, then there is a uniform constant $c>0$ such that for all $d>0$,

$$\mathbb{P}(S_\delta\cap K(d)=\varnothing)>c,$$

whenever $\delta>0$ is sufficiently small compared to d. (S_δ is considered as a curve whereby the intersection is well defined.) To see this, note that for $\delta>0$ sufficiently small the probability of the event that S_δ reaches $\{z:\operatorname{Im} z>2d\}$ without hitting $K(d)$ is uniformly bounded from below by some strictly positive constant independent of d. By the strong Markov property and a gambler's ruin estimate for simple random walk, conditional on this event, there is a strictly positive probability (independent of d) that S_δ never returns to $\{z:\operatorname{Im} z\leqslant d\}$. This implies the stated inequality. See also [7].

Proof of Proposition 4.2. Let us fix $m\in\mathbb{N}$ to be determined later. We define $d_k=2^{-k}r$ for $k=0,1,...,m$. Note that $d_{k+1}<d_k$. For each $k=1,...,m$ define the dyadic half-annulus

$$A^k=\{x+iy: d_k\leqslant \max\{|x|,y\}\leqslant d_{k-1} \text{ and } y\geqslant 0\}$$

and then set

$$B^k=\{x+iy: |x|\leqslant y \text{ and } 0\leqslant |x|\leqslant d_{k-1}\}\cap A^k,$$
$$C^k=\{x+iy: h_\beta(|x|)\leqslant y\leqslant |x| \text{ and } 0\leqslant |x|\leqslant d_{k-1}\}\cap A^k,$$

and

$$D^k=\overline{A^k\setminus(B^k\cup C^k)}.$$

Next, let $E^k=\partial C^k\cap\partial D^k=\partial\mathcal{C}_\beta\cap A^k$ and $F^k=\left(\frac{3}{4}\partial A^k\right)\cap A^k$. Finally we set $G^k=\partial B^k\cap\partial C^k$. We use the subscripts $-$ and $+$ to distinguish components with negative and positive real parts.

Let T_k be the first time that S_δ has used an edge intersecting F^k. Notice that $T_{k-1}>T_k$. Given $S_\delta(T_k)$, let J_k^- be the event that $S_\delta[T_k,T_{k-1}]$ uses an edge intersecting E_-^k before exiting A^k, and let J_k^+ be defined in the same way with E_-^k

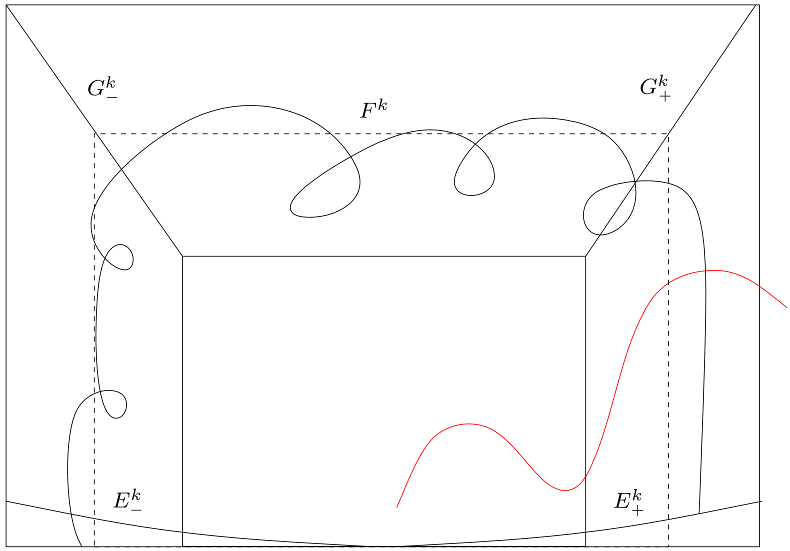

Figure 1. Sketch for the proof of Proposition 4.2. The probability of this event is bounded below by c/k^{β}, $\beta<1$

replaced by E^k_+. We claim that there is a constant $c>0$ only depending on β and r such that for $k=1, ..., m$,

$$\min\{\mathbb{P}\,(J_k^-), \mathbb{P}\,(J_k^+)\} > \frac{c}{k^{\beta}} \tag{4.3}$$

whenever $\delta>0$ is small enough. The proposition is a consequence of this fact and the (strong) Markov property of random walk excursion. Indeed, by assumption there is always a lattice path in K_δ separating the vertex $S_\delta(T_k)$ from at least one of E^k_- and E^k_+ in A^k. Hence

$$\mathbb{P}(S_\delta[T_k, T_{k-1}] \cap K_\delta \neq \varnothing \mid S_\delta(T_k)) \geqslant \min\{\mathbb{P}(J_k^-), \mathbb{P}(J_k^+)\}.$$

