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Infima of superharmonic functions

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Abstract. Let Ω be a Greenian domain in $\mathbb{R}^d$, $d\geq 2$, or—more generally—let Ω be a connected $\mathcal{P}$-Brelot space satisfying axiom D, and let u be a numerical function on Ω, $u\not\equiv\infty$, which is locally bounded from below. A short proof yields the following result: The function u is the infimum of its superharmonic majorants if and only if each set $\{x:u(x)>t\}$, $t\in\mathbb{R}$, differs from an analytic set only by a polar set and $\int u\,d\mu_x^V\leq u(x)$, whenever V is a relatively compact open set in Ω and $x\in V$.

The purpose of this paper is to provide a *short* proof for a *full* characterization of infima of superharmonic functions in terms of mean-value inequalities and a weak measurability property (Theorem 2). Its background is the publication [7], where the main results Theorem 1.1 and Theorem 1.5 can be stated as follows.

Theorem. *Let Ω be a Greenian domain in $\mathbb{R}^d$, $d\geq 2$, and $u\colon \Omega\to(-\infty,\infty]$, $u\not\equiv\infty$, be such that u is locally bounded from below and, for every $t\in\mathbb{R}$, the set $\{x:u(x)>t\}$ is analytic. Then the following statements are equivalent*:

(i) *the greatest lower semicontinuous minorant $\hat{u}$ of u is superharmonic, and the set $\{x:\hat{u}(x)<u(x)\}$ is polar*;

(ii) *u is the infimum of its superharmonic majorants*;

(iii) *for all $x\in\Omega$ and Jensen measures μ for x, $\int u\,d\mu\leq u(x)$.*

Let us recall that a Jensen measure for x with respect to Ω is a (Radon) measure μ, supported on a compact set in Ω, such that every superharmonic function u on Ω satisfies $\int u\,d\mu\leq u(x)$. In particular, for every relatively compact open V in Ω and every $x\in V$, the harmonic measure μ_x^V is a Jensen measure for x. The proof of [8, Theorem 1.4] shows that (i) already follows, if (iii) is replaced by

(iv) *for all relatively compact regular sets V in Ω and all $x\in V$, $\int u\,d\mu_x^V\leq u(x)$.*

For a recent discussion of the relation between Jensen measures and harmonic measures see [10].

The proof for (i) ⇒ (ii) is short and straightforward (it can be reduced to four lines; see the proof of Theorem 2) and the implication (ii) ⇒ (iii) is almost trivial. However, the proof for the implication (iii) ⇒ (i) given in [7] (or the proof for (iv) ⇒ (i) in [8]) is quite involved and uses a duality theorem for Jensen measures which is based on an abstract duality theorem by D. A. Edwards.

Clearly, (iii) implies that u is nearly hyperharmonic and hence $\hat{u}$ is superharmonic (see [3] and [5]). Moreover, it is well known that (ii) implies the polarity of the set $\{x:\hat{u}(x)<u(x)\}$ (see [6] or, for example, [2, Theorem 5.7.1]). So the (only) interesting part in the theorem above is the fact that already (iii) implies that the set $\{x:\hat{u}(x)<u(x)\}$ is polar.

We shall present a short proof for this implication (using harmonic measures only so that we might dispense with general Jensen measures; see also [1]) and, at the same time, remove the a priori measurability assumption on u. The latter is possible, since both (i) and (ii) already provide some regularity of the function u (it differs from a lower semicontinuous function only on a polar set, and it is finely upper semicontinuous, respectively).

In the sequel let Ω be as in the theorem above or, more generally, let Ω be a connected (metrizable) $\mathcal{P}$-Brelot space satisfying the axiom of domination. We introduce the following (weak) measurability property for numerical functions u on Ω.

(QA) For every $t\in\mathbb{R}$, the set $E:=\{x:u(x)>t\}$ is *quasi-analytic*, that is, there exists an analytic set A in Ω such that the symmetric difference $E\triangle A$ is polar.

If u is a numerical function on Ω having property (QA) and μ is a Jensen measure for $x\in\Omega$, then u is μ-measurable, since μ does not charge polar sets in $\Omega\setminus\{x\}$.

Lemma 1. *If u and v are numerical functions on Ω such that $u=v$ quasi-everywhere and v is finely lower or finely upper semicontinuous, then u satisfies* (QA) (*even with $G_{\delta\sigma}$-sets A*).

