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An extension property of the Bourgain–Pisier construction

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Abstract. It is proved that the natural embedding of a separable Banach space X into the corresponding Bourgain–Pisier space extends $\mathcal{L}_\infty$-valued operators.

1. Introduction

In the Banach space setting, extension of operators trace back to the famous Hahn–Banach extension theorem. After that, Lindenstrauss and Pełczyński [8] proved that $C(K)$-valued operators from a subspace of c_0 can be extended to the whole c_0. Since then, the topic—extension of $C(K)$-valued operators—has been studied in several papers by Johnson and Zippin; see [4], [9], [10], [11] and [12]. Very recently, Kalton in [6] and [7] gave several new important examples of extension. There is only one result, due to Johnson and Zippin [5, Corollary 3.1], about extension into arbitrary $\mathcal{L}_\infty$-spaces. This is, replacing $C(K)$-spaces by a larger class, the $\mathcal{L}_\infty$-spaces. We provide a new non-trivial example: The embedding of a separable Banach space X into a certain Bourgain–Pisier space constructed ad hoc has the property to extend $\mathcal{L}_\infty$-valued operators. Recall that in the paper [1], Bourgain and Pisier showed that for every separable Banach space X and $\lambda>1$, X can be embedded into some $\mathcal{L}_{\infty,\lambda}$-space, we denote for the rest $\mathcal{L}_{\infty,\lambda}(X)$ in such a way that the corresponding quotient space $\mathcal{L}_{\infty,\lambda}(X)/X$ has the Schur and Radon–Nikodym properties. This class of spaces has been the starting point for many counterexamples. But, why does the Bourgain–Pisier construction fit with the extension of $\mathcal{L}_{\infty,\mu}$-valued operators? There are two reasons behind this. The first one is that $\mathcal{L}_{\infty,\lambda}(X)$-spaces are made by amalgamating universal pieces. These pieces receive the name of push-out spaces (see the definition below) and behave quite

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well with respect to the extension of operators. This fact was observed in [3]. The second reason is connected with a parameter in the construction. People familiar with the Bourgain–Pisier construction know that there is some freedom in choosing some of the parameters involved. One of them is the η appearing in the so-called η-admissible embeddings, see [1, p. 114]. This parameter η is connected not only with Schur and Radon–Nikodym properties but with extension of operators too. The role of η could be described as follows: For a given operator $T\colon X\to\mathcal{L}_{\infty,\mu}$ of finite rank, η controls a certain contraction of T in a such way that an extension is not spoiled in norm by μ. There must necessarily be a balance between μ and η.

We present two special ways to assemble a Bourgain–Pisier space. The first one replaces η by a suitable sequence $\{\eta_n\}_{n=1}^{\infty}$. This constructions produces, for a fixed $\lambda>1$, a Bourgain–Pisier space $\mathcal{L}_{\infty,\lambda}(X)$ which is universal with respect to the extension of $\mathcal{L}_{\infty,\lambda}$-valued operators. In this construction the norm of the operator is not preserved. The second assembly provides isometric extension but fails to be universal in the sense mentioned above. In this second case, only one specific choice of η is required and not a whole sequence.

To make the paper self-contained we give some basic definitions: The push-out space and its universal property, see [2, p. 6]: Let $T_1\colon V\to X$ and $T_2\colon V\to Y$ be two operators and Δ be the closure of the set $D=\{(T_1v,-T_2v):v\in V\}\subset X\oplus Y$. The *push-out space* or simply push-out of (T_1,T_2) is the quotient space $PO=(X\oplus_1 Y)/\Delta$ endowed with its natural quotient norm. The push-out has the following *universal property*: Given two operators $\alpha\colon Y\to E$ and $\beta\colon X\to E$ such that $\alpha T_2=\beta T_1$ there exists a unique operator $\gamma\colon PO\to E$ such that $\gamma j_Y=\alpha$ and $\gamma j_X=\beta$ (by j_Y and j_X we mean the natural maps $Y\to PO$ and $X\to PO$, respectively). Moreover,

$$\|\gamma\|\le\max\{\|\alpha\|,\|\beta\|\}.$$

This last estimate will control the norm of the extension. To finish let us also recall the definition of an $\mathcal{L}_{\infty,\lambda}$-space. We say that Z is an $\mathcal{L}_{\infty,\lambda}$-space if and only if the following holds: For every finite-dimensional subspace F of Z one may find a further finite-dimensional subspace G of Z such that $F\subseteq G$ and $d(G,\ell_\infty^{\dim G})\le\lambda$.

2. The extension property

The following is our first result.

