Ark. Mat., **51** (2013), 315–328
DOI: 10.1007/s11512-011-0161-5

Möbius homogeneous hypersurfaces with two distinct principal curvatures in S^{n+1}

Tongzhu Li, Xiang Ma and Changping Wang

Abstract. The purpose of this paper is to classify the Möbius homogeneous hypersurfaces with two distinct principal curvatures in S^{n+1} under the Möbius transformation group. Additionally, we give a classification of the Möbius homogeneous hypersurfaces in S^4.

1. Introduction

A diffeomorphism $\phi\colon S^{n+1}\to S^{n+1}$ is said to be a *Möbius transformation* if ϕ takes round n-spheres into round n-spheres. The Möbius transformations form a transformation group, which is called the *Möbius transformation group of* S^{n+1} and denoted by $M(S^{n+1})$. It is well known that for $n\geq 2$ the Möbius group $M(S^{n+1})$ coincides with the conformal group $C(S^{n+1})$. In [11], Wang introduced a complete Möbius invariant system for a submanifold $x\colon M^m\to S^{n+1}$, and obtained a congruence theorem for hypersurfaces in S^{n+1} (see also [1]). Recently some special hypersurfaces in S^{n+1}, for example, the Möbius isoparametric hypersurfaces, the Blaschke isoparametric hypersurfaces and so on, have been extensively studied in the context of Möbius geometry (see, for instance, [3]–[6]).

Another special hypersurface is the Möbius homogeneous hypersurface. A hypersurface $x\colon M^n\to S^{n+1}$ is said to be a *Möbius homogeneous hypersurface* if for any two points $p,q\in M^n$, there exists a Möbius transformation $\phi\in M(S^{n+1})$ such that $\phi\circ x(M^n)=x(M^n)$ and $\phi\circ x(p)=x(q)$. Standard examples of Möbius homogeneous hypersurfaces are images of (Euclidean) homogeneous hypersurfaces in S^{n+1} under Möbius transformations. But there are some examples of Möbius homogeneous hypersurfaces which cannot be obtained in this way. In [9], Sulanke constructed a Möbius homogeneous surface, which is the image of the inverse of the stereographic projection $\sigma\colon \mathbb{R}^3\to S^3$ of a cylinder over a logarithmic spiral in $\mathbb{R}^2$, and classified Möbius homogeneous surfaces in S^3 under the Möbius transformation group.

Our goal is to classify the Möbius homogeneous hypersurfaces with two distinct principal curvatures in S^{n+1} under the Möbius transformation group. Let H^{n+1} be the hyperbolic space

$$H^{n+1}=\{(y_0,\vec{y}_1)\in\mathbb{R}^+\times\mathbb{R}^{n+1}\mid\langle y,y\rangle=-y_0^2+\vec{y}_1\cdot\vec{y}_1=-1\}.$$

We can define the conformal map $\tau\colon H^{n+1}\to S^{n+1}$ by

$$\tau(y)=\left(\frac{1}{y_0},\frac{\vec{y}_1}{y_0}\right),\quad y=(y_0,\vec{y}_1)\in H^{n+1}.$$

The inverse of the stereographic projection $\sigma\colon\mathbb{R}^{n+1}\to S^{n+1}$ is defined by

$$\sigma(u)=\left(\frac{1-|u|^2}{1+|u|^2},\frac{2u}{1+|u|^2}\right).$$

The conformal maps σ and τ assign any hypersurface in $\mathbb{R}^{n+1}$ or H^{n+1} to a hypersurfaces in S^{n+1}. In [7], the authors proved that the Möbius invariants on $f\colon M^n\to\mathbb{R}^{n+1}$ and $f\colon M^n\to H^{n+1}$ are the same as the Möbius invariants on $\sigma\circ f\colon M^n\to S^{n+1}$ and $\tau\circ f\colon M^n\to S^{n+1}$, respectively. Next we give an example of a Möbius homogeneous hypersurface, which is a higher-dimensional version of Sulanke's example.

Example 1.1. Let $\gamma\colon I\to\mathbb{R}^2$ be the logarithmic spiral given by

$$\gamma(s)=(\sin se^{cs},\cos se^{cs}),\quad c>0.$$

The cylinder in $\mathbb{R}^{n+1}$ over $\gamma(s)$ is defined by

$$f(\gamma,\mathrm{id})\colon I\times\mathbb{R}^{n-1}\longrightarrow\mathbb{R}^{n+1},$$

where $\mathrm{id}\colon\mathbb{R}^{n-1}\to\mathbb{R}^{n-1}$ is the identity mapping. We call the hypersurface f a *logarithmic spiral cylinder.*

We give a characteristic of logarithmic spiral cylinders as follows.

Theorem 1.2. *Let $x\colon M^n\to S^{n+1}$ be a Möbius homogeneous hypersurface with two distinct principal curvatures. If the Möbius form $C\neq0$, then x is Möbius equivalent to the image of σ of a logarithmic spiral cylinder.*

We need to point out that the logarithmic spiral cylinder is of constant Möbius sectional curvature $K=-|C|^2$. Our main results are as follows.

