Verdier specialization via weak factorization

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Abstract. Let $X \subset V$ be a closed embedding, with $V \setminus X$ nonsingular. We define a constructible function $\psi_{X,V}$ on X, agreeing with Verdier's specialization of the constant function $\mathbf{1}_V$ when X is the zero-locus of a function on V. Our definition is given in terms of an embedded resolution of X; the independence of the choice of resolution is obtained as a consequence of the weak factorization theorem of Abramovich–Karu–Matsuki–Włodarczyk. The main property of $\psi_{X,V}$ is a compatibility with the specialization of the Chern class of the complement $V \setminus X$. With the definition adopted here, this is an easy consequence of standard intersection theory. It recovers Verdier's result when X is the zero-locus of a function on V.

Our definition has a straightforward counterpart $\Psi_{X,V}$ in a motivic group. The function $\psi_{X,V}$ and the corresponding Chern class $c_{\mathrm{SM}}(\psi_{X,V})$ and motivic aspect $\Psi_{X,V}$ all have natural 'monodromy' decompositions, for any $X \subset V$ as above.

The definition also yields an expression for Kai Behrend's constructible function when applied to (the singularity subscheme of) the zero-locus of a function on V.

1. Introduction

Consider a family $\pi \colon V \to D$ over the open disk, satisfying a suitable condition of local triviality over $D \setminus \{0\}$. In [33], J.-L. Verdier defines a 'specialization morphism' for constructible functions, producing a function $\sigma_*(\varphi)$ on the central fiber X of the family for every constructible function φ on V. The key property of this specialization morphism is that it commutes with the construction of Chern classes of constructible functions in the sense of MacPherson [23]; cf. Theorem 5.1 in Verdier's note. The specialization morphism for constructible functions is induced from a morphism at the level of constructible sheaves $\mathcal{F}$, by taking alternating sums of ranks for the corresponding complex of nearby cycles $R\Psi_\pi \mathcal{F}$.

The main purpose of this note is to give a more direct description of the specialization morphism (in the algebraic category, over algebraically closed fields of characteristic 0), purely in terms of constructible functions and of resolution of singularities, including an elementary proof of the basic compatibility relation with

Chern classes. We will assume that V is nonsingular away from X, and focus on the case of the specialization of constant functions; by linearity and functoriality properties, this suffices in order to determine σ_* in the situation considered by Verdier. On the other hand, the situation we consider is more general than the specialization template recalled above: we define a constructible function $\psi_{X,V}$ for *every* proper closed subscheme X of a variety V (such that $V \setminus X$ is nonsingular), which agrees with Verdier's specialization of the constant function $\mathbf{1}_V$ when X is the fiber of a morphism from V to a nonsingular curve.

The definition of $\psi_{X,V}$ (Definition 2.1) is straightforward, and can be summarized as follows. Let $w \colon W \to V$ be a proper birational morphism such that W is nonsingular, and $D = w^{-1}(X)$ is a divisor with normal crossings and nonsingular components, and for which $w|_{W \setminus D}$ is an isomorphism. Then define $\psi_{D,W}(p)$ to be m if p is on a single component of D of multiplicity m, and 0 otherwise; and let $\psi_{X,V}$ be the push-forward of $\psi_{D,W}$ to X.

Readers who are familiar with Verdier's paper [33] should recognize that this construction is implicit in Section 5 of that paper, if X is the zero-locus of a function on V. Our contribution is limited to the realization that the weak factorization theorem of [1] may be used to adopt this prescription as a *definition,* that the properties of this function follow directly from the standard apparatus of intersection theory, and that this approach extends the theory beyond the specialization situation considered by Verdier (at least in the algebraic case). Denoting by $c_{\mathrm{SM}}(\cdot)$ the Chern–Schwartz–MacPherson class of a constructible function, we prove the following theorem.

Theorem I. *Let $i \colon X \hookrightarrow V$ be an effective Cartier divisor. Then*

$$c_{\mathrm{SM}}(\psi_{X,V}) = i^* c_{\mathrm{SM}}(\mathbf{1}_{V \setminus X}).$$

An expression for $i^* c_{\mathrm{SM}}(\mathbf{1}_{V \setminus X})$ in terms of the basic ingredients needed to define $\psi_{X,V}$ as above may be given as soon as $i \colon X \to V$ is a regular embedding (cf. Remark 3.5). In fact, with suitable positions, Theorem I holds for arbitrary closed embeddings $X \subset V$ (Theorem 3.3).

Theorem I reproduces Verdier's result when X is a fiber of a morphism from V to a nonsingular curve; in that case (but not in general) $\mathbf{1}_{V \setminus X}$ may be replaced with $\mathbf{1}_V$, as in Verdier's note. The definition of $\psi_{X,V}$ is clearly compatible with smooth maps, and in particular the value of $\psi_{X,V}$ at a point p may be computed after restricting to an open neighborhood of p. Thus, Verdier's formula for the specialization function in terms of the Euler characteristic of the intersection of a nearby fiber with a ball (Section 4 in [33]) may be used to compute $\psi_{X,V}$ if X is a divisor in V, over $\mathbb{C}$.

From the definition it is clear that the function $\psi_{X,V}$ is birationally invariant in the following weak sense.

Theorem II. *Let $\pi \colon V' \to V$ be a proper birational morphism, let $X' = \pi^{-1}(X)$, and assume that π restricts to an isomorphism $V' \setminus X' \to V \setminus X$. Then*

$$\pi_*(\psi_{X',V'}) = \psi_{X,V}.$$

(*Here, π_* is the push-forward of constructible functions.*)

In fact, the whole specialization *morphism* commutes with arbitrary proper maps, at least in Verdier's specialization situation ([33], Corollary 3.6). It would be desirable to establish this fact for the morphism induced by $\psi_{X,V}$ for arbitrary X, by the methods used in this paper.

The definition summarized above yields a natural decomposition of the constructible function $\psi_{V,X}$ (and hence of its Chern class $i^* c_{\mathrm{SM}}(\mathbf{1}_{V \setminus X})$) according to the multiplicities of some of the exceptional divisors, see Remarks 2.5 and 3.6. In the specialization situation, this decomposition matches the one induced on the Milnor fiber by monodromy, as follows from the description of the latter in [2]. As Schürmann pointed out to me, an analogous description in the more general case considered here may be found in [32], Theorem 3.2.

In the basic specialization situation, in which X is the zero-scheme of a function f on V and V is nonsingular, let Y be the *singularity subscheme* of X (i.e., the 'critical scheme' of f). One can define a constructible function μ on X by

$$(1) \qquad \mu = (-1)^{\dim V}(\mathbf{1}_X - \psi_{X,V}).$$

In this case (and over $\mathbb{C}$), the function $\psi_{X,V}$ agrees with Verdier's specialization function χ (here we use notation as in [26], cf. especially Proposition 5.1). The function μ is 0 outside of Y, so may be viewed as a constructible function on Y. In fact, it has been observed (cf., e.g., [4] and [12]) that the function μ is a specific linear combination of local Euler obstructions, and in particular it is determined by the scheme Y and can be generalized to arbitrary schemes. Kai Behrend denotes this generalization ν_Y in [12]. The definition of $\psi_{X,V}$ given in this paper yields an alternative computation of μ when Y is the critical scheme of a function, and Theorem II describes the behavior through modifications along X of this function: if $\pi \colon V' \to V$ is as in Theorem II, then

$$(2) \qquad \nu_Y = \pi_*(\nu_{Y'}) + (-1)^{\dim V}(\mathbf{1}_Y - \pi_*(\mathbf{1}_{Y'}))$$

provided that V and V' are nonsingular, X and hence $X' = \pi^{-1}(X)$ are hypersurfaces, and Y and Y' are their singularity subschemes. As $\psi_{X,V}$ is defined for

arbitrary $X \subset V$, there may be a generalization of (1) linking Behrend's function and $\chi_{X,V}$ when X is not necessarily a hypersurface; it would be interesting to have statements analogous to (2) holding for more general X.

In Section 5 we comment on the relation between $c_{\mathrm{SM}}(\psi_{X,V})$ and the 'weighted Chern–Mather class' of the singularity subscheme Y of a hypersurface X; the degree of this class is a Donaldson–Thomas type invariant ([12], Section 4.3). We also provide an explicit formula for the function μ in terms of a resolution of the hypersurface X. It would be interesting to extend these results to the non-hypersurface case.

In a different vein, J. Schürmann has considered the iteration of the specialization operator over a set of generators for a complete intersection $X \subset V$ ([29], Definition 3.6). It would be a natural project to compare Schürmann's definition (which depends on the order of the generators) with our definition of $\psi_{X,V}$ (which is independent of the order, and may be extended to arbitrary $X \subset V$). Schürmann also points out that the *deformation to the normal cone* may be used to reduce an arbitrary $X \subset V$ to a specialization situation; this strategy was introduced in [34], and is explained in detail in [28], Section 1. Again, it would be interesting to establish the precise relation between the resulting specialization morphism and the function $\psi_{X,V}$ studied here.