Theorem 2.1. *Let X be a separable Banach space and fix $\lambda>1$ and $\varepsilon>0$. Then one may construct a Bourgain–Pisier space associated with X, namely $\mathcal{L}_{\infty,\lambda}(X)$, with the following extension property: every operator $T\colon X\to\mathcal{L}_{\infty,\lambda}$ extends to an operator $\widehat{T}\colon\mathcal{L}_{\infty,\lambda}(X)\to\mathcal{L}_{\infty,\lambda}$ with $\|\widehat{T}\|\le(1+\varepsilon)\|T\|$.*

Proof. For fixed $\lambda>1$ and $\varepsilon>0$, consider a sequence $\{\varepsilon_n\}_{n=1}^{\infty}$ satisfying:
(1) $0<\varepsilon_{n+1}<\varepsilon_n$ for all $n\in\mathbb{N}$;
(2) $1+\varepsilon_n<\lambda$ for all $n\in\mathbb{N}$;
(3) $\prod_{n=1}^{\infty}(1+\varepsilon_n)\le 1+\varepsilon$.
Once the sequence $\{\varepsilon_n\}_{n=1}^{\infty}$ has been fixed, define for every $n\in\mathbb{N}$,

$$\eta_n := \frac{1+\varepsilon_n}{\lambda}.$$

Claim. *The expression $\lambda^{-1}<\eta_n<1$ holds for every $n\in\mathbb{N}$.*

The proof of the claim is trivial by using properties (1) and (2) of the selected sequence $\{\varepsilon_n\}_{n=1}^{\infty}$.

We proceed now as in the Bourgain–Pisier construction. Write X as the closure of an increasing sequence of finite-dimensional X_n, and let $i_n\colon X_n\to X$ be the inclusion.

1*st piece of* $\mathcal{L}_{\infty,\lambda}(X)$. For $1+\varepsilon_1=\lambda\eta_1$, let $s_1\colon S_1\to\ell_\infty^{a(1)}$ be a subspace such that there is a $(1+\varepsilon_1)$-isomorphism $u_1\colon S_1\to X_1$ with $\|u_1\|\le\eta_1$ and $\|u_1^{-1}\|\le\lambda$. Form the push-out of s_1 and i_1u_1 to obtain a Banach space E_1, an isometric embedding $j_1\colon X\to E_1$ and an embedding $\tilde{u}_1\colon \ell_\infty^{a(1)}\to E_1$ making a commutative square, namely, $j_1i_1u_1=\tilde{u}_1s_1$:

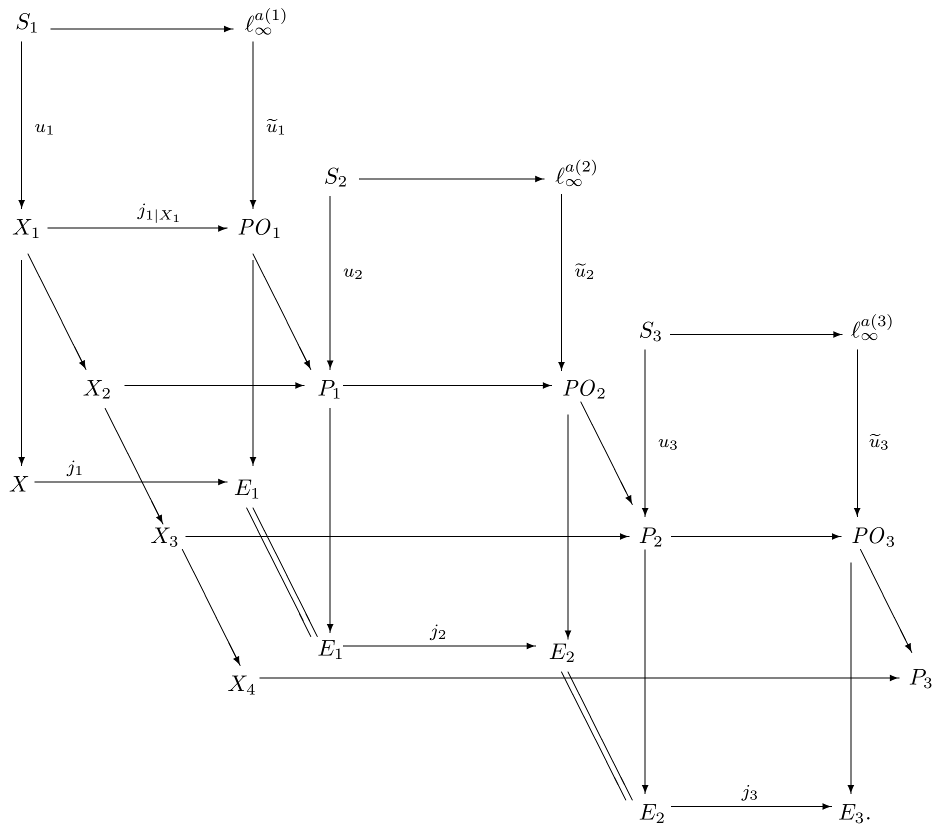
$$\begin{array}{ccc} S_1 & \xrightarrow{\ s_1\ } & \ell_\infty^{a(1)} \\ \big\downarrow{\scriptstyle u_1} & & \big\downarrow{\scriptstyle \tilde{u}_1} \\ X & \xrightarrow[\ j_1\ ]{} & E_1. \end{array}$$

We let PO_1 be the subspace of E_1 that is the push-out of s_1 and u_1, i.e.

$$\begin{array}{ccc} S_1 & \xrightarrow{\ s_1\ } & \ell_\infty^{a(1)} \\ \big\downarrow{\scriptstyle u_1} & & \big\downarrow{\scriptstyle \tilde{u}_1} \\ X_1 & \xrightarrow[\ j_1\ ]{} & PO_1. \end{array}$$

In this case, PO_1 is λ-isomorphic to $\ell_\infty^{a(1)}$ and we have our first piece.