Since

$$\{S_\delta^r \cap K_\delta = \varnothing\} \subset \bigcap_{k=1}^{m} \{S_\delta[T_k, T_{k-1}] \cap K_\delta = \varnothing\},$$

the Markov property and (4.3) imply that

$$\mathbb{P}(S_\delta^r \cap K_\delta = \varnothing) \leqslant \prod_{k=1}^m \mathbb{P}(S_\delta[T_k, T_{k-1}] \cap K_\delta = \varnothing \mid S_\delta(T_k)) \leqslant \prod_{k=1}^m \Big(1 - \frac{c}{k^\beta}\Big) < \varepsilon,$$

if $m<\infty$ is chosen sufficiently large and $\delta>0$ sufficiently small. Here we have used that $\beta<1$.

It remains to prove (4.3). Let $\widetilde{S}_\delta$ be random walk excursion started from $\lfloor -(d_k+d_{k-1})/2 \rfloor_\delta + \delta i$, and let p_k denote the probability that $\widetilde{S}_\delta$ uses an edge intersecting E_+^k before exiting A^k. If δ is small enough, this probability is positive. By symmetry and the Markov property, it is enough to prove that p_k is bounded from below by c/k^β, where $c>0$ may depend on β and r. By Lemma 4.1 the probability that $\widetilde{S}_\delta$ hits G_-^k in a point with x-coordinate in the middle third of $[-d_{k-1}, -d_k]$ before exiting A^k is bounded from below by a constant $c_1>0$ independent of $\delta>0$ small enough and k. By the Markov property and (2.2) using the fact that $\operatorname{Im} w/\operatorname{Im} z \asymp 1$ for any $z, w \in B^k$, the same holds with G_-^k replaced by G_+^k. Finally, since $\operatorname{Im} w/\operatorname{Im} z \geqslant c/k^\beta |\log r|^\beta$ for $z \in G_+^k$ and $w \in E_+^k$, using (2.2) and the Markov property once again, we see that

$$p_k > \frac{c/|\log r|^\beta}{k^\beta},$$

where $c>0$ only depends on β whenever $\delta>0$ is small enough. This proves (4.3) and completes the proof. $\square$

To prove Proposition 1.2 we need to show that the initial part of the LERW excursion path from a point close to 0 satisfies the weak cone separation condition from Proposition 4.2 with large probability. It would be desirable to prove this directly; we shall however use Theorem 1.1 from [17] (in a slightly modified form). As before, let $h_\beta(x) = x/|\log x|^\beta$ and let $\mathcal{C}(h_\beta) = \big\{x+iy : y > h_\beta(|x|) \text{ and } |x| < \tfrac{1}{2}\big\}$. Set $\beta(\kappa) = 1/(8/\kappa - 2)$.

Lemma 4.3. (Schramm–Zhou) *Let γ be the standard chordal* SLE(κ) *path, $\kappa \in (0,4)$ and let $\beta > \beta(\kappa)$. For each $\varepsilon>0$ there exists an $r>0$ such that*

$$\mathbb{P}\left(\gamma(0, T_r] \subset \mathcal{C}(h_\beta)\right) \geqslant 1 - \varepsilon, \tag{4.4}$$

where $T_r = \inf\{t \geqslant 0 : |\gamma(t)| \geqslant r\}$.

Note in particular that $\beta(2) = \frac{1}{2} < 1$.