Theorem 1.3. *Let $x\colon M^n \to S^{n+1}$ be a Möbius homogeneous hypersurface with two distinct principal curvatures. Then x is Möbius equivalent to one of the following hypersurfaces*:

(1) *the standard torus $S^k(r) \times S^{n-k}\big(\sqrt{1-r^2}\big)$, $1 \le k \le n-1$;*

(2) *the images of σ of the standard cylinder $S^k(1) \times \mathbb{R}^{n-k} \subset \mathbb{R}^{n+1}$, $1 \le k \le n-1$;*

(3) *the images of τ of $S^k(r) \times H^{n-k}\big(\sqrt{1+r^2}\big)$, $1 \le k \le n-1$;*

(4) *the image of σ of a logarithmic spiral cylinder.*

In [10], Wang classified the Möbius homogeneous hypersurfaces with three distinct principal curvatures in S^4, which consists of two categories. One is the 1-parameter family of isoparametric hypersurfaces with three principal curvatures, which is a tube of constant radius over a standard Veronese embedding of RP^2 into S^4 (see [2]). Another is the images of σ of the cone over the 1-parameter family of isoparametric tori in S^3. Thus combining Theorem 1.3, we have the following results.

Corollary 1.4. *Let $x\colon M^3 \to S^4$ be a Möbius homogeneous hypersurface. Then x is Möbius equivalent to one of the following hypersurfaces*:

(1) *the round sphere $S^3 \subset S^4$;*

(2) *the standard torus $S^1(r) \times S^2\big(\sqrt{1-r^2}\big)$;*

(3) *the images of σ of the standard cylinder $S^k(1) \times \mathbb{R}^{3-k} \subset \mathbb{R}^4$, $1 \le k \le 2$;*

(4) *the images of τ of $S^k(r) \times H^{3-k}\big(\sqrt{1+r^2}\big)$, $1 \le k \le 2$;*

(5) *the image of σ of a logarithmic spiral cylinder;*

(6) *the image of σ of the cone over the Clifford torus $S^1(r) \times S^1\big(\sqrt{1-r^2}\big)$;*

(7) *the tube of constant radius over a standard Veronese embedding of RP^2 into S^4.*

We organize the paper as follows. In Section 2, we give the elementary facts about Möbius geometry for hypersurfaces in S^{n+1} needed in this paper. In Section 3, we construct some Möbius homogeneous hypersurfaces in S^{n+1}, and give the proofs of Theorems 1.2 and 1.3.

2. Möbius invariants for hypersurfaces in S^{n+1}

In this section, we recall some facts about the Möbius transformation group and define Möbius invariants of hypersurfaces in S^{n+1}. For details we refer to [11].

Let $\mathbb{R}^{n+3}_1$ be the Lorentz space, i.e., $\mathbb{R}^{n+3}$ with the inner product $\langle \cdot, \cdot \rangle$ defined by

$$\langle x, y \rangle = -x_0 y_0 + x_1 y_1 + \ldots + x_{n+2} y_{n+2}$$

for $x=(x_0,x_1,...,x_{n+2}), y=(y_0,y_1,...,y_{n+2})\in\mathbb{R}^{n+3}$.

Let $\mathrm{O}(n+2,1)$ be the Lorentz group of $\mathbb{R}^{n+3}_1$ defined by

$$\mathrm{O}(n+2,1)=\{T\in \mathrm{GL}(\mathbb{R}^{n+3})\mid {}^tTI_1T=I_1\},$$

where ${}^tT$ denotes the transpose of T and $I_1=\left(\begin{smallmatrix}-1&0\\0&I\end{smallmatrix}\right)$.

Let

$$C^{n+2}_+=\{y=(y_0,y_1)\in\mathbb{R}\times\mathbb{R}^{n+2}\mid \langle y,y\rangle=0 \text{ and } y_0>0\}\subset\mathbb{R}^{n+3}_1,$$

and $\mathrm{O}^+(n+2,1)$ denote the subgroup of $\mathrm{O}(n+2,1)$ defined by

$$\mathrm{O}^+(n+2,1)=\{T\in\mathrm{O}(n+2,1)\mid T(C^{n+2}_+)=C^{n+2}_+\}.$$

Lemma 2.1. ([8]) *Let $T=\left(\begin{smallmatrix}w&u\\v&B\end{smallmatrix}\right)\in\mathrm{O}(n+2,1)$. Then $T\in\mathrm{O}^+(n+2,1)$ if and only if $w>0$.*

It is well known that the subgroup $\mathrm{O}^+(n+2,1)$ is isomorphic to the Möbius transformation group $M(S^{n+1})$. In fact, for any

$$T=\begin{pmatrix}w&u\\v&B\end{pmatrix}\in\mathrm{O}^+(n+2,1),$$

we can define the Möbius transformation $L(T)\colon S^{n+1}\to S^{n+1}$ by

$$L(T)(x)=\frac{Bx+u}{vx+w},\quad x={}^t(x_1,...,x_{n+2})\in S^{n+1}.$$

Then the map $L\colon \mathrm{O}^+(n+2,1)\to M(S^{n+1})$ is a group isomorphism.