We include in Section 2.3 a brief discussion of a 'motivic' invariant $\Psi_{X,V}$, also defined for any closed embedding $X \subset V$ into a variety, still assumed for simplicity to be nonsingular outside of X. This invariant can be defined in the quotient of the Grothendieck ring of varieties by the ideal generated by the class of a torus $\mathbb{T} = \mathbb{A}^1 - \mathbb{A}^0$, or in a more refined relative ring over X. The definition is again extremely simple, when given in terms of a resolution in which the inverse image of X is a divisor with normal crossings; the proof that the invariant is well-defined also follows from the weak factorization theorem. As its constructible function counterpart, the class $\Psi_{X,V}$ admits a natural 'monodromy' decomposition (although a Milnor fiber is not defined in general in the situation we consider), see Remark 2.14. When V is nonsingular and X is the zero-locus of a function on V, $\Psi_{X,V}$ is a poor man's version of the Denef–Loeser motivic Milnor/nearby fiber ([17], Section 3.5); it is defined in a much coarser ring, but it carries information concerning the topological Euler characteristic and some other Hodge-type data. We note that $\Psi_{X,V}$ is *not* the image of the limit of the *naive* motivic zeta function of Denef–Loeser, since it does carry multiplicity information, while $Z^{\mathrm{naive}}(T)$ discards it (see for example [17], Corollary 3.3). The limit of the (non-naive) Denef–Loeser motivic zeta function $Z(T)$ encodes the multiplicity and much more as actual monodromy information, and in this sense it lifts the information carried by our $\Psi_{X,V}$. It would be

interesting to define and study an analogous lift for more general closed embeddings $X \subset V$, and possibly allowing V to be singular along X.

The approach of [15] could be used to unify the constructions of $\psi_{X,V}$ and $\Psi_{X,V}$ given in this paper, and likely extend them to other environments, but we will not pursue such generalizations here since our aim is to keep the discussion at the simplest possible level. Likewise, 'celestial' incarnations of the Milnor fiber (in the spirit of [6] and [8]) will be discussed elsewhere.

In our view, the main advantages of the approach taken in this paper are the simplicity afforded by the use of the weak factorization theorem and the fact that the results have a straightforward interpretation for any closed embedding $X \subset V$, whether arising from a specialization situation or not. These results hold with identical proofs over any algebraically closed field of characteristic zero. We note that the paper [32] of van Proeyen and Veys also deals with arbitrary closed embeddings with nonsingular complements, as in this note. A treatment of Verdier specialization over arbitrary algebraically closed fields of characteristic zero, also using only the standard apparatus of intersection theory, was given by Kennedy in [22] by relying on the Lagrangian viewpoint introduced by C. Sabbah [27]. Fu [19] gives a description of Verdier's specialization in terms of normal currents.

We were motivated to take a new look at Verdier's specialization because of applications to string-theoretic identities (cf. [10], Section 4). Also, Verdier specialization offers an alternative approach to the main result of [9]. The main reason to allow V to have singularities along X is that this typically is the case for specializations arising from pencils of hypersurfaces in a linear system, as in these applications. See Section 4 for a few simpler examples illustrating this point.

2. The definition

Our schemes are separated, of finite type over an algebraically closed field k of characteristic 0. The characteristic restriction is due to the use of resolution of singularities and the main result of [1], as well as the theory of Chern–Schwartz–MacPherson classes. (Cf. [21] and [7] for discussions of the theory in this generality.)

2.1. Definition of $\psi_{X,V}$

Definition 2.1. Let V be a variety, and let $i\colon X \hookrightarrow V$ be a closed embedding. We assume that $V \setminus X$ is nonsingular and nonempty. The *constructible function* $\psi_{X,V}$ on X is defined as follows:
   – Let $w\colon W \to V$ be a proper birational morphism; let $D = w^{-1}(X)$, and $d = w|_D$:

$$
\begin{array}{ccc}
D & \overset{j}{\hookrightarrow} & W \\
{\scriptstyle d}\downarrow & & \downarrow{\scriptstyle w} \\
X & \underset{i}{\hookrightarrow} & V.
\end{array}
$$

We assume that W is nonsingular, D is a divisor with normal crossings and nonsingular components D_l in W, and w restricts to an isomorphism $W \setminus D \to V \setminus X$. (Such a w exists, by resolution of singularities.) Let m_l be the multiplicity of D_l in D.
   – We define a constructible function $\psi_{D,W}$ on D by letting $\psi_{D,W}(p) = m_l$ for $p \in D_l$, $p \notin D_k$, $k \neq l$, and $\psi_{D,W}(p) = 0$ for $p \in D_l \cap D_k$, any $k \neq l$.
   – Then let $\psi_{X,V} := d_*(\psi_{D,W})$.

We remind the reader that the push-forward of constructible functions is defined as follows: For any scheme S, denote by $\mathbf{1}_S$ the function with value 1 along S, and 0 outside of S. If S is a subvariety of D, and $x \in X$, $d_*(\mathbf{1}_S)(x)$ equals $\chi(d^{-1}(x) \cap S)$. By linearity, this prescription defines $d_*(\varphi)$ for every constructible function φ on D. Here, χ denotes the topological Euler characteristic for $k = \mathbb{C}$; see [22] or [7] for the extension to algebraically closed fields of characteristic 0.
Of course we have to verify that the definition of $\psi_{X,V}$ given in Definition 2.1 does not depend on the choice of $w\colon W \to V$.

Lemma 2.2. *With notation as above, the function $\psi_{X,V}$ is independent of the choice of $w\colon W \to V$.*

Proof. The weak factorization theorem of [1] reduces the verification to the following fact:
   – Let W, D and $\psi_{D,W}$ as above.
   – Let $\pi\colon \widetilde{W} \to W$ be the blow-up of W along a center $Z \subseteq D$ that meets D with normal crossings, and let $\widehat{D} = \pi^{-1}(D)$.
   – Then $\pi_*(\psi_{\widehat{D},\widetilde{W}}) = \psi_{D,W}$, where $\psi_{\widehat{D},\widetilde{W}}$ and $\psi_{D,W}$ are defined by the prescription for divisors with normal crossings given in Definition 2.1.

Recall that Z meets D with normal crossings if at each point z of Z there is an analytic system of parameters $x_1,...,x_n$ for D at z such that Z is given by $x_1=...=x_{r+1}=0$, and D is given by a monomial in the x_l's. The divisor $\widehat{D}=\pi^{-1}(D)$ is then a divisor with normal crossings, cf. Lemma 2.4 in [5]. The divisor $\widehat{D}$ consists of the proper transforms $\widetilde{D}_l$ of the components D_l, appearing with the same multiplicity m_l, and of the exceptional divisor E, appearing with multiplicity $\sum_{D_l \supset Z} m_l$. It is clear that $\pi_*(\psi_{\widehat{D},\widetilde{W}})$ agrees with $\psi_{D,W}$ away from Z; we have to verify that the functions match at all $z \in Z$. The fiber of $E=\pi^{-1}(Z)$ over z is a projective space of dimension $r=\mathrm{codim}_Z W-1$; we have to analyze the intersection of the rest of $\pi^{-1}(D)$ (that is, of the proper transforms $\widetilde{D}_l$) with this projective space.

Now there are two kinds of points $z \in Z$: either z is in exactly one component D_l, or it is in the intersection of several components. In the first case, $\widetilde{D}_l$ is the unique component of $\pi^{-1}(D)$ other than E meeting the fiber F of E over z. By definition, $\psi_{\widehat{D},\widetilde{W}}=m_l$ on the complement of $F \cap \widetilde{D}_l$ in $\widetilde{D}_l$, and $\psi_{\widehat{D},\widetilde{W}}=0$ along $F \cap \widetilde{D}_l$. Thus, $\pi_*(\psi_{\widehat{D},\widetilde{W}})(z)$ equals

$$m_l\chi(F \setminus (F \cap \widetilde{D}_l))+0\chi(F \cap \widetilde{D}_l) = m_l = \psi_{D,W}(z),$$

since $F \cong \mathbb{P}^r$ and $F \cap \widetilde{D}_l \cong \mathbb{P}^{r-1}$. This is as it should be. If z is in the intersection of two or more components D_l, then $\psi_{D,W}(z)=0$, so we have to verify that $\pi_*(\psi_{\widehat{D},\widetilde{W}})(z)=0$. Again there are two possibilities: either one of the components containing z does not contain Z, and then the whole fiber $F \cong \mathbb{P}^r$ is contained in the proper transform of that component, as well as in E; or all the components D_l containing z contain Z. In the first case, the value of $\psi_{\widehat{D},\widetilde{W}}$ is zero on the whole fiber F, because F is in the intersection of two components of $\widehat{D}$; so the equality is clear in this case.