Remark 4.4. Actually, $\mathcal{C}(h_\beta)$ is not the optimal weak cone implied by the result from [17]. Let $g_\beta(x) = |\log x|^{2\beta}/(|\log x|^{2\beta}+1)$, and set

$$\mathcal{C}'_\beta = \{z : \operatorname{Re} z = x g_\beta(|x|), \operatorname{Im} z > h_\beta(|x|) g_\beta(|x|) \text{ and } |x| < r\}.$$

Then $\mathcal{C}'_{\beta(\kappa)}$ is the sharp weak cone for SLE(κ). However, since $g_\beta \leqslant 1$ and $h_\beta(|x|)$ is convex we see that $\mathcal{C}'_\beta \subset \mathcal{C}(h_\beta)$.

Proof of Proposition 1.2. Let $\varepsilon>0$ be given. Let r' be such that (4.4) holds with ε as given and $\beta=\frac{2}{3}>\frac{1}{2}=\beta(2)$. Let $r:=\min\{r',\varepsilon\}/2$. Let d'' be the constant whose existence is asserted by Proposition 4.2 and which corresponds to ε, $r/\sqrt{2}$ and $\beta=\frac{2}{3}$. (That is, $d''=d_0=d_0\big(\varepsilon, r/\sqrt{2}, \frac{2}{3}\big)$ in the notation of Proposition 4.2.) By Corollary 3.3 there exist d^* and d' with $0<d'\leqslant d^*\leqslant d''<r$ such that if $y<d'$ and η is the radial SLE(2) path in $\mathbb{H}$ from ∞ to iy, then

$$\mathbb{P}(\eta \cap \mathcal{A}(d^*, r) \subset \mathcal{C}(h_{3/4})) \geqslant 1-\varepsilon, \tag{4.5}$$

where $\mathcal{A}(a,b)=\{z\colon a<|z|<b\}$. Here we used that $\mathcal{C}(h_{2/3}) \subsetneq \mathcal{C}(h_{3/4})$.

Let S^y_δ be random walk excursion on $\delta\mathbb{Z}^2$ from a lattice point closest to iy. Set $L^y_\delta := L\{S^y_\delta\}[0,T_r]$ and consider the event $\mathcal{E}_\delta$ that $L^y_\delta \cap \mathcal{A}(d^*, r) \subset \mathcal{C}(h_{4/5})$. By the convergence of radial LERW to radial SLE(2) (Corollary 2.5) and (4.5), since $\mathcal{C}(h_{3/4}) \subsetneq \mathcal{C}(h_{4/5})$, there exists $\delta_0=\delta_0(y)>0$ such that

$$\mathbb{P}(\mathcal{E}_\delta) > 1-2\varepsilon, \tag{4.6}$$

whenever $\delta<\delta_0$. Let S_δ be random walk excursion from 0, and let $\mathcal{F}_\delta$ be the event that S_δ hits L^y_δ before hitting $\partial D(0,r)$. By Proposition 4.2 we see that $\mathbb{P}(\mathcal{F}^c_\delta|\mathcal{E}_\delta)<\varepsilon$, since $y\leqslant d''$, and this holds uniformly in δ small enough. Hence in view of (4.6) we get that

$$\mathbb{P}(\mathcal{F}_\delta) > 1-3\varepsilon$$

for all δ small enough. We now apply Wilson's algorithm (Lemma 2.1) to see that we may couple LERW excursion from 0 with LERW excursion from iy (both on $\delta\mathbb{Z}^2$, with δ small enough) so that with probability at least $1-3\varepsilon$ the paths *agree* outside the disc $D(0,r)\subset D(0,\varepsilon)$. In view of Lemma 2.2 this completes the proof. □

5. Proof of Theorem 1.1

We now combine the results from the previous sections to prove our main result, Theorem 1.1. For the moment, let $R>1$ be fixed and let $\varepsilon>0$ and $T<\infty$ be given. It will be enough to show that when $\delta>0$ is sufficiently small we can couple γ and γ_δ so that