Let $x\colon M^n\to S^{n+1}$ be a hypersurface without umbilical point, and e_{n+1} be the unit normal vector field. Let II and H be the second fundamental form and the mean curvature of x, respectively. The Möbius position vector $Y\colon M^n\to\mathbb{R}^{n+3}_1$ of x is defined by

$$Y=\rho(1,x),\quad \rho^2=\frac{n}{n-1}(\|II\|^2-nH^2).$$

Theorem 2.2. ([11]) *Two hypersurfaces $x,\tilde{x}\colon M^n\to S^{n+1}$ are Möbius equivalent if and only if there exists $T\in\mathrm{O}^+(n+2,1)$ such that $\widetilde{Y}=YT$.*

It follows immediately from Lemma 2.1 that

$$g=\langle dY,dY\rangle=\rho^2\,dx\cdot dx$$

is a Möbius invariant, which is called the Möbius metric of x (see [11]).

Let Δ be the Laplacian operator with respect to g. We define

$$N = -\frac{1}{n}\Delta Y - \frac{1}{2n^2}\langle \Delta Y, \Delta Y\rangle Y.$$

Then we have

$$\langle Y, Y\rangle = 0, \quad \langle N, Y\rangle = 1 \quad \text{and} \quad \langle N, N\rangle = 0.$$

Let $\{E_1, ..., E_n\}$ be a local orthonormal basis for (M^n, g) with the dual basis $\{\omega_1, ..., \omega_n\}$, and write $Y_i = E_i(Y)$. Then we have

$$\langle Y_i, Y\rangle = \langle Y_i, N\rangle = 0 \text{ and } \langle Y_i, Y_j\rangle = \delta_{ij}, \quad 1 \le i, j \le n.$$

We define the conformal Gauss map

$$G = (H, Hx + e_{n+1}).$$

By direct computation, we have

$$\langle G, Y\rangle = \langle G, N\rangle = \langle G, Y_i\rangle = 0 \quad \text{and} \quad \langle G, G\rangle = 1.$$

Then $\{Y, N, Y_1, ..., Y_n, G\}$ forms a moving frame in $\mathbb{R}^{n+3}_1$ along M^n. We use the following range of indices in this section: $1 \le i, j, k, l \le n$. We can write the structure equations as

$$\begin{aligned}
dY &= \sum_{i=1}^{n} Y_i \omega_i, \\
dN &= \sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij}\omega_i Y_j + \sum_{i=1}^{n} C_i \omega_i G, \\
dY_i &= -\sum_{j=1}^{n} A_{ij}\omega_j Y - \omega_i N + \sum_{j=1}^{n} \omega_{ij} Y_j + \sum_{j=1}^{n} B_{ij}\omega_j G, \\
dG &= -\sum_{i=1}^{n} C_i \omega_i Y - \sum_{i=1}^{n}\sum_{j=1}^{n} \omega_j B_{ij} Y_i,
\end{aligned}$$

where ω_{ij} is the connection form of the Möbius metric g, and $\omega_{ij} + \omega_{ji} = 0$. The tensors $A = \sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij}\omega_i \otimes \omega_j$, $C = \sum_{i=1}^{n} C_i \omega_i$ and $B = \sum_{i=1}^{n}\sum_{j=1}^{n} B_{ij}\omega_i \otimes \omega_j$ are called the *Blaschke tensor*, the *Möbius form* and the *Möbius second fundamental form* of x, respectively. The eigenvalues of (B_{ij}) are called the *Möbius principal curvatures* of x. The covariant derivatives of C_i, A_{ij} and B_{ij} are defined by

$$\sum_{j=1}^{n} C_{i,j}\omega_j = dC_i + \sum_{j=1}^{n} C_j \omega_{ji},$$

$$\sum_{k=1}^{n} A_{ij,k}\omega_k = dA_{ij} + \sum_{k=1}^{n} A_{ik}\omega_{kj} + \sum_{k=1}^{n} A_{kj}\omega_{ki},$$

$$\sum_{k=1}^{n} B_{ij,k}\omega_k = dB_{ij} + \sum_{k=1}^{n} B_{ik}\omega_{kj} + \sum_{k=1}^{n} B_{kj}\omega_{ki}.$$

The integrability conditions for the structure equations are given by

$$(1) \qquad A_{ij,k} - A_{ik,j} = B_{ik}C_j - B_{ij}C_k,$$

$$(2) \qquad C_{i,j} - C_{j,i} = \sum_{k=1}^{n} (B_{ik}A_{kj} - B_{jk}A_{ki}),$$

$$(3) \qquad B_{ij,k} - B_{ik,j} = \delta_{ij}C_k - \delta_{ik}C_j, \qquad \sum_{j=1}^{n} B_{ij,j} = -(n-1)C_i,$$

$$(4) \qquad R_{ijkl} = B_{ik}B_{jl} - B_{il}B_{jk} + \delta_{ik}A_{jl} + \delta_{jl}A_{ik} - \delta_{il}A_{jk} - \delta_{jk}A_{il},$$

$$(5) \qquad R_{ij} := \sum_{k=1}^{n} R_{ikjk} = -\sum_{k=1}^{n} B_{ik}B_{kj} + (\operatorname{tr}\mathbf{A})\delta_{ij} + (n-2)A_{ij},$$

$$(6) \qquad \sum_{i=1}^{n} B_{ii} = 0, \quad \sum_{i=1}^{n}\sum_{j=1}^{n} B_{ij}^2 = \frac{n-1}{n} \quad \text{and} \quad \operatorname{tr} A = \sum_{i=1}^{n} A_{ii} = \frac{1}{2n}(1+n^2 s),$$

where R_{ijkl} denotes the curvature tensor of g and $s=(1/n(n-1))\sum_{i=1}^{n}\sum_{j=1}^{n} R_{ijij}$ is the normalized Möbius scalar curvature. When $n\geq 3$, we know that all coefficients in the structure equations are determined by $\{g, B\}$ and we have the following theorem.