In the second case, let $D_1,...,D_e$ be the components of D containing Z; no other component of D contains z, by assumption. The proper transforms $\widetilde{D}_l$ meet the fiber $F \cong \mathbb{P}^r$ along e hyperplanes meeting with normal crossings, $1<e\leq r+1$. The value of $\psi_{\widehat{D},\widetilde{W}}$ along F is then $m_1+...+m_e$ along the complement U of these hyperplanes, and 0 along these hyperplanes. Thus,

$$\pi_*(\psi_{\widehat{D},\widetilde{W}})(z) = (m_1+...+m_e)\chi(U).$$

The proof will be complete if we show that $\chi(U)=0$; and this is done in the elementary lemma that follows. $\square$

Lemma 2.3. *Let $H_1,...,H_e$ be hyperplanes in $\mathbb{P}^r$ meeting with normal crossings, with $1\leq e\leq r+1$, and let $U=\mathbb{P}^r \setminus (H_1\cup...\cup H_e)$. Then $\chi(U)=1$ for $e=1$, and $\chi(U)=0$ for $e=2,...,r+1$.*

Proof. The hyperplanes may be assumed to be coordinate hyperplanes, and hence U may be described as the set of $(x_0 : ... : x_r)$ such that the first e coordinates are nonzero. As the first coordinate is nonzero, we may set it to be 1, and view the rest as affine coordinates. It is then clear that $U \cong \mathbb{A}^{r+1-e} \times \mathbb{T}^{e-1}$, where $\mathbb{T} = \mathbb{A} \setminus \{0\}$ is the 1-dimensional torus. The statement is then clear, since $\chi(\mathbb{T}) = 0$. □

Remark 2.4. By definition, the value of $\psi_{X,V}$ at a point $p \in X$ is

$$\psi_{X,V} = \sum_l m_l \chi(D_l^\circ \cap w^{-1}(p)),$$

where $D_l^\circ = D_l \setminus \bigcup_{k \neq l} D_k$. In the complex hypersurface case, this equals the Euler characteristic $\chi(F_\theta)$ of the Milnor fiber, by formula (2) in Theorem 1 of [2]. An analogous interpretation holds in general, as may be established by using Theorem 3.2 in [32].

Remark 2.5. Let $\alpha \colon \mathbb{Z} \to \mathbb{Z}$ be any function. The argument proving Lemma 2.2 shows that one may define a constructible function $\psi_{X,V}^\alpha$ on X as the push-forward of the function $\psi_{D,W}^\alpha$ with value $\alpha(m_l)$ on $D_l \setminus \bigcup_{k \neq l} D_k$ and 0 on intersections, for D and W as in Definition 2.1.

For $\alpha \neq$ identity we do not have an interpretation for $\psi_{X,V}^\alpha$ (or for the corresponding Chern class, cf. Remark 3.6). Letting ε_m be the function that is 1 at $m \in \mathbb{Z}$ and 0 at all other integers, we have a decomposition of the identity as $\sum_{m=-\infty}^\infty m\varepsilon_m$, and hence a distinguished decomposition

$$\psi_{X,V} = \sum_{m=-\infty}^\infty m\psi_{X,V}^{\varepsilon_m}.$$

The individual terms in this decomposition are clearly preserved by the morphisms considered in Propositions 2.6 and 2.7. In the hypersurface case they can be related to the monodromy action on the Milnor fiber, as the nonzero contributions arise from the nonzero eigenspaces of monodromy (cf. [2], Theorem 4; and Theorem 3.2 in [32] for generalizations to the non-hypersurface case). From this perspective, the piece $\psi_{X,V}^{\varepsilon_1}$ may be thought of as the 'unipotent part' of $\psi_{X,V}$.

2.2. Basic properties and examples

Theorem II from the introduction is an immediate consequence of the definition of $\psi_{X,V}$.

Proposition 2.6. *Let V' be a variety, and let $\pi\colon V'\to V$ be a proper morphism. Let $X'=\pi^{-1}(X)$, and assume that π restricts to an isomorphism $V'\setminus X'\to V\setminus X$. Then*

$$\pi_*(\psi_{V',X'})=\psi_{V,X}.$$

Indeed, a resolution for the pair $X'\subset V'$ as in Definition 2.1 is also a resolution for the pair $X\subset V$. Another immediate consequence of the definition is the behavior with respect to smooth maps.

Proposition 2.7. *Let U be a variety, and let $\eta\colon U\to V$ be a smooth morphism. Then*

$$\psi_{\eta^{-1}(X),U}=\eta^*(\psi_{X,V}).$$

(Pull-backs of constructible functions are defined by composition.) Indeed, in this case one can construct compatible resolutions.

For example, the value of $\psi_{X,V}$ at a point $p\in X$ may be computed by restricting to an open neighborhood of p.

Example 2.8. If X and V are nonsingular varieties, then $\psi_{X,V}=\operatorname{codim}_X V\cdot\mathbf{1}_X$.

Indeed, if X and V are nonsingular, then we may let $w\colon W\to V$ be the blow-up of V along X; D is the exceptional divisor, and the push-forward of $\mathbf{1}_D$ equals $\operatorname{codim}_X V\cdot\mathbf{1}_X$ because the fibers of d are projective spaces $\mathbb{P}^{\operatorname{codim}_X V-1}$.

Example 2.9. The function $\psi_{X,V}$ depends on the scheme structure on X. For example, let $X\subset\mathbb{P}^2$ be the scheme defined by (x^2,xy), consisting of a line L with an embedded point at $p\in L$. Then $\psi_{X,\mathbb{P}^2}$ equals 1 along $L\setminus\{p\}$, while $\psi_{X,\mathbb{P}^2}(p)=2$. (Blow-up at p, then apply Definition 2.1. In terms of the decomposition in Remark 2.5, $\psi_{X,\mathbb{P}^2}$ is the sum of $\psi_{X,\mathbb{P}^2}^{\varepsilon_1}=\mathbf{1}_{L\setminus p}$ and $2\psi_{X,\mathbb{P}^2}^{\varepsilon_2}=2\cdot\mathbf{1}_p$.) Thus, $\psi_{X,\mathbb{P}^2}=\mathbf{1}_L+\mathbf{1}_p$, while of course $\psi_{L,\mathbb{P}^2}=\mathbf{1}_L$.

Example 2.10. Let X be the reduced scheme supported on the union of three non-coplanar concurrent lines in $\mathbb{P}^3$; for example, we may take (xy,xz,yz) as a defining ideal. Then $\psi_{X,\mathbb{P}^3}=2$ along the three components, and $\psi_{X,\mathbb{P}^3}(p)=0$ at the point of intersection p.

To see this, blow-up $\mathbb{P}^3$ at p first, and then along the proper transforms of the three lines, to produce a morphism $w\colon W\to\mathbb{P}^3$ as in Definition 2.1; $D=w^{-1}(X)$ consists of 4 components, one of which dominates p. This component is a $\mathbb{P}^2$ blown up at three points; the complement in this component of the intersection with the three other exceptional divisors has Euler characteristic 0, so the push-forward of $\psi_{D,W}$ equals 0 at p.

If the three concurrent lines are coplanar, say with ideal $(xy(x+y), z)$, then the value of $\psi_{X,\mathbb{P}^3}$ at the intersection point is -2.

That the value of the function is 2 away from the point of intersection is clear a priori in both cases by Example 2.8, since as observed above we may restrict to an open set avoiding the singularity and use Example 2.8.

2.3. A motivic invariant

Definition 2.1 has a counterpart in a quotient of the naive Grothendieck group of varieties $K(\mathrm{Var}_k)$. Recall that this is the group generated by isomorphism classes of k-varieties, modulo the relations $[S]=[S-T]+[T]$ for every closed embedding $T\subseteq S$; setting $[S_1]\cdot[S_2]=[S_1\times S_2]$ makes $K(\mathrm{Var}_k)$ into a ring. Denote by $\mathbb{L}$ the class of $\mathbb{A}^1$ in $K(\mathrm{Var}_k)$, and by $\mathbb{T}=\mathbb{L}-1$ the class of the multiplicative group of k.

We let $\mathcal{M}^{\mathbb{T}}$ denote the quotient $K(\mathrm{Var}_k)/(\mathbb{T})$. Every pair $X\subset V$ as in Definition 2.1 (that is, with V being a variety and $V\setminus X$ being nonsingular) determines a well-defined element $\Psi_{X,V}$ of $\mathcal{M}^{\mathbb{T}}$, as follows:

 – let D, d, W and w be as in Definition 2.1;
 – for every component D_l of D, let D_l° be the complement $D_l\setminus\bigcup_{k\neq l} D_k$;
 – then $\Psi_{X,V}=\sum_l m_l[D_l^\circ]$, where m_l is the multiplicity of D_l in D.