Proof. Obviously, it suffices to show that v satisfies (QA). We first suppose that v is finely lower semicontinuous, and fix $t\in\mathbb{R}$. Then $V:=\{x:v(x)>t\}$ is finely open. Hence the quasi-Lindelöf property implies that there exists a countable union A of compact sets and a polar set P such that $V=A\cup P$ (see [9, Section 1.XI.11] and [4, II.4.1, VI.5.14]).

Next, let us assume that v is finely upper semicontinuous, and hence $-v$ is finely lower semicontinuous. So, for each $n\in\mathbb{N}$, there exists a countable union A_n

of compact sets and a polar set P_n such that $\{x:-v(x)>-t-1/n\}=A_n\cup P_n$. Then $A:=\bigcup_{n=1}^{\infty} A_n^c$ is a $G_{\delta\sigma}$-set and $\{x:v(x)>t\}=\bigcup_{n=1}^{\infty}(A_n\cup P_n)^c\subset A$. Therefore, the set $A\setminus\{x:v(x)>t\}$ is polar, since it is contained in the polar set $\bigcup_{n=1}^{\infty} P_n$. □

Theorem 2. *Let $u\colon \Omega\to(-\infty,\infty]$ be such that $u\not\equiv\infty$ and u is locally lower bounded. Then the following statements are equivalent*:

(1) *$\hat{u}$ is superharmonic and $\hat{u}=u$ quasi-everywhere*;

(2) *u is the infimum of its superharmonic majorants*;

(3) (QA) *holds and $\int u\,d\mu\le u(x)$ for all $x\in\Omega$ and Jensen measures μ for x*;

(4) (QA) *holds and $\int u\,d\mu_x^V\le u(x)$ for all relatively compact open $V\subset\Omega$ and all $x\in V$.*

Proof. (1)⇒(2) Let $x\in\Omega$ be such that $u(x)<\infty$, and let $\varepsilon>0$. There exists a superharmonic function w on Ω such that $w\ge 0$, $w(x)=u(x)-\hat{u}(x)+\varepsilon$, and $w(y)=\infty$ whenever $y\in\Omega$, $y\ne x$ and $\hat{u}(y)<u(y)$. Let $v=\hat{u}+w$. Then v is superharmonic on Ω, $v\ge u$ and $v(x)=u(x)+\varepsilon$.

(2)⇒(3) Clearly, u is finely upper semicontinuous. So the assumption (QA) holds, by Lemma 1. Let μ be a Jensen measure for $x\in\Omega$. Then $\int u\,d\mu\le\int v\,d\mu\le v(x)$ whenever v is a superharmonic majorant of u. Thus $\int u\,d\mu\le u(x)$.

(3)⇒(4) This is trivial.

(4)⇒(1) Let w_0 be a continuous superharmonic function on Ω, $0<w_0<\infty$. Let us suppose first that $u\le Mw_0$ for some $M>0$. Then $\hat{u}$ is superharmonic, since the function u is nearly hyperharmonic.

The set $\{x:\hat{u}(x)<u(x)\}$ is the union of the sets $\{x:\hat{u}(x)\le s\}\cap\{x:s<u(x)\}$, s rational, where the sets $\{x:\hat{u}(x)\le s\}$ are closed and the sets $\{x:s<u(x)\}$ are quasi-analytic. Hence there exists an analytic set A and a polar set P in Ω such that

$$\{x:\hat{u}(x)<u(x)\}=A\triangle P.$$

To prove that $\hat{u}=u$ quasi-everywhere, it clearly suffices to show that all compact subsets of A are polar.

So let us fix a compact subset K of A. Let U be a relatively compact open neighborhood of K in Ω and set $V:=U\setminus K$. The functions $g\colon x\mapsto\int\hat{u}\,d\mu_x^V$ and $h\colon x\mapsto\int u\,d\mu_x^V$ are harmonic on V, and $g\le h$. Moreover, $h\le u$, by assumption, and hence $h\le\hat{u}$, by the continuity of h. By the axiom of domination, g is the greatest harmonic minorant of $\hat{u}$ on V. So $g=h$. Fixing $x\in V$, we conclude that $\mu_x^V(K\setminus P)=0$, since $\hat{u}\le u$ on Ω and $\hat{u}<u$ on $K\setminus P$. Knowing that μ_x^V does not charge polar sets, we see that $\mu_x^V(K)=0$, and hence

$$\inf\{s(x):s \text{ is superharmonic on } U \text{ and } s\ge 1_K w_0\}=:{}^U R_{w_0}^K(x)=\int_K w_0\,d\mu_x^V=0$$

(see, for example, [4, VI.2.9]). So the lower semicontinuous regularization w of ${}^U R^K_{w_0}$ is a positive superharmonic function on U which vanishes on $U \setminus K$. By ellipticity, $w=0$ on U. Thus K is polar.