Theorem 2.3. ([11]) *Two hypersurfaces $x\colon M^n \to S^{n+1}$ and $\tilde{x}\colon M^n \to S^{n+1}$, $n\geq 3$, are Möbius equivalent if and only if there exists a diffeomorphism $\varphi\colon M^n \to M^n$, which preserves the Möbius metric g and the Möbius second fundamental form B.*

Using the stereographic projection $\sigma^{-1}\colon S^{n+1} \to \mathbb{R}^{n+1} \cup \{\infty\}$, the Möbius invariants of the hypersurface $f = \sigma^{-1} \circ x\colon M^n \to \mathbb{R}^{n+1}$ and the Euclidean invariants of f are related by [7] as follows:

$$B_{ij} = \rho^{-1}(h_{ij} - H\delta_{ij}),$$

$$(7) \quad C_i = -\rho^{-2}\left[e_i(H) + \sum_{j=1}^{n} (h_{ij} - H\delta_{ij})e_j(\log\rho)\right],$$

$$A_{ij} = -\rho^{-2}[\mathrm{Hess}_{ij}(\log\rho) - e_i(\log\rho)e_j(\log\rho) - Hh_{ij}] - \tfrac{1}{2}\rho^{-2}(|\nabla \log\rho|^2 + H^2)\delta_{ij},$$

where Hess_{ij} and ∇ are the Hessian matrix and the gradient with respect to $I = df \cdot df$, respectively, and H is the mean curvature of f. Let $\{e_1, ..., e_n\}$ be an orthonormal basis for (M^n, I) and let the dual basis be $\{\theta_1, ..., \theta_n\}$. Then

$$A = \rho^2 \sum_{i=1}^n \sum_{j=1}^n A_{ij}\theta_i \otimes \theta_j, \quad B = \rho^2 \sum_{i=1}^n \sum_{j=1}^n B_{ij}\theta_i \otimes \theta_j \quad \text{and} \quad C = \rho \sum_{i=1}^n C_i \theta_i.$$

Clearly the number of distinct Möbius principal curvatures is the same as that of its distinct Euclidean principal curvatures. Let $k_1, ..., k_n$ be the principal curvatures of f, and $\{\lambda_1, ..., \lambda_n\}$ be the corresponding Möbius principal curvatures. Let e_{n+1} be the unit normal vector field of f. Then the curvature sphere of principal curvature k_i is

$$\xi_i = \lambda_i Y + \xi = \left(\frac{1+|f|^2}{2}k_i + f\cdot e_{n+1}, \frac{1-|f|^2}{2}k_i - f\cdot e_{n+1}, k_i f + e_{n+1}\right),$$

where Y and ξ are the Möbius position vector and the conformal Gauss map of f, respectively, given by

$$Y = \rho\left(\frac{1+|f|^2}{2}, \frac{1+|f|^2}{2}, f\right), \quad \rho^2 = \frac{n}{n-1}(\|II\|^2 - nH^2), \quad \text{and}$$

$$\xi = \left(\frac{1+|f|^2}{2}H + f\cdot e_{n+1}, \frac{1-|f|^2}{2}H - f\cdot e_{n+1}, Hf + e_{n+1}\right).$$

If $\langle \xi_i, (1, -1, 0, ..., 0)\rangle = 0$, then $k_i = 0$. This means that the curvature sphere of principal curvature k_i is a hyperplane in $\mathbb{R}^{n+1}$.

3. The Möbius homogeneous hypersurface in S^{n+1}

In this section we give some examples of the Möbius homogeneous hypersurface, after which we prove our main Theorem 1.3.

Let $x\colon M^n \to S^{n+1}$ be a Möbius homogeneous hypersurface. We define

$$\Pi = \{\phi \in M(S^{n+1}) \mid \phi \circ x(M^n) = x(M^n)\}.$$

Then Π is a subgroup of the Möbius group $M(S^{n+1})$, and the hypersurface x is the orbit of the subgroup Π. Thus the Möbius invariants on the hypersurface x are constant. Next we give some examples of Möbius homogeneous hypersurfaces.

Example 3.1. Let

$$\Pi = \left\{\begin{pmatrix} 1 & 0 & 0 \\ 0 & \mathrm{O}(k+1) & 0 \\ 0 & 0 & \mathrm{O}(n-k+1) \end{pmatrix}\right\} \subset \mathrm{O}^+(n+2, 1).$$

Then Π is a subgroup of $\mathrm{O}^+(n+2,1)$.