The argument given in Lemma 2.2 proves that $\Psi_{X,V}$ is well defined in the quotient $\mathcal{M}^{\mathbb{T}}$. Indeed, the argument applies verbatim to show that the class of the fiber of a blow-up over $z\in Z$ agrees mod $\mathbb{T}$ with the class of z; Lemma 2.3 is replaced by the analogous result in $\mathcal{M}^{\mathbb{T}}$:

Lemma 2.11. *Let $H_1,...,H_e$ be hyperplanes in $\mathbb{P}^r$ meeting with normal crossings, with $1\leq e\leq r+1$, and let $U=\mathbb{P}^r\setminus(H_1\cup...\cup H_e)$. Then $[U]=1\in\mathcal{M}^{\mathbb{T}}$ for $e=1$, and $[U]=0\in\mathcal{M}^{\mathbb{T}}$ for $e=2,...,r+1$.*

The proof of Lemma 2.3 implies this statement, as it shows that $U\cong\mathbb{A}^{r+1-e}\times\mathbb{T}^{e-1}$. It is also evident that $\Psi_{X,V}$ satisfies the analogues of Propositions 2.6 and 2.7. Since $\chi(\mathbb{T})=0$, the information carried by an element of $\mathcal{M}^{\mathbb{T}}$ suffices to compute topological Euler characteristics, but is considerably more refined: for example, the series obtained by setting $v=u^{-1}$ in the Hodge–Deligne polynomial can be recovered from the class in $\mathcal{M}^{\mathbb{T}}$.

Example 2.12. Let $X\subseteq\mathbb{P}^3$ be a cone over a smooth plane curve $C\subseteq\mathbb{P}^2$ of degree m. Blowing up at the vertex p yields a resolution as needed in Definition 2.1; the function $\psi_{X,\mathbb{P}^3}$ is immediately computed to be 1 outside of the vertex and

$m(\chi(\mathbb{P}^2)-\chi(C))=(m-1)^3+1$ at p; the class $\Psi_{X,\mathbb{P}^3}$ equals $[C]+m(3-[C])\in\mathcal{M}^{\mathbb{T}}$. For $m>2$, this is not the class of a constant.

If X is the scheme of zeros of a nonzero function $f\colon V\to k$, and V is nonsingular, then a *motivic Milnor fiber* ψ_f was defined and studied by J. Denef and F. Loeser, see [18], Section 3. The motivic Milnor fiber ψ_f is defined in a ring $\mathcal{M}_{k,\mathrm{loc}}^{\mathrm{mon}}$ analogous to the ring $K(\mathrm{Var}_k)$ considered above, but localized at $\mathbb{L}$ and including monodromy information. The much naiver $\Psi_{X,V}$ generalizes to arbitrary pairs $X\subset V$ (with $V\setminus X$ nonsingular) a small part of the information carried by the Denef–Loeser motivic Milnor fiber in the specialization case.

Remark 2.13. It is probably preferable to work in the *relative* Grothendieck ring of varieties over X, moding out by classes of varieties $Z\to X$ which fiber in tori. Lemma 2.11 shows that the resulting class $\Psi_{X,V}^{\mathrm{rel}}$ is also well defined. For $p\in X$, the fiber of $\Psi_{X,V}^{\mathrm{rel}}$ over p is well defined as a class in $\mathcal{M}^{\mathbb{T}}$, so it has an Euler characteristic, which clearly equals $\psi_{X,V}(p)$. Thus, the constructible function $\psi_{X,V}$ may be recovered from the relative class $\Psi_{X,V}^{\mathrm{rel}}$. This point of view is developed fully for Chern classes and more in [14] and [15]. (Cf. [17] and [13] for the relative viewpoint on the Denef–Loeser motivic Milnor fiber.)

Remark 2.14. There is a class $\Psi_{X,V}^{\alpha}$ (resp. $\Psi_{X,V}^{\mathrm{rel},\alpha}$) for every function $\alpha\colon\mathbb{Z}\to\mathbb{Z}$, defined by $\sum_l\alpha(m_l)[D_l^\circ]$ on a resolution. In particular, there is a decomposition

$$\Psi_{X,V}=\sum_{m=-\infty}^{\infty}m\Psi_{X,V}^{\varepsilon_m},$$

with ε_m as in Remark 2.5. As observed in that remark, in the hypersurface case this decomposition can be related to the monodromy decomposition. If X is the zero scheme of a function on V, and V is nonsingular, then the limit of the *naive motivic zeta function* $Z^{\mathrm{naive}}(T)$ of Denef–Loeser ([17], Corollary 3.3) lifts $\Psi_{X,V}^1$, where $\alpha\equiv1$ is the constant function 1. In fact, the corresponding expression

$$\sum_{|I|>0}(-\mathbb{T})^{|I|-1}[D_I^\circ]$$

(where $D_I=\bigcap_{l\in I}D_l$, and $D_I^\circ=D_I\setminus\bigcup_{l\notin I}D_l$) may be verified to be a well-defined element of $K(\mathrm{Var}_k)$ for any $X\subset V$ such that $V\setminus X$ is nonsingular, and with D as in Definition 2.1. This also follows from the weak factorization theorem; no localization is necessary. Also, this expression is clearly preserved by morphisms which restrict to the identity on $V\setminus X$ (as in Theorem II). Thus, $\Psi_{X,V}^1$ admits a natural lift to $K(\mathrm{Var}_k)$ in general. We do not know whether $\Psi_{X,V}$ itself admits a natural lift to $K(\mathrm{Var}_k)$.

3. Compatibility with Chern classes

3.1. Specialization in the Chow group and Chern–Schwartz–MacPherson classes

Recall ([30], [31] and [23]) that every constructible function φ determines a class in the Chow group, satisfying good functoriality properties and the normalization restriction of agreeing with the total Chern class of the tangent bundle if applied to the constant function $\mathbf{1}$ over a nonsingular variety. We call this function the *Chern–Schwartz–MacPherson* (CSM) class of φ, $c_{\mathrm{SM}}(\varphi)$. The functoriality may be expressed as follows: if $f\colon X\to Y$ is a proper morphism, and φ is a constructible function on X, then $f_*(c_{\mathrm{SM}}(\varphi))=c_{\mathrm{SM}}(f_*(\varphi))$. The simplest instance of this property is the fact that for every complete variety X (nonsingular or otherwise), the degree of $c_{\mathrm{SM}}(\mathbf{1}_X)$ equals the Euler characteristic of X.

We are particularly interested in the CSM classes of the function $\psi_{X,V}$ defined in Section 2, and of the function $\mathbf{1}_{V\setminus X}=\mathbf{1}_V-\mathbf{1}_X$, with value 0 on X and 1 along the complement of X.

In the situation considered by Verdier (and more generally when X is, e.g., a Cartier divisor in V), there is a natural way to specialize classes defined on V or $V\setminus X$ to X. We consider the following definition of the specialization of a specific class, for arbitrary $X\subset V$.

Definition 3.1. Let V be a variety, $X\subset V$ be a closed embedding, and assume that $V\setminus X$ is nonsingular. Let $\tilde{v}\colon \widetilde{V}\to V$ be the blow-up of V along X, let $\iota\colon E\to \widetilde{V}$ be the exceptional divisor, and let $e\colon E\to X$ be the induced map. We define the 'specialization of $c_{\mathrm{SM}}(\mathbf{1}_{V\setminus X})$ to X' to be the class

$$\sigma_{X,V}(c_{\mathrm{SM}}(\mathbf{1}_{V\setminus X})):=e_*\iota^*(c_{\mathrm{SM}}(\mathbf{1}_{\widetilde{V}\setminus E}))\in A_*X.$$

The blow-up $\widetilde{V}\to V$ in Definition 3.1 may be replaced by any proper birational morphism $v'\colon V'\to V$ dominating the blow-up and restricting to an isomorphism on $V'\setminus(v')^{-1}(X)$; this follows easily from the projection formula and the functoriality of CSM classes.

When X is a Cartier divisor in V, this specialization is the ordinary pull-back, and could be defined for arbitrary classes in V.

Lemma 3.2. *Let* $i\colon X\hookrightarrow V$ *be a Cartier divisor. Then* $\sigma_{X,V}(c_{\mathrm{SM}}(\mathbf{1}_{V\setminus X}))=i^*c_{\mathrm{SM}}(\mathbf{1}_{V\setminus X})$.

Proof. In this case we may take $\widetilde{V}=V$ and $E=X$, and let e be the identity. $\square$

If $X \subset V$ is *not* a Cartier divisor, it is not clear how to extend Definition 3.1 to arbitrary classes; but this is not needed for the results in this paper.

By Lemma 3.2, Theorem I from the introduction follows from the following result.