$$\mathbb{P}(\rho_R(\gamma,\gamma_\delta)>\varepsilon)<\varepsilon. \tag{5.1}$$

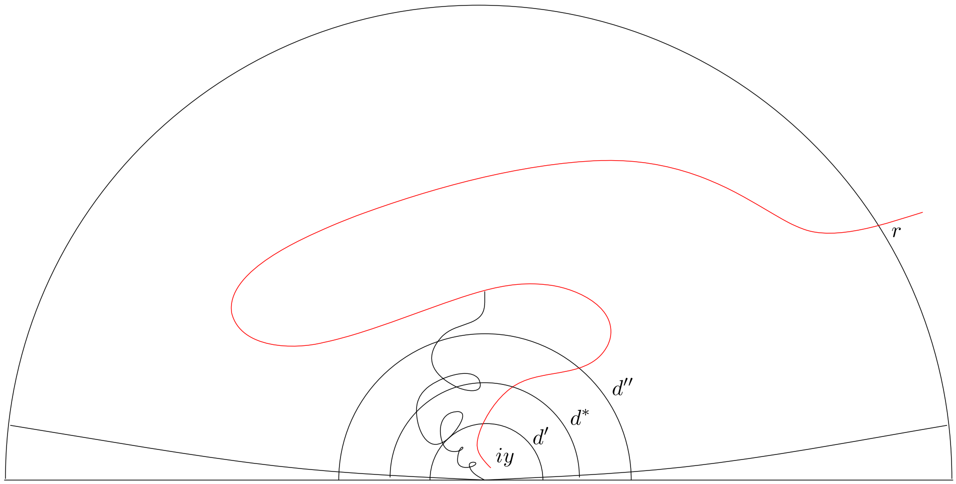

Figure 2. Sketch for the proof of Proposition 1.2. Since the loop-erased path is contained in a weak cone, it is quickly hit by the random walk. Note that $r<\varepsilon\ll R$

Indeed, assume that such a coupling exists and that γ and γ_δ are parameterized by capacity. There is a uniform constant $c<\infty$ such that (see [7])

$$\operatorname{diam}(\gamma[0,t]) \leqslant c\Big(\sqrt{t}+\sup_{0\leqslant s\leqslant t}|W_s|\Big),$$

where $W=\sqrt{2}B$ is the scaled Brownian motion driving γ. Consequently we can choose $R_0=R_0(T,\varepsilon)<\infty$ such that

$$\mathbb{P}\left(\gamma[0,2T]\subset D(0,R_0)\right)>1-\varepsilon.$$

Let $\mathcal{E}_\delta$ be the event that $\rho_{R_0}(\gamma_\delta,\gamma)\leqslant\varepsilon$ and $\gamma[0,2T]\subset D(0,R_0)$. Whenever $\delta>0$ is sufficiently small we have $\mathbb{P}\left(\mathcal{E}_\delta^c\right)<2\varepsilon$. By taking δ smaller if necessary, on $\mathcal{E}_\delta$, there is a reparameterization φ_δ of γ with the property that

$$\sup_{0\leqslant t\leqslant T}|\gamma_\delta(t)-\gamma(\varphi_\delta(t))|=o(1)$$

as $\delta\to0$. This estimate implies that

$$2\varphi_\delta(t)=\operatorname{hcap}(\gamma[0,\varphi_\delta(t)])=\operatorname{hcap}(\gamma_\delta[0,t])+o(1)=2t+o(1).$$

Here we used that if the paths converge in the Hausdorff metric, then the corresponding uniformizing conformal mappings and their derivatives converge uniformly on compact sets. Hence, using a modulus of continuity estimate for γ in the capacity parameterization (see [7]) we see that whenever δ is small enough the ρ-distance of $\gamma(t)$, $t\in[0,T]$, and $\gamma_\delta(t)$, $t\in[0,T]$, is smaller than ε on $\mathcal{E}_\delta$, and the probability of this event is at least $1-2\varepsilon$.

It remains to prove the existence of a coupling such that (5.1) holds. Let $R>1$ be fixed. By Proposition 1.2 we can find $d_0>0$ so that when $d\leqslant d_0$ and z is a lattice point closest to di whenever $0<\delta<\delta_0(d)$ there exists a coupling of γ_δ and γ_δ^z such that

$$\mathbb{P}\,(\rho_{3R}(\gamma_\delta,\gamma_\delta^z)>\varepsilon)<\varepsilon.$$