The standard torus $x\colon S^k(r)\times S^{n-k}\big(\sqrt{1-r^2}\big)\to S^{n+1}$ is a Möbius homogeneous hypersurface. It is the orbit of the subgroup $L(\Pi)\subset M(S^{n+1})$ acting on the point

$$p=\big(r,\underbrace{0,\dots,0}_{k},\sqrt{1-r^2},\underbrace{0,\dots,0}_{n-k}\big)\in S^{n+1}.$$

Example 3.2. Let

$$\Pi=\left\{\begin{pmatrix}\mathrm{O}^+(n-k,1) & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & \mathrm{O}(k+1)\end{pmatrix}\right\}\subset \mathrm{O}^+(n+2,1).$$

Then Π is a subgroup of $\mathrm{O}^+(n+2,1)$.

Let $H^{n-k}\big(\sqrt{1+r^2}\big)\times S^k(r)$ be the isoparametric hypersurface in H^{n+1}. The hypersurface

$$\tau\big(H^{n-k}\big(\sqrt{1+r^2}\big)\times S^k(r)\big)\subset S^{n+1}$$

is a Möbius homogeneous hypersurface. It is the orbit of the subgroup $L(\Pi)\subset M(S^{n+1})$ acting on the point

$$p=\left(\underbrace{0,\dots,0}_{n-k},\frac{1}{\sqrt{1+r^2}},\frac{r}{\sqrt{1+r^2}},\underbrace{0,\dots,0}_{k}\right)\in S^{n+1}.$$

Example 3.3. Let

$$\Pi=\left\{\begin{pmatrix}1+\frac12|u|^2 & -\frac12|u|^2 & u_1 & \dots & u_{n-k} & 0\\ \frac12|u|^2 & 1-\frac12|u|^2 & u_1 & \dots & u_{n-k} & 0\\ u_1 & -u_1 & 1 & \dots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ u_{n-k} & -u_{n-k} & 0 & \dots & 1 & 0\\ 0 & 0 & 0 & \dots & 0 & \mathrm{O}(k+1)\end{pmatrix}\right\}\subset \mathrm{O}^+(n+2,1).$$

Then Π is a subgroup of $\mathrm{O}^+(n+2,1)$.

Let $\mathbb{R}^{n-k}\times S^k\big(\sqrt2\big)$ be the isoparametric hypersurface in $\mathbb{R}^{n+1}$. The hypersurface

$$\sigma\big(\mathbb{R}^{n-k}\times S^k\big(\sqrt2\big)\big)\subset S^{n+1}$$

is a Möbius homogeneous hypersurface. It is the orbit of the subgroup $L(\Pi)\subset M(S^{n+1})$ acting on the point

$$p=\big(\tfrac13,\underbrace{0,\dots,0}_{n-k},\tfrac23\sqrt2,\underbrace{0,\dots,0}_{k}\big)\in S^{n+1}.$$

Example 3.4. Let $f(s,u_1,...,u_{n-1})=(\sin se^{cs},\cos se^{cs},u_1,...,u_{n-1})\in\mathbb{R}^{n+1}$, and

$$\Pi=\left\{\begin{pmatrix} 1+\frac{1}{2}|f|^2 & -\frac{1}{2}|f|^2 & \sin se^{cs} & \cos se^{cs} & u_1 & \dots & u_{n-1} \\ \frac{1}{2}|f|^2 & 1-\frac{1}{2}|f|^2 & \sin se^{cs} & \cos se^{cs} & u_1 & \dots & u_{n-1} \\ \sin se^{cs} & -\sin se^{cs} & 1 & 0 & 0 & \dots & 0 \\ \cos se^{cs} & -\cos se^{cs} & 0 & 1 & 0 & \dots & 0 \\ u_1 & -u_1 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ u_{n-1} & -u_{n-1} & 0 & 0 & 0 & \dots & 1 \end{pmatrix}\right\}\subset \mathrm{O}^+(n+2,1).$$

Then Π is a subgroup of $\mathrm{O}^+(n+2,1)$.

The logarithmic spiral cylinder

$$f(s,u_1,...,u_{n-1})=(\sin se^{cs},\cos se^{cs},u_1,...,u_{n-1})\in\mathbb{R}^{n+1}$$

is a Möbius homogeneous hypersurface in $\mathbb{R}^{n+1}$. The hypersurface $\sigma\circ f$ is a Möbius homogeneous hypersurface in S^{n+1}. It is the orbit of the subgroup $L(\Pi)\subset M(S^{n+1})$ acting on the point $p=(1,0,...,0)\in S^{n+1}$.

Let $x\colon M^n\to S^{n+1}$, $n\geq 3$, be a Möbius homogeneous hypersurface with two distinct principal curvatures. We denote by b_1 and b_2 the Möbius principal curvatures, whose multiplicities are k and $n-k$, respectively. Using (6), we get

$$b_1=\frac{1}{n}\sqrt{\frac{(n-1)(n-k)}{k}}\quad\text{and}\quad b_2=-\frac{1}{n}\sqrt{\frac{(n-1)k}{n-k}}.$$

First we assume that the Möbius form $C=0$. Since the Möbius principal curvatures are constant, x is a Möbius isoparametric hypersurface. In [5], the authors classified Möbius isoparametric hypersurfaces with two distinct principal curvatures in S^{n+1}. Using [5], we have the following result.