Theorem 3.3. *Let V be a variety, $X \subset V$ be a closed embedding, and assume that $V \setminus X$ is nonsingular. With notation as above,*

$$c_{\mathrm{SM}}(\psi_{X,V}) = \sigma_{X,V}(c_{\mathrm{SM}}(\mathbf{1}_{V \setminus X})).$$

Proof. First consider the case in which $V = W$ is nonsingular, and $X = D$ is a divisor with normal crossings and nonsingular components D_l, appearing with multiplicity m_l. Let $D_l^\circ \subseteq D_l$ be the complement of $\bigcup_{k \neq l} D_k$ in D_l. By the definition of $\psi_{D,W}$, and by the linearity of the c_{SM} operator,

$$c_{\mathrm{SM}}(\psi_{D,W}) = \sum_l m_l c_{\mathrm{SM}}(\mathbf{1}_{D_l^\circ}).$$

The key to the computation is the fact that the CSM class of the complement of a divisor with normal crossings in a nonsingular variety is given by the Chern class of a corresponding logarithmic twist of the tangent bundle; for this, see e.g. (*) at the top of p. 4002 in [3]. As D_l° is the complement of $\bigcup_{k \neq l} D_k \cap D_l$, a normal crossing divisor in D_l, we have (omitting evident pull-backs)

$$c_{\mathrm{SM}}(\mathbf{1}_{D_l^\circ}) = \frac{c(TD_l)}{\prod_{k \neq l}(1+D_k)} \cap [D_l] = \frac{c(TW)}{\prod_k(1+D_k)} \cap [D_l].$$

Therefore

$$c_{\mathrm{SM}}(\psi_{D,W}) = \frac{c(TW)}{\prod_k(1+D_k)} \cap \sum_l m_l[D_l] = \frac{c(TW)}{\prod_k(1+D_k)} \cap [D].$$

Since $c_{\mathrm{SM}}(\mathbf{1}_{W \setminus D}) = (c(TW)/\prod_k(1+D_k)) \cap [W]$, this shows that

$$(3) \qquad c_{\mathrm{SM}}(\psi_{D,W}) = j^* c_{\mathrm{SM}}(\mathbf{1}_{W \setminus D}),$$

where $j \colon D \hookrightarrow W$ is the inclusion. This is the statement in the normal crossing case.

Now assume that $X \subset V$ is any closed embedding, and apply (3) to D and W as in Definition 2.1, observing that any such $w \colon W \to V$ must factor through the

blow-up $\widetilde{V}$ along X. Let $w'\colon W\to\widetilde{V}$ and $d'\colon D\to E$ be the induced morphisms, so that $w=\tilde{v}\circ w'$ and $d=e\circ d'$:

By the covariance of c_{SM}, we have

$$c_{\mathrm{SM}}(\psi_{E,\widetilde{V}})=c_{\mathrm{SM}}(d'_*\psi_{D,W})=d'_*c_{\mathrm{SM}}(\psi_{D,W})=d'_*j^*c_{\mathrm{SM}}(\mathbf{1}_{W\setminus D})$$

$$\overset{\dagger}{=}\iota^*w'_*c_{\mathrm{SM}}(\mathbf{1}_{W\setminus D})=\iota^*c_{\mathrm{SM}}(w'_*(\mathbf{1}_{W\setminus D}))=\iota^*c_{\mathrm{SM}}(\mathbf{1}_{\widetilde{V}\setminus E}),$$

where $\overset{\dagger}{=}$ holds by Theorem 6.2 in [20] (note that $j^*=j^!=\iota^!$ as both D and E are Cartier divisors). We have $w'_*(\mathbf{1}_{W\setminus D})=\mathbf{1}_{V'\setminus E}$ since w' is an isomorphism off D. Finally, using Proposition 2.6 and again the covariance of c_{SM}:

$$c_{\mathrm{SM}}(\psi_{X,V})=c_{\mathrm{SM}}(e_*\psi_{E,\widetilde{V}})=e_*c_{\mathrm{SM}}(\psi_{E,\widetilde{V}})=e_*\iota^*c_{\mathrm{SM}}(\mathbf{1}_{\widetilde{V}\setminus E})=\sigma_{X,V}(c_{\mathrm{SM}}(\mathbf{1}_{V\setminus X}))$$

as claimed. $\square$

Example 3.4. Let $X\subseteq\mathbb{P}^2$ be the line L with embedded point p considered in Example 2.9. By Theorem 3.3,

$$\sigma_{X,\mathbb{P}^2}(c_{\mathrm{SM}}(\mathbf{1}_{\mathbb{P}^2\setminus X}))=c_{\mathrm{SM}}(\psi_{X,\mathbb{P}^2})=c_{\mathrm{SM}}(\mathbf{1}_L+\mathbf{1}_p)=[L]+3[p].$$

(This may of course be verified by hand using Definition 3.1.) Here, $c_{\mathrm{SM}}(\mathbf{1}_{\mathbb{P}^2\setminus X})=[\mathbb{P}^2]+2[\mathbb{P}^1]+[\mathbb{P}^0]$. Restricting this class to L gives $[L]+2[p]$; the specialization $\sigma_{X,\mathbb{P}^2}$ picks up an extra $[p]$ due to the embedded point.

Remark 3.5. By Lemma 3.2, if $i\colon X\to V$ embeds X as a Cartier divisor in V, then $\sigma_{X,V}$ acts as the pull-back to X, so that in this case Theorem 3.3 states that

$$c_{\mathrm{SM}}(\psi_{X,V})=i^*c_{\mathrm{SM}}(\mathbf{1}_{V\setminus X}).$$

This is not the case if $i\colon X\to V$ is a regular embedding of higher codimension. If the embedding is regular, then the exceptional divisor $E\subset\widetilde{V}$ may be identified with the

projective normal bundle to X, and as such it has a universal quotient bundle $\mathcal{N}$. Tracing the argument proving Theorem 3.3 shows that

$$i^* c_{\mathrm{SM}}(\mathbf{1}_{V \setminus X}) = e_*(c_{\mathrm{top}}(\mathcal{N}) \cap c_{\mathrm{SM}}(\psi_{E, \widetilde{V}}));$$

and in fact

$$i^* c_{\mathrm{SM}}(\mathbf{1}_{V \setminus X}) = d_*\left(c_{\mathrm{top}}\left(\frac{d^* N_X V}{N_D V} \right) \cap c_{\mathrm{SM}}(\psi_{D, W}) \right)$$

for any W and D, etc. as in Definition 2.1.

Warning. When $X \subset V$ is a regular embedding, so that the blow-up morphism $\widetilde{V} \to V$ is a local complete intersection morphism (cf. Section 6.7 in [20]) one may be tempted to define a specialization of classes from V to X by pulling back to the blow-up, restricting to the exceptional divisor, and pushing forward to X. However, the pull-back of $c_{\mathrm{SM}}(\mathbf{1}_{V \setminus X})$ may not agree with $c_{\mathrm{SM}}(\mathbf{1}_{\widetilde{V} \setminus E})$: the blow-up of $V = \mathbb{P}^2$ at $X = \{$a point$\}$ already gives a counterexample. (Corollary 4.4 in [11] provides a condition under which a similar pull-back formula does hold.) Thus, this operation does not agree in general with the class defined in Definition 3.1.

Remark 3.6. Since $\psi_{X,V}$ admits a distinguished decomposition $\sum_{m=-\infty}^{\infty} m \psi_{X,V}^{\varepsilon_m}$ (see Remark 2.5), so does the specialization of $c_{\mathrm{SM}}(\mathbf{1}_{V \setminus X})$:

$$\sigma_{X,V}(c_{\mathrm{SM}}(\mathbf{1}_{V \setminus X})) = \sum_{m=-\infty}^{\infty} m \, c_{\mathrm{SM}}(\psi_{X,V}^{\varepsilon_m}).$$

In the classical case of specializations along the zero set of a function, this means that every eigenspace of monodromy carries a well-defined piece of the Chern class, a fact we find intriguing, and which we are not sure how to interpret for more general $X \subset V$. In fact, any $\alpha \colon \mathbb{Z} \to \mathbb{Z}$ determines a class $c_{\mathrm{SM}}(\psi_{X,V}^{\alpha})$; Theorem 3.3 gives an interpretation of this class for $\alpha =$ identity. It would be interesting to interpret this class for more general α.