In the general case, we apply the preceding result to $u_m := \min\{u, mw_0\}$, $m \in \mathbb{N}$. For every $m \in \mathbb{N}$, the function u_m satisfies (4), and $\hat{u}_m = \min\{\hat{u}, mw_0\}$. In particular, $\hat{u}_m \uparrow \hat{u}$ and $\{x : \hat{u}_m(x) < u_m(x)\} \uparrow \{x : \hat{u}(x) < u(x)\}$ as $m \to \infty$. Thus (1) holds. □

Remark 3. (a) If there exists a nonpolar set A in Ω which is inner polar and measurable with respect to all harmonic measures, then the assumption (QA) in (4) cannot be omitted. To see this it suffices to consider the function $u := 1_A$. Indeed, such a set A has no interior points. Hence $\hat{u}=0$, and the set $\{x : \hat{u}(x) < u(x)\} = A$ is nonpolar. But, for all relatively compact open sets V in Ω and $x \in V$,

$$\int u \, d\mu_x^V = \mu_x^V(A) = \sup\{\mu_x^V(K) : K \text{ compact in } A\} = 0 \le u(x).$$

(b) It is easily seen (directly or noting that (1) is a local statement) that we may replace (4) by

(4′) (QA) *holds and there exists a covering of* Ω *by open sets* Ω_i, $i \in I$, *such that* $\int u \, d\mu_x^V \le u(x)$ *for all relatively compact open* $V \subset \overline{V} \subset \Omega_i$ *for some* $i \in I$, and all $x \in V$.

However, it is certainly not sufficient to consider sets V from an arbitrary base $\mathcal{V}$ of open sets in Ω. Indeed, let $\Omega := \mathbb{R}^d$, $d \ge 3$. For $1 \le j \le d$, let A_j be the set of all $x = (x_1, ..., x_d)$ such that x_j is rational, and let $A := \bigcup_{j=1}^d A_j$, $u := 1_{A^c}$ and $\mathcal{V} := \left\{\prod_{j=1}^d (\alpha_j, \beta_j) : \alpha_j \text{ and } \beta_j \text{ are rational}\right\}$. For every $V \in \mathcal{V}$, $u=0$ on the boundary of V, and hence $\int u \, d\mu_x^V = 0 \le u(x)$, $x \in V$. Of course, $\hat{u}=0$ and $\{x : \hat{u}(x) = u(x)\}$ is the Lebesgue null set A so that $\{x : \hat{u}(x) < u(x)\} = A^c$ is far from being polar.

References

1. Alakhrass, M., *Superharmonic and Multiply Superharmonic Functions and Jensen Measures in Axiomatic Brelot Spaces*, Ph.D. Thesis, McGill University, Montreal, QC, 2009.
2. Armitage, D. H. and Gardiner, S. J., *Classical Potential Theory*, Springer Monographs in Mathematics, Springer, London, 2001.
3. Bauer, H., *Harmonische Räume und ihre Potentialtheorie*, Springer, Berlin, 1966.
4. Bliedtner, J. and Hansen, W., *Potential Theory—An Analytic and Probabilistic Approach to Balayage*, Universitext, Springer, Berlin, 1986.
5. Brelot, M., *Lectures on Potential Theory* (*Notes by K. N. Gowrisankaran and M. K. Venkatesha Murthy*), Lectures on Mathematics **19**, Tata Institute of Fundamental Research, Bombay, 1967.

6. Cartan, H., Théorie du potentiel newtonien: énergie, capacité, suites de potentiels, *Bull. Soc. Math. France* **73** (1945), 74–106.
7. Cole, B. J. and Ransford, T. J., Subharmonicity without semicontinuity, *J. Funct. Anal.* **147** (1997), 420–442.
8. Cole, B. J. and Ransford, T. J., Jensen measures and harmonic measures, *J. Reine Angew. Math.* **541** (2001), 29–53.
9. Doob, J. L., *Classical Potential Theory and Its Probabilistic Counterpart*, Springer, New York, 1984.
10. Hansen, W. and Netuka, I., Jensen measures in potential theory, to appear in *Potential Anal.*, doi:10.1007/s11118-011-9247-8.

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