Proposition 3.5. ([5]) *Let $x\colon M^n\to S^{n+1}$ be a hypersurface with two distinct principal curvatures. If the Möbius form $C=0$, then x is Möbius equivalent to an open part of one of the following Möbius isoparametric hypersurface in S^{n+1}:*

(1) *the standard torus $S^k(r)\times S^{n-k}\big(\sqrt{1-r^2}\big)$ in S^{n+1}, $1\leq k\leq n-1$;*
(2) *the image of σ of the standard cylinder $S^k(1)\times\mathbb{R}^{n-k}\subset\mathbb{R}^{n+1}$, $1\leq k\leq n-1$;*
(3) *the image of τ of $S^k(r)\times H^{n-k}\big(\sqrt{1+r^2}\big)$ in H^{n+1}, $1\leq k\leq n-1$.*

Remark 3.6. From Examples 3.1–3.3, we know that the hypersurfaces given in Proposition 3.5 are Möbius homogeneous hypersurfaces.

Next we assume the Möbius form $C \neq 0$.

Theorem 3.7. *Let $x: M^n \to S^{n+1}$ be a Möbius homogeneous hypersurface with two distinct principal curvatures. If the Möbius form $C \neq 0$, then x is Möbius equivalent to the image of σ of a logarithmic spiral cylinder. Moreover, the logarithmic spiral cylinder is of constant Möbius sectional curvature $K = -|C|^2$.*

Proof. We can choose a local orthonormal basis $\{E_1, ..., E_n\}$ with respect to the Möbius metric g of x such that

$$(B_{ij}) = \mathrm{diag}(b_1, ..., b_1, b_2, ..., b_2).$$

Claim. *One of the principal curvatures must be simple.*

Proof of Claim. We assume that the multiplicities of both of the principal curvatures are greater than one. Using

$$dB_{ij} + \sum_{k=1}^{n} B_{kj}\omega_{ki} + \sum_{k=1}^{n} B_{ik}\omega_{kj} = \sum_{k=1}^{n} B_{ij,k}\omega_k,$$

we obtain that

$$\begin{aligned} B_{ij,l} &= 0, \quad 1 \leq i, j \leq k \text{ and } 1 \leq l \leq n, \\ B_{\alpha\beta,l} &= 0, \quad k+1 \leq \alpha, \beta \leq n \text{ and } 1 \leq l \leq n. \end{aligned} \tag{8}$$

Since the multiplicities of both of the principal curvatures are greater than one, from (8) we have that

$$\begin{aligned} C_j &= B_{ii,j} - B_{ij,i} = 0, \qquad 1 \leq i, j \leq k \text{ and } i \neq j, \\ C_\alpha &= B_{\beta\beta,\alpha} - B_{\alpha\beta,\beta} = 0, \quad k+1 \leq \alpha, \beta \leq n \text{ and } \alpha \neq \beta. \end{aligned}$$

Thus the Möbius form $C = 0$, which is in contradiction with the assumption that $C \neq 0$. This proves the claim. □

Under the local orthonormal basis $\{E_1, ..., E_n\}$,

$$(B_{ij}) = \mathrm{diag}\left(\frac{n-1}{n}, -\frac{1}{n}, ..., -\frac{1}{n}\right). \tag{9}$$

In this section we make use of the following convention on the ranges of indices:

$$1 \leq i, j, k \leq n \quad \text{and} \quad 2 \leq \alpha, \beta, \gamma \leq n.$$

Since $B_{\alpha\beta}=(-1/n)\delta_{\alpha\beta}$, we can rechoose a local orthonormal basis $\{E_1,...,E_n\}$ with respect to the Möbius metric g such that

$$(B_{ij})=\operatorname{diag}\left(\frac{n-1}{n},-\frac{1}{n},...,-\frac{1}{n}\right)\quad\text{and}\quad (A_{ij})=\begin{pmatrix} A_{11} & A_{12} & A_{13} & ... & A_{1n}\\ A_{21} & a_2 & 0 & ... & 0\\ A_{31} & 0 & a_3 & ... & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ A_{n1} & 0 & 0 & ... & a_n\end{pmatrix}.$$

Let $\{\omega_1,...,\omega_n\}$ be the dual basis, and $\{\omega_{ij}\}$ be the connection forms. Using

$$dB_{ij}+\sum_{k=1}^{n}B_{kj}\omega_{ki}+\sum_{k=1}^{n}B_{ik}\omega_{kj}=\sum_{k=1}^{n}B_{ij,k}\omega_k$$

and (3), we get that

$$\begin{aligned} &B_{1\alpha,\alpha}=-C_1;\quad \text{and}\quad B_{ij,k}=0,\ \text{otherwise;}\\ &\omega_{1\alpha}=-C_1\omega_\alpha\quad\text{and}\quad C_\alpha=0.\end{aligned}\tag{10}$$