3.2. Specialization to the central fiber of a morphism

To interpret Theorem 3.3 in terms of standard specializations, consider the template situation in which V fibers over a nonsingular curve T, and X is the fiber over a marked point $0 \in T$:

$$\begin{array}{ccc} X & \lhook\joinrel\longrightarrow & V \\ \downarrow & & \downarrow{\scriptstyle v} \\ \{0\} & \lhook\joinrel\longrightarrow & T. \end{array}$$

Further assume for simplicity that the restriction of v to $V \setminus X$ is a trivial fibration:

$$V \setminus X \cong (T \setminus \{0\}) \times V_t,$$

with nonsingular ('general') fiber V_t. (In particular, $V \setminus X$ is nonsingular, as in Section 3.1.) Here X is a Cartier divisor in V, so the specialization of Definition 3.1 agrees with ordinary pull-back (Lemma 3.2). We can use this pull-back to specialize classes from the general fiber V_t to the special fiber $v^{-1}(0) = X$, as follows. The pull-back $i^* \colon A_* V \to A_* X$ factors through $A_*(V \setminus X)$ via the basic exact sequence of Proposition 1.8 in [20]: indeed, for $\alpha \in A_* X$, $i^* i_*(\alpha) = c_1(N_X V) \cap \alpha = 0$ as $N_X V$ is trivial. We may then define a specialization morphism by composing this morphism with flat pull-back:

$$\sigma_* \colon A_* V_t \longrightarrow A_*(V \setminus X) \longrightarrow A_* X.$$

Corollary 3.7. *In the specialization situation just described,*

$$c_{\mathrm{SM}}(\psi_{X,V}) = \sigma_*(c(TV_t) \cap [V_t]).$$

Proof. Since the normal bundle of V_t in $V \setminus X$ is trivial, and abusing notation slightly, $c(TV_t) \cap [V_t] = c(T(V \setminus X) \cap [V_t])$. Hence, by definition of σ_* we have

$$\sigma_*(c(TV_t) \cap [V_t]) = i^*(\tau),$$

where τ is any class in V whose restriction to $A_*(V \setminus X)$ equals $c(T(V \setminus X) \cap [V \setminus X])$. We claim that $\tau = c_{\mathrm{SM}}(\mathbf{1}_{V \setminus X})$ is such a class; it then follows that

$$\sigma_*(c(TV_t) \cap [V_t]) = i^*(c_{\mathrm{SM}}(\mathbf{1}_{V \setminus X})) = \sigma_{X,V}(c_{\mathrm{SM}}(\mathbf{1}_{V \setminus X})) = c_{\mathrm{SM}}(\psi_{X,V}),$$

by Theorem 3.3 (and Lemma 3.2). Our claim is essentially immediate if V itself is nonsingular, but as we are only assuming $V \setminus X$ to be nonsingular we have to do a bit of work. Again consider a resolution $w \colon W \to V$ as in Definition 2.1. Let $w' \colon W \setminus D \to V \setminus X$ be the restriction (an isomorphism by hypothesis). Also, let $i' \colon V \setminus X \to V$ and $j' \colon W \setminus D \to W$ be the open inclusions. Thus we have the fiber diagram

$$
\begin{array}{ccc}
W \setminus D & \stackrel{j'}{\hookrightarrow} & W \\
{\scriptstyle w'}\downarrow & & \downarrow{\scriptstyle w} \\
V \setminus X & \stackrel{i'}{\hookrightarrow} & V
\end{array}
$$

with i' and j' flat, and w and w' proper. We have

$$c(T(V \setminus X)) \cap [V \setminus X] = w'_* c(T(W \setminus D)) \cap [W \setminus D]$$

$$\overset{(1)}{=} w'_* (j')^* c(TW(-\log D)) \cap [W]$$

$$\overset{(2)}{=} (i')^* w_* c(TW(-\log D)) \cap [W]$$

$$\overset{(3)}{=} (i')^* w_* c_{\mathrm{SM}}(\mathbf{1}_{W \setminus D})$$

$$= (i')^* c_{\mathrm{SM}}(\mathbf{1}_{V \setminus X})$$

as claimed. Equality (1) holds by definition of $TW(-\log D) = (\Omega^1_W(\log D))^\vee$; equality (2) follows from Proposition 1.7 in [20]; and equality (3) is again the computation of the CSM class of the complement of a divisor with normal crossings mentioned in the proof of Theorem 3.3. $\square$

Summarizing, in the strong specialization situation detailed above, $\psi_{X,V}$ is the constructible function on the central fiber X corresponding via MacPherson's natural transformation to the specialization of the Chern class of the general fiber.

Remark 3.8. The hypothesis of triviality of the family away from $0 \in T$ is not necessary, if specialization is interpreted appropriately. Let $v \colon V \to T$ be a morphism to a nonsingular curve, and let $X = v^{-1}(0)$ as above; assume our blanket hypothesis that $V \setminus X$ is nonsingular, but no more. As V is nonsingular away from X, we may assume that $v|_{V \setminus X}$ is smooth, after replacing T with a neighborhood of 0 in T. The fiber V_t is then smooth for all $t \neq 0$ in T, and since $N_{V_t} V$ is trivial, we get

$$c_{\mathrm{SM}}(\mathbf{1}_{V_t}) = c(V_t) \cap [V_t] = V_t \cdot (c(T(V \setminus X)) \cap [V \setminus X]);$$

since $c_{\mathrm{SM}}(\mathbf{1}_V)$ restricts to $c(T(V \setminus X)) \cap [V \setminus X]$ on $V \setminus X$, this gives

$$(4) \qquad\qquad c_{\mathrm{SM}}(\mathbf{1}_{V_t}) = V_t \cdot c_{\mathrm{SM}}(\mathbf{1}_V).$$

This computation fails for $t = 0$, as the fiber X over 0 is (possibly) singular. However, the classes $c_{\mathrm{SM}}(\mathbf{1}_V)$ and $c_{\mathrm{SM}}(\mathbf{1}_{V \setminus X})$ have the same specialization to X: indeed, their difference is supported on X, and $N_X V$ is trivial in the case considered here. Thus, in this case Theorem 3.3 gives

$$(5) \qquad\qquad c_{\mathrm{SM}}(\psi_{X,V}) = X \cdot c_{\mathrm{SM}}(\mathbf{1}_V).$$

Comparing (4) and (5), we may still view $\psi_{X,V}$ as the limit of the constant $\mathbf{1}_{V_t}$ as $t \to 0$ (in terms of the information carried by c_{SM} classes).

Remark 3.9. We can also consider specializations arising from maps $v\colon V\to T$, allowing T to be a nonsingular variety of arbitrary dimension with a marked point 0, and $X=v^{-1}(0)$ be a local complete intersection of codimension $\dim T$, with trivial normal bundle. We still assume $V\setminus X$ to be nonsingular. Using the formula in Remark 3.5, the specialization $i^* c_{\mathrm{SM}}(\mathbf{1}_{V\setminus X})$ may be written as

$$i^* c_{\mathrm{SM}}(\mathbf{1}_{V\setminus X}) = d_*((-D)^{\dim T - 1} \cdot c_{\mathrm{SM}}(\psi_{D,W})),$$

where W, D and d are as in Definition 2.1. Indeed, since $N_X V$ is trivial, then

$$c_{\mathrm{top}}\left(\frac{d^* N_X V}{N_D V}\right) = \text{term of codimension } \dim T - 1 \text{ in } \frac{1}{1+D} = (-D)^{\dim T - 1}.$$

Note that in this situation $d_*(-D)^{\dim T - 1} \cdot [D] = [X]$: indeed, the Segre class of D in W pushes forward to the Segre class of X in V, by the birational invariance of Segre classes.

As mentioned in the introduction, Schürmann has considered specialization to a complete intersection, by iterating applications of Verdier specialization ([29], Definition 3.6). It would be interesting to establish a precise relation between Schürmann's specialization and the formula given above.

4. Example: pencils of curves

4.1. Pencils of hypersurfaces

Pencils of hypersurfaces give rise to specializations, as follows.

Consider a pencil of hypersurfaces in a linear system in a nonsingular variety V'. Let the pencil be defined by the equation

$$F + tG = 0,$$

where F and G are elements of the system, and $t \in k$. Assume that for any $t \neq 0$ in a neighborhood of 0, $V_t = \{p \mid F(p) + tG(p) = 0\}$ is nonsingular. We can interpret this datum as a specialization by letting V be the correspondence

$$V = \{(p,t) \in V' \times \mathbb{A}^1 \mid F(p) + tG(p) = 0\}.$$

This is endowed with a projection $v\colon V \to T = \mathbb{A}^1$; after removing a finite set of $t \neq 0$ from $\mathbb{A}^1$ (that is, those $t \neq 0$ for which the fiber $F + tG = 0$ is singular), we reach the standard situation considered in Section 3.2 (Remark 3.8):

    – T is a nonsingular curve, and $0 \in T$;
    – $v\colon V \to T$ is a surjective morphism, and $X = v^{-1}(0)$;
    – $V \setminus X$ is nonsingular, and $v|_{V\setminus X}$ is smooth.

Note that V may be singular along X; in fact, the singularities of V are contained in the base locus of the pencil and in the singular locus of X.

As proven in Section 3,

$$c_{\mathrm{SM}}(\psi_{X,V}) = X \cdot c_{\mathrm{SM}}(\mathbf{1}_V),$$

for $\psi_{X,V}$ as in Definition 2.1, and this class can be viewed as the limit as $t \to 0$ of the classes $c(TV_t) \cap [V_t]$ of the general fibers. For example, the degree $\int c_{\mathrm{SM}}(\psi_{X,V})$ equals $\int c(TV_t) \cap [V_t] = \chi(V_t)$, the Euler characteristic of the general fiber.

The following are immediate consequences of the definition.

– Let $p \in X$ be a point at which X (and hence also V) is nonsingular. Then $\psi_{X,V}(p) = 1$.

– More generally, let p be a point such that there exists a neighborhood U of p in V such that U is nonsingular, and $U \cap X$ is a normal crossing divisor in $U \cap V$. Then $\psi_{X,V}(p) = 0$ if p is in the intersection of two or more components of X, and $\psi_{X,V}(p) = m$ if p is in a single component of X, of multiplicity m.