Corollary 2.5 implies that there exists $\delta_1=\delta_1(d)>0$ such that whenever $\delta<\delta_1$ there is a coupling of γ_δ^z and $\gamma^{z,\infty}$ such that

$$\mathbb{P}\,(\rho_{3R}(\gamma_\delta^z,\gamma^{z,\infty})>\varepsilon)<\varepsilon.$$

Consequently, as long as $\delta<\min\{\delta_0(d),\delta_1(d)\}$ there is a coupling of γ_δ and $\gamma^{z,\infty}$ such that

$$\mathbb{P}\,(\rho_{2R}(\gamma_\delta,\gamma^{z,\infty})>2\varepsilon)<2\varepsilon. \tag{5.2}$$

Next, using Corollary 3.3, we can choose $0<d'\leqslant d_0$ small enough so that whenever $|z'|\leqslant d'$ there will exist a coupling of γ and $\gamma^{z',\infty}$ such that

$$\mathbb{P}\,(\rho_{2R}(\gamma^{z',\infty},\gamma)>\varepsilon)<\varepsilon. \tag{5.3}$$

Finally, by combining (5.2) and (5.3) it follows that whenever $\delta<\min\{\delta_0(d'),\delta_1(d')\}$ we can couple γ_δ with γ in such a way that

$$\mathbb{P}\,(\rho_R(\gamma_\delta,\gamma)>3\varepsilon)<3\varepsilon.$$

This concludes the proof. $\square$

6. Other domains

In this section we discuss how to extend our argument for the half-plane to a certain class of simply connected domains. We stress that we do not give a proof, but rather a sketch of how one could try to proceed.

Consider a bounded simply connected domain $D\subset\mathbb{H}$ with two marked distinct boundary points x and y. We assume that ∂D is locally analytic around x and y, and that x lies in an open interval $I\subset\mathbb{R}$ such that $\mathbb{R}\cap\partial D=I$. This means that we can reflect D in I to obtain the simply connected domain $\widetilde{D}=D\cup\overline{D}\cup I$. For each $\delta>0$, the intersection of $\partial\widetilde{D}$ with the closed faces of $\delta\mathbb{Z}^2$ partitions the plane and we let $\widetilde{D}_\delta\subset\widetilde{D}$ be the component containing the origin. Set $D_\delta=\widetilde{D}_\delta\cap\mathbb{H}$. Then $\widetilde{D}_\delta$ and D_δ are grid domains in the terminology of [10], that is, they are simply connected domains with boundary contained in the edge set of $\delta\mathbb{Z}^2$. Let V_δ and $\widetilde{V}_\delta$ be the interior vertices in D_δ and $\widetilde{D}_\delta$, respectively. Note that D_δ converges to D in the Carathéodory sense, as $\delta\to 0$. We choose a sequence $y_\delta\in\partial V_\delta$, accessible by simple random walk from 0, such that $y_\delta\to y$ as $\delta\to 0$. Define $\varphi_\delta\colon D_\delta\to\mathbb{H}$ to be the conformal mapping such that $\varphi_\delta(0)=0$, $\varphi_\delta(y_\delta)=\infty$, and $|\varphi_\delta'(0)|=1$. We extend φ_δ to $\widetilde{\varphi}_\delta$ (defined on $\widetilde{D}_\delta$) by Schwarz reflection. Set $u_\delta:=\lfloor\varphi_\delta^{-1}(i)\rfloor_\delta$ and let $\widehat{\varphi}_\delta\colon D_\delta\to\mathbb{H}$ be the conformal mapping normalized by $\widehat{\varphi}_\delta(y_\delta)=\infty$ and $\widehat{\varphi}_\delta(u_\delta)=i$. Then for each $\delta>0$ we can write $\widehat{\varphi}_\delta=c_1\varphi_\delta+c_2$, where c_1 and c_2 are real-valued and satisfy $|c_1-1|\asymp|c_2|\asymp\delta$.

Let γ_δ be loop-erased random walk excursion on V_δ from 0 to y_δ and for $z\in D_\delta$ let γ_δ^z be loop-erased random walk excursion on V_δ from $\lfloor z\rfloor_\delta$ to y_δ, provided $\lfloor z\rfloor_\delta\in V_\delta$. A natural extension of our main result is to show that the law of the curve $\varphi_\delta(\gamma_\delta)$ converges weakly, as $\delta\to 0$, to the law of the chordal SLE(2) path in the sense of Theorem 1.1.