Since the vector field E_1 is an eigenvector of the Möbius second fundamental form B, we have

$$C_1=\mathit{constant}\neq 0\quad\text{and}\quad A_{11}=\mathit{constant}.\tag{11}$$

Using $\sum_{j=1}^{n}C_{i,j}\omega_j=dC_i+\sum_{j=1}^{n}C_j\omega_{ji}$ and (10), we get that

$$C_{i,j}=0,\quad i\neq j.\tag{12}$$

Combining (2) and (12) we obtain that

$$A_{1\alpha}=0.\tag{13}$$

Using (10),

$$d\omega_{1\alpha}=-dC_1\wedge\omega_\alpha-C_1\,d\omega_\alpha=-dC_1\wedge\omega_\alpha-C_1^2\omega_1\wedge\omega_\alpha-C_1\sum_{\gamma=1}^{n}\omega_\gamma\wedge\omega_{\gamma\alpha},$$

and $d\omega_{1\alpha}-\sum_{j=1}^{n}\omega_{1j}\wedge\omega_{j\alpha}=-\frac{1}{2}\sum_{k=1}^{n}\sum_{l=1}^{n}R_{1\alpha kl}\omega_k\wedge\omega_l$, we get that

$$R_{1\alpha1\alpha}=-C_1^2.\tag{14}$$

Since $R_{1\alpha1\alpha}=-(n-1)/n^2+a_1+a_\alpha=-C_1^2$, we thus have

$$a_2=a_3=...=a_n=\mathit{constant}.\tag{15}$$

Using

$$dB_{ij}+\sum_{k=1}^{n}B_{kj}\omega_{ki}+\sum_{k=1}^{n}B_{ik}\omega_{kj}=\sum_{k=1}^{n}B_{ij,k}\omega_k$$

and (1), we get that

$$A_{1\alpha,\alpha}=-\frac{1}{n}C_1 \quad \text{and} \quad (a_1-a_2)\omega_{1\alpha}=A_{1\alpha,\alpha}\omega_\alpha. \tag{16}$$

From (16), we know that $a_1\neq a_2$ and

$$\omega_{1\alpha}=\frac{A_{1\alpha,\alpha}}{a_1-a_2}\omega_\alpha=\frac{C_1}{n(a_2-a_1)}\omega_\alpha. \tag{17}$$

Combining (17) and (10), we have

$$a_2=a_1-\frac{1}{n}. \tag{18}$$

Using (4) and (18), we get that

$$R_{\alpha\beta\alpha\beta}=-C_1^2, \quad \alpha\neq\beta. \tag{19}$$

Since $A_{ij}=\text{diag}(a_1,a_2,...,a_2)$, from (3), (14) and (19), we know that (M^n,g) is of constant Möbius sectional curvature $K=-C_1^2=-|C|^2$. We define

$$F=-\frac{1}{n}Y+\xi, \quad X_1=-C_1Y-Y_1 \quad \text{and} \quad P=-a_2Y+N+C_1X_1+\frac{1}{n}F.$$

Clearly F is the curvature sphere of the Möbius principal curvature $b_2=-1/n$ of multiplicity $n-1$. Then

$$\langle F,X_1\rangle=0,\ \langle F,P\rangle=0,\ \langle X_1,P\rangle=0,\ \langle F,F\rangle=\langle X_1,X_1\rangle=1 \text{ and } \langle P,P\rangle=0. \tag{20}$$

From the structure equations of x we derive that

$$\begin{array}{rclrcl} E_1(F) &=& X_1, & E_\alpha(F) &=& 0,\\ E_1(X_1) &=& P-F, & E_\alpha(X_1) &=& 0,\\ E_1(P) &=& C_1P, & E_\alpha(P) &=& 0. \end{array} \tag{21}$$

Thus the subspace $V=\text{span}\{F,X_1,P\}$ is fixed along M^n, and P determines a fixed direction. Hence up to a Möbius transformation we can write

$$\begin{aligned} &P=\nu(1,-1,0,...,0), \quad \nu\in C^\infty(U),\\ &V=\text{span}\{F,X_1,P\}\\ &\quad=\text{span}\{(1,-1,0,...,0),(0,0,1,0,...,0),(0,0,0,1,0,...,0)\}\subset\mathbb{R}_1^{n+3}. \end{aligned}$$

Assume that $f = \sigma^{-1} \circ x \colon M^n \to \mathbb{R}^{n+1}$ has principal curvatures $k_1, k_2, ..., k_2$. Since

$$\langle P, F \rangle = \langle (1, -1, 0, ..., 0), F \rangle = 0 \quad \text{and} \quad \langle X_1, P \rangle = 0,$$

from (7) we get that

$$k_2 = 0 \quad \text{and} \quad C_1 \rho + E_1(\rho) = 0, \text{ i.e., } E_1(\log \rho) = -C_1. \tag{22}$$

From the definitions of F, X_1 and P, we get that $Y_\alpha \perp V$. Thus $\langle P, Y_\alpha \rangle = 0$. Therefore

$$E_\alpha(\rho) = 0, \quad \text{i.e.,} \quad E_\alpha(\log \rho) = 0. \tag{23}$$