Since $\psi_{X,V}(p) = 1$ at nonsingular points of X, $\mathbf{1}_X - \psi_{X,V}$ is supported on the singular locus of X. This function has a compelling interpretation, see (1) in the introduction and Section 5. The following example illustrates a typical situation.

Example 4.1. Let X be a reduced hypersurface with isolated singularities in a nonsingular variety V', and assume that a general element V_t of the linear system of X is nonsingular, and avoids the singularities of X. Then $(-1)^{\dim V}(1 - \psi_{X,V}(p))$ equals the Milnor number $\mu_X(p)$ of X at p.

Indeed, as the matter is local, after a resolution we may assume that X is complete and p is its only singularity. Consider the pencil through X and a general element V_t of its linear system. Then by the linearity of c_{SM}, and since $\mathbf{1}_X - \psi_{X,V}$ is 0 away from p,

$$(-1)^{\dim V}(1 - \psi_{X,V}(p)) = (-1)^{\dim V} \int c_{\mathrm{SM}}(\mathbf{1}_X(p) - \psi_{X,V}(p))$$

$$= (-1)^{\dim V} \int c_{\mathrm{SM}}(\mathbf{1}_X - \psi_{X,V})$$

$$= (-1)^{\dim V} \left(\int c_{\mathrm{SM}}(\mathbf{1}_X) - \int c_{\mathrm{SM}}(\psi_{X,V}) \right).$$

Now $\int c_{\mathrm{SM}}(\mathbf{1}_X) = \chi(X)$ (as recalled in Section 3.1), and $\int c_{\mathrm{SM}}(\psi_{X,V}) = \int c_{\mathrm{SM}}(V_t) = \chi(V_t)$ as observed above. Thus,

$$(-1)^{\dim V}(1 - \psi_{X,V}(p)) = (-1)^{\dim V}(\chi(X) - \chi(V_t)),$$

and this is well known to equal the Milnor number of X at p (see, e.g., [24], Corollary 1.7).

The above formula is a particular case of the formula $\mu = (-1)^{\dim V}(\mathbf{1}_X - \psi_{X,V})$ mentioned in the introduction, and discussed further in Section 5.

Near points of X away from the base locus, V is trivially isomorphic to V', and hence it is itself nonsingular; the specialization function is then computed directly by an embedded resolution of X. Along the base locus, V itself may be singular, and it will usually be necessary to resolve V first in order to apply Definition 2.1 (or Proposition 2.6).

In the following subsections we illustrate this process in a few simple examples, for pencils of curves.

4.2. Singular points on curves

Let X be a curve with an ordinary multiple point p of multiplicity m, and assume that this point is not in the base locus of the pencil. As pointed out above, $\psi_{X,V}(p)$ may be computed by considering the embedded resolution of X over p. This may be schematically represented as:
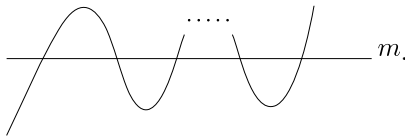
We have one exceptional divisor, a $\mathbb{P}^1$, meeting the proper transform of the curve m times and appearing with multiplicity m. As the Euler characteristic of the complement of the m points of intersection in the exceptional divisor is $2-m$, we have that $\psi_{X,V}(p) = m(2-m)$.

More generally, let X be a plane curve with an isolated singular point p, again assumed to be away from the base locus of the pencil. If the embedded resolution of X has exceptional divisors D_i over p, D_i appears with multiplicity m_i, and meets the rest of the full transform of X at r_i points, then

$$\psi_{X,V}(p) = \sum_i m_i(2-r_i);$$

indeed, each D_i is a copy of $\mathbb{P}^1$, and the Euler characteristic of the complement of r_i points in $\mathbb{P}^1$ is $2-r_i$. It follows (cf. Proposition 4.1) that the Milnor number of p is $1 - \sum_i m_i(2-r_i)$, yielding a quick proof of this well-known formula (see [16],

Section 8.5, Lemma 3, for a discussion of the geometry underlying this formula over $\mathbb{C}$).

For example, the resolution graph of an ordinary cusp is:
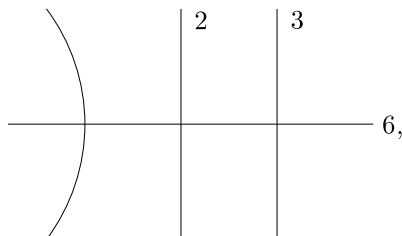
where the numbers indicate the multiplicity of the exceptional divisors. It follows that $\psi_{X,V}(p) = 2(2-1) + 3(2-1) + 6(2-3) = -1$ at an ordinary cusp.

4.3. Cuspidal curve, cusp in the base locus

Typically, V is *singular* at base points of the system at which X is singular. Subtleties in the computation of the specialization function arise precisely because of this phenomenon.

Again consider a pencil centered at an ordinary cusp p, but such that the general element of the pencil is nonsingular at p and tangent to X at p:
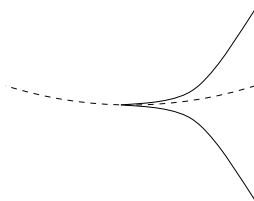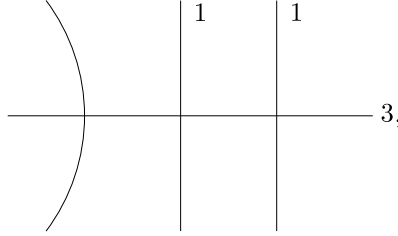

A local description for the correspondence V near p is

$$(y^2 - x^3) - ty = 0.$$

This may be viewed as a singular hypersurface in $\mathbb{A}^3_{(x,y,t)}$ and is resolved by a single blow-up at $(0,0,0)$ (as the reader may check). One more blow-up produces a divisor with normal crossings as needed in Definition 2.1, with the same resolution graph

as in Section 4.2:

but with different multiplicities, as indicated. This does not affect the value of $\psi_{X,V}(p)$:

$$\psi_{X,V}(p) = 1(2-1) + 1(2-1) + 3(2-3) = -1.$$

In fact, it is clear 'for specialization reasons' that the value of $\psi_{X,V}(p)$ at an isolated singularity p of X is the same whether p is in the base locus of the pencil or not, since (as pointed out at the beginning of this section) the Euler characteristic of X weighted according to $\psi_{X,V}$ must equal the Euler characteristic of the general element of the pencil, and this latter is unaffected by the intersection of X with the general element. It is however interesting that the geometry of the resolution *is* affected by the base locus of the pencil: the total space of the specialization is smooth near p if the cusp is not in the base locus (as in Section 4.2), while it is singular if the cusp is in the base locus (as in this subsection). Any difference in the normal crossing resolution due to these features must compensate and produce the same value for $\psi_{X,V}(p)$. The next example will illustrate that this is not necessarily the case for nonisolated singularities.

Note also that the 'distinguished decomposition' of $\psi_{X,V}$, or of its motivic counterpart, do tell these two situations apart: for example, with notation as in Remark 2.5, $\psi_{X,V}^{\varepsilon_3} = \mathbf{1}_p$ for the cuspidal curve in Section 4.2 (cusp p away from the base locus), while $\psi_{X,V}^{\varepsilon_3} = -\mathbf{1}_p$ for the cuspidal curve in this subsection (cusp on the base locus).

4.4. Nonisolated singularities

Let V' be a nonsingular surface, and $X \subset V'$ be a (possibly multiple, reducible) curve. Consider the pencil between X and a nonsingular curve Y meeting a component of multiplicity $m \geq 1$ in X transversally at a general point p. View this as a specialization, as explained in Section 4.1. Then $\psi_{X,V}(p) = 1$, regardless of the multiplicity m.

To verify this, we may choose analytic coordinates (x, y) so that p is the origin, X is given by x^{n+1} for $n = m - 1$, and Y is $y = 0$; the correspondence V is given by

$$x^{n+1} - yt = 0$$

in coordinates (x, y, t).

If $n = 0$, then both X and V are nonsingular, and $\psi_X(p) = 1$ as seen above. If $n > 0$, V has an A_n singularity at the origin, and its resolution is classically well known: the exceptional divisors are $\mathbb{P}^1$'s, linked according to the A_n diagram,

$$\circ\!\!-\!\!-\!\!-\!\!\circ\!\!-\!\!-\!\!-\!\!...\!\!-\!\!-\!\!\circ.$$

The only work needed here is to keep track of the multiplicities of the components in the inverse image of $t = 0$. The reader may check that the pull-back of $t = 0$ in the resolution consists of the proper transform of the original central fiber, with multiplicity $n + 1$, and of a chain of smooth rational curves, with decreasing multiplicities:
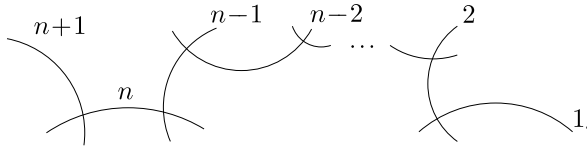
The contribution to $\psi_{X,V}(p)$ of all but the right-most component in this diagram is 0, because the Euler characteristic of the complement of 2 points in $\mathbb{P}^1$ is 0. The right-most component contributes 1, since it appears with multiplicity 1 and it contributes by the Euler characteristic of the complement of one point in $\mathbb{P}^1$, that is 1. Thus $\psi_{X,V}(p) = 1$, as claimed.