Loop-erased random walk from an interior point, by Lawler–Schramm–Werner's theorem, is known to converge to radial SLE. (Recall, however, that their result is for the loop-erasure of an unconditioned walk. Consequently one will have to be careful regarding tightness. Instead of conditioning on a fixed exiting point one could, e.g., condition the simple random walk generating the loop-erased random walk to exit at some (small) part J_δ of the boundary containing y_δ with harmonic measure uniformly bounded away from 0.) Since the convergence of radial SLE to chordal SLE can be used without modification, it remains to check that an analogue of the hitting estimate Proposition 1.2 holds on V_δ.

Let $R>0$. For each $\varepsilon>0$ there exists $d_0>0$ such that the following holds. Assume that z satisfies $|z|\leqslant d_0$ and that $\varphi_\delta(\lfloor z\rfloor_\delta)\in\mathcal{C}_\beta$ for all δ small enough. Then there exist $\delta_0>0$ and whenever $\delta<\delta_0$ a coupling of γ_δ with γ_δ^z such that

$$\mathbb{P}(\rho_R(\varphi_\delta(\gamma_\delta),\varphi_\delta(\gamma_\delta^z))>\varepsilon)<\varepsilon.$$

The level lines of $\operatorname{Im}\varphi_\delta(z)$ play a similar part as the level lines of $\operatorname{Im} z$ for random walk excursion in $\mathbb{H}$. Indeed, recall that random walk excursion in D_δ is obtained by weighting simple random walk by $z\mapsto H_{D_\delta}(z,y_\delta)$. Let

$$\lambda_\delta(z)=\frac{H_{D_\delta}(z,y_\delta)}{H_{D_\delta}(u_\delta,y_\delta)}$$

be the normalized discrete harmonic measure. It was proved in [10, Proposition 2.2] that $\lambda_\delta(z)$ converges for z away from the boundary, as $\delta\to 0$, to a conformally invariant version of the Poisson kernel: for each $\varepsilon>0$ there exists a $\delta_0>0$ such that if $|\psi_\delta(z)|<1-\varepsilon$ and $\delta<\delta_0$ then

$$\left|\lambda_\delta(z)-\frac{1-|\psi_\delta(z)|^2}{|\psi_\delta(z)-\psi_\delta(y_\delta)|^2}\right|<\varepsilon,$$

where $\psi_\delta\colon D_\delta\to\mathbb{D}$ is a conformal mapping normalized by $\psi_\delta(u_\delta)=0$ and $\psi'_\delta(u_\delta)>0$.

After expressing this using the mapping $\widehat{\varphi}$ to the half-plane instead, we see that λ_δ is close to $\operatorname{Im}\widehat{\varphi}_\delta(z)=\operatorname{Im}\varphi_\delta(z)+O(\delta)$ for small δ and z away from the boundary of D_δ. Now, using discrete Schwarz reflection we can extend λ_δ to $\tilde{\lambda}_\delta$ defined on $\widetilde{V}_\delta$ and using again the result from [10] this function converges away from the boundary of $\widetilde{D}_\delta$ to $\operatorname{Im}\widetilde{\varphi}_\delta(z)$, as $\delta\to 0$. A consequence is that we can write the transition probabilities for random walk excursion on V_δ as

$$\mathbb{P}^z(S_\delta(1)=w)=p_\delta(z,w)\frac{\operatorname{Im}\varphi_\delta(w)+\eta_\delta(\varphi_\delta(w))}{\operatorname{Im}\varphi_\delta(z)+\eta_\delta(\varphi_\delta(z))}, \tag{6.1}$$

where $|\eta_\delta(z)|\to 0$ as $\delta\to 0$ uniformly in z away from the boundary of $\widetilde{D}_\delta$. To prove a version of the Beurling-type estimate Proposition 4.2, we can use/adapt Corollary 3.14 from [3], which is an analog of Lemma 4.1 for this situation. We map the random walk excursion to the half-plane and argue as before to prove the hitting estimate (4.3) using (6.1) and convergence of simple random walk to Brownian motion.

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Received June 29, 2010
published online June 11, 2011