Let $\{e_i = \rho E_i \,|\, 1 \le i \le n\}$. Then $\{e_1, ..., e_n\}$ is an orthonormal basis of TM^n with respect to the first fundamental form $I = df \cdot df$. Let $\{\widetilde{\omega}_1, ..., \widetilde{\omega}_n\}$ be its dual basis and $\{\widetilde{\omega}_{ij}\}$ be the corresponding connection forms. Since $g = \rho^2 I$, it is well known that

$$\widetilde{\omega}_{ij} = \omega_{ij} + e_i(\log \rho)\omega_j - e_j(\log \rho)\omega_i.$$

Thus from (10) and (23) we get $\widetilde{\omega}_{1\alpha} = 0$. Therefore $f = \sigma^{-1} \circ x \colon M^n \to \mathbb{R}^{n+1}$ is Möbius equivalent to a hypersurface given by

$$f(s, \mathrm{id}) = (\gamma(s), \mathrm{id}) \colon I \times \mathbb{R}^{n-1} \longrightarrow \mathbb{R}^{n+1},$$

where $\mathrm{id} \colon \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ is the identity mapping and $\gamma(s) \subset \mathbb{R}^2$ is a regular curve. Let I and II denote, respectively, the first fundamental form and the second fundamental form of the hypersurface f. Then

$$I = ds^2 + I_{\mathbb{R}^{n-1}} \quad \text{and} \quad II = k\, ds^2,$$

where $k(s)$ is the geodesic curvature of γ, and $I_{\mathbb{R}^{n-1}}$ is the standard Euclidean metric of $\mathbb{R}^{n-1}$. So we have $(h_{ij}) = \mathrm{diag}(k, 0, ..., 0)$, $H = k/n$ and $\rho = k$. Thus the Möbius metric g of the hypersurface f is

$$g = \rho^2 I = k^2 (ds^2 + I_{\mathbb{R}^{n-1}}).$$

The coefficients of the Möbius form of f with respect to an orthonormal frame $\{E_1, ..., E_n\}$ can be obtained as follows using (7):

$$C_1 = -\frac{k_s}{k^2} \quad \text{and} \quad C_2 = ... = C_n = 0.$$

Since C_1 is constant, $k = 1/C_{1s}$ and the regular curve $\gamma(s) = (\sin s e^{C_1 s}, \cos s e^{C_1 s})$ is a logarithmic spiral. Thus we finish the proof of Theorems 1.2 and 3.7. □

Using Proposition 3.5 and Theorem 1.2 we finish the proof of Theorem 1.3.

Acknowledgments. Li was partially supported by the grant No. 10801006 of NSFC, while Ma and Wang were partially supported by the grant No. 10771005 of NSFC.

References

1. Akivis, M. V. and Goldberg, V. V., A conformal differential invariant and the conformal rigidity of hypersurfaces, *Proc. Amer. Math. Soc.* **125** (1997), 2415–2424.
2. Cartan, É., Sur des familles remarquables d'hypersurfaces isoparamétriques dans les espaces sphériques, *Math. Z.* **45** (1939), 335–367.
3. Guo, Z., Li, H. and Wang, C., The Möbius characterizations of Willmore tori and Veronese submanifolds in the unit sphere, *Pacific J. Math.* **241** (2009), 227–242.
4. Hu, Z. J. and Li, D., Möbius isoparametric hypersurfaces with three distinct principal curvatures, *Pacific J. Math.* **232** (2007), 289–311.
5. Li, H., Liu, H., Wang, C. and Zhao, G., Moebius isoparametric hypersurfaces in S^{n+1} with two distinct principal curvatures, *Acta Math. Sin.* (*Engl. Ser.*) **18** (2002), 437–446.
6. Li, X. and Zhang, F., On the Blaschke isoparametric hypersurfaces in the unit sphere, *Acta Math. Sin.* (*Engl. Ser.*) **25** (2009), 657–678.
7. Liu, H., Wang, C. and Zhao, G., Möbius isotropic submanifolds in S^n, *Tohoku Math. J.* **53** (2001), 553–569.
8. O'Neil, B., *Semi-Riemannian Geometry*, Academic Press, New York, 1983.
9. Sulanke, R., Möbius geometry V: Homogeneous surfaces in the Möbius space S^3, in *Topics in Differential Geometry* (*Debrecen, 1984*), vol. **2**, pp. 1141–1154, Colloq. Math. Soc. János Bolyai **46**, North-Holland, Amsterdam, 1988.
10. Wang, C., Möbius geometry for hypersurfaces in S^4, *Nagoya Math. J.* **139** (1995), 1–20.
11. Wang, C., Möbius geometry of submanifolds in S^n, *Manuscripta Math.* **96** (1998), 517–534.

Tongzhu Li
Department of Mathematics
Beijing Institute of Technology
Beijing 100081
China
litz@bit.edu.cn

Changping Wang
LMAM, School of Mathematical Sciences
Peking University
Beijing 100871
China
cpwang@math.pku.edu.cn

Xiang Ma
LMAM, School of Mathematical Sciences
Peking University
Beijing 100871
China
maxiang@math.pku.edu.cn

Received April 24, 2011
published online January 4, 2012