Note that $\psi_{X,V} = m$ at a general point of a component of multiplicity m in X. Thus, the effect of p being in the base locus is to erase the multiplicity information, provided that the general element Y of the pencil meets X transversally at p.

This is in fact an instance of a general result, proven over $\mathbb{C}$ in all dimensions and for arbitrarily singular X by A. Parusiński and P. Pragacz in Proposition 5.1 of [26]. The proof given in this reference uses rather delicate geometric arguments (for example, it relies on the fact that a Whitney stratification satisfies Thom's a_f condition, [25]). It would be worthwhile constructing a direct argument from the definition of $\psi_{X,V}$ given in this paper, and valid over any algebraically closed field of characteristic 0.

5. Weighted Chern–Mather classes, and a resolution formula

5.1. Weighted Chern–Mather classes

Let V be a nonsingular variety, and let $Y \subset V$ be a closed subscheme. In [4] we have considered the *weighted Chern–Mather class* of Y,

$$c_{\mathrm{wMa}}(Y) := \sum_{i=1}^{\infty} (-1)^{\dim Y - \dim Y_i} m_i j_{i*} c_{\mathrm{Ma}}(Y_i),$$

where Y_i are the supports of the components of the normal cone of Y in V, m_i is the multiplicity of the component over Y_i, and c_{Ma} denotes the ordinary Chern–Mather class. Also, $\dim Y$ is the largest dimension of a component of Y. Up to a sign, c_{wMa} is the same as the *Aluffi class* of [12]. With the definition given above, if Y is irreducible and reduced, then $c_{\mathrm{wMa}}(Y)$ equals $c_{\mathrm{Ma}}(Y)$; in particular, $c_{\mathrm{wMa}}(Y) = c(TY) \cap [Y]$ if Y is nonsingular. (However, with this choice of sign the contribution of a component Y_i depends on the dimension of the largest component of Y.)

Consider the case in which Y is the *singularity subscheme* of a hypersurface X in V: if f is a local generator for the ideal of X, then the ideal of Y in V is locally generated by f and the partial derivatives of f.

Proposition 5.1. *Let X be a hypersurface in a nonsingular variety V, and let Y be its singularity subscheme. Then*

(6) $$(-1)^{\dim Y} c_{\mathrm{wMa}}(Y) = (-1)^{\dim V} c_{\mathrm{SM}}(\mathbf{1}_X - \psi_{X,V})$$

in $A_* X$.

Since $c_{\mathrm{SM}}(\psi_{X,V})$ admits a natural multiplicity decomposition (Remark 3.6), so do the class $c_{\mathrm{wMa}}(Y)$ and its degree, a Donaldson–Thomas type invariant ([12], Proposition 4.16). That is, monodromy induces a decomposition of these invariants.

Remark 5.2. The relation in Proposition 5.1 is a c_{SM}-counterpart of the identity $\mu = (-1)^{\dim V}(\mathbf{1}_X - \psi_{X,V})$ mentioned in the introduction, and is equivalent to Theorem 1.5 in [4]. The identity amounts to the relation of μ with the Euler characteristic of the perverse sheaf of vanishing cycles. It goes at least as far back as [2], Theorem 4; it also follows from Proposition 5.1 in [26], and is equation (4) in [12]. However, these references work over $\mathbb{C}$, and it seems appropriate to indicate a proof of (6) which is closer in spirit to the content of this paper.

Proof. By Theorem 3.3,

$$c_{\mathrm{SM}}(\psi_{X,V}) = X \cdot c_{\mathrm{SM}}(\mathbf{1}_{V \setminus X})$$

(cf. Lemma 3.2). Thus

$$\begin{aligned}
c_{\mathrm{SM}}(\mathbf{1}_X - \psi_{X,V}) &= c_{\mathrm{SM}}(\mathbf{1}_X) - X \cdot c_{\mathrm{SM}}(\mathbf{1}_V) + X \cdot c_{\mathrm{SM}}(\mathbf{1}_X) \\
&= (1+X) \cdot c_{\mathrm{SM}}(\mathbf{1}_X) - c(TV|_X) \cap [X] \\
&= c(\mathcal{L}) \cap \left(c_{\mathrm{SM}}(\mathbf{1}_X) - \frac{c(TV|_X)}{1+X} \cap [X] \right)
\end{aligned}$$

with $\mathcal{L} = \mathcal{O}(X)$. The class $(c(TV|_X)/(1+X)) \cap [X]$ is the class of the virtual tangent bundle to X, denoted $c_{\mathrm{F}}(X)$ in [4]. Thus,

$$c_{\mathrm{SM}}(\mathbf{1}_X - \psi_{X,V}) = c(\mathcal{L}) \cap (c_{\mathrm{SM}}(X) - c_{\mathrm{F}}(X)).$$

By Theorem 1.2 in [4], it follows that

$$c_{\mathrm{SM}}(\mathbf{1}_X - \psi_{X,V}) = (-1)^{\dim V - \dim Y} c_{\mathrm{wMa}}(Y),$$

which is the statement. $\square$

Both sides of the formula in Proposition 5.1 make sense in a more general situation than the case in which Y is the singularity subscheme of a hypersurface X: the class $c_{\mathrm{wMa}}(Y)$ is defined for arbitrary subschemes of a nonsingular variety, and Definition 2.1 also gives a meaning to $\psi_{X,V}$ for any subscheme X of a nonsingular variety. Or in other terms, both sides of the identity $\mu = (-1)^{\dim V}(\mathbf{1}_X - \psi_{X,V})$ admit compelling generalizations: Definition 2.1 does not require X to be a hypersurface, and $c_{\mathrm{wMa}}(Y)$ is the class corresponding to a constructible function ν_Y defined for all Y as a specific combination of local Euler obstructions (see Proposition 1.5 in [4]). This function has garnered some interest in Donaldson–Thomas theory, and is currently commonly known as *Behrend's function.* It is tempting to guess that a statement closely related to Proposition 5.1 may still hold for *any* Y and a suitable choice of $X \subset V$ with singularities along Y. (Of course $\mathbf{1}_X$ should be replaced by $(\mathrm{codim}_X V)\mathbf{1}_X$ for the right-hand side to be supported on the singularities of X, see Example 2.8.)

5.2. Resolutions

We end with an expression for the function μ for the singularity subscheme Y of a hypersurface X in terms of a resolution of X. For this we need to work with $\mathbb{Q}$-valued constructible functions; we do not know if a similar statement can be given over $\mathbb{Z}$.

Assume that V is nonsingular, $X \subset V$ is a hypersurface, and $Y \subseteq X$ is the singularity subscheme. Consider a morphism $w \colon W \to V$ as in Definition 2.1. Thus, $w^{-1}(X)$ is a divisor D with normal crossings and nonsingular components D_l, $l \in L$; m_l denotes the multiplicity of D_l in D, and $d \colon D \to X$ is the restriction of w.

The relative canonical divisor of w is a combination $\sum_l \mu_l D_l$ of components of D.

For $K \subseteq L$, we let

$$D_K^\circ = \left(\bigcap_{k \in K} D_k \right) \Big\backslash \left(\bigcup_{l \notin K} D_l \right).$$

Proposition 5.3. *With notation as above, μ is given by*

$$(-1)^{\dim X} d_* \left(\sum_{l \in L} \left(m_l - \frac{1}{1+\mu_l} \right) \mathbf{1}_{D_l^\circ} - \sum_{\substack{K \subseteq L \\ |K| \geq 2}} \frac{1}{\prod_{k \in K}(1+\mu_k)} \mathbf{1}_{D_K^\circ} \right).$$

Proof. As discussed above, $\mu = (-1)^{\dim X}(\psi_{X,V} - \mathbf{1}_X)$. The given formula follows immediately from

$$\psi_{X,V} = d_* \sum_{l \in L} m_l \mathbf{1}_{D_l^\circ},$$

which is a restatement of Definition 2.1, and

$$\mathbf{1}_X = d_* \left(\sum_{\substack{K \subseteq L \\ |K| \geq 1}} \frac{1}{\prod_{k \in K}(1+\mu_k)} \mathbf{1}_{D_K^\circ} \right),$$

which follows from a small generalization of Section 4.4.3 in [17] (cf. Theorem 2.1 and the proof of Theorem 3.1 in [5]). $\square$

Again, it is tempting to guess that a similar expression may exist for Behrend's function of an arbitrary subscheme Y of a nonsingular variety V, for a suitable choice of a corresponding pair $X \subset V$.

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