2*nd piece of* $\mathcal{L}_{\infty,\lambda}(X)$. Next we form the push-out of the restriction of j_1 to X_1 and the inclusion $X_1\to X_2$. We call this new push-out space P_1 (endowed with the norm of E_1). For the next step, take $s_2\colon S_2\to\ell_\infty^{a(2)}$ to be a subspace such that there is a $(1+\varepsilon_2)$-isomorphism $u_2\colon S_2\to P_1$ with $\|u_2\|\le\eta_2$ and $\|u_2^{-1}\|\le\lambda$. Form the push-out of s_2 and the composition $S_2\to P_1\to E_1$. This yields a Banach space E_2 with an isometric embedding $j_2\colon E_1\to E_2$ and an embedding $\tilde{u}_2\colon \ell_\infty^{a(2)}\to E_2$ making a commutative square. We let PO_2 be the push-out of s_2 and u_2, a subspace of E_2 λ-isomorphic to $\ell_\infty^{a(2)}$ and we get our second piece. Form then the push-out of the restriction $j_2\colon X_2\to PO_2$ and the embedding $X_2\to X_3$. This new push-out space is P_2 (endowed with the norm of E_2), and the process can continue. For the nth step take $s_n\colon S_n\to\ell_\infty^{a(n)}$ to be a subspace such that there is a $(1+\varepsilon_n)$-isomorphism $u_n\colon S_n\to P_{n-1}$ with $\|u_n\|\le\eta_n$ and $\|u_n^{-1}\|\le\lambda$. The following diagram might be useful to understand the first steps in the construction,



The resulting $\mathcal{L}_{\infty,\lambda}(X)$ superspace is the inductive limit

$$PO_1 \longrightarrow PO_2 \longrightarrow PO_3 \longrightarrow ...,$$

while the embedding $j\colon X\to\mathcal{L}_{\infty,\lambda}(X)$ is given by $j(x)=j_n\dots j_1(x)$ when $x\in X_n$. A couple of remarks here are necessary. First, the resulting space is indeed an $\mathcal{L}_{\infty,\lambda}$-space: the condition $\|u_n^{-1}\|\le\lambda$ ensures that $d(\ell_\infty^{\dim PO_n}, PO_n)\le\lambda$ ([2, Lemma 1.3.b]). Secondly, the corresponding quotient space $\mathcal{L}_{\infty,\lambda}(X)/X$ has Schur and Radon–Nikodym properties. This follows from the following trivial observation:

By the very definition the morphism $j_1\colon X\to E_1$ is η_1-admissible and $j_n\colon E_{n-1}\to E_n$ is η_n-admissible for each $n>1$. In particular, since $\eta_n<\eta_1<1$ (for every $n>1$), all the morphisms j_n involved are η_1-admissible and we are in position to apply [1, Theorem 1.6] which allows us to conclude the argument exactly as in [1, Theorem 2.1].

We proceed to prove the extension property. Pick an operator $T\colon X\to\mathcal{L}_{\infty,\lambda}$. There is no loss of generality to assume it has norm 1. Denote by T_n the restriction of T to X_n. Take T_1 and consider $T_1u_1\colon S_1\to\mathcal{L}_{\infty,\lambda}$. This last composition clearly admits an extension $T^1\colon \ell_\infty^{a(1)}\to\mathcal{L}_{\infty,\lambda}$ with

$$\|T^1\|\le\lambda\|T_1u_1\|\le\lambda\eta_1=1+\varepsilon_1.$$

The universal property of the push-out for the couple (T_1,T^1) gives us a commuting operator $T_p^1\colon PO_1\to\mathcal{L}_{\infty,\lambda}$ with norm

$$\|T_p^1\|\le\max\{\|T_1\|,\|T^1\|\}\le 1+\varepsilon_1.$$

We may again apply the universal property of the push-out to the couple (T_2,T_p^1). We find properly an operator $\widehat{T}_1\colon P_1\to\mathcal{L}_{\infty,\lambda}$ with norm $1+\varepsilon_1$. For the next step, consider $\widehat{T}_1u_2\colon S_2\to\mathcal{L}_{\infty,\lambda}$ and extend to an operator T^2 on $\ell_\infty^{a(2)}$ with

$$\|T^2\|\le\lambda\|\widehat{T}_1u_2\|\le\lambda(1+\varepsilon_1)\eta_2=(1+\varepsilon_1)(1+\varepsilon_2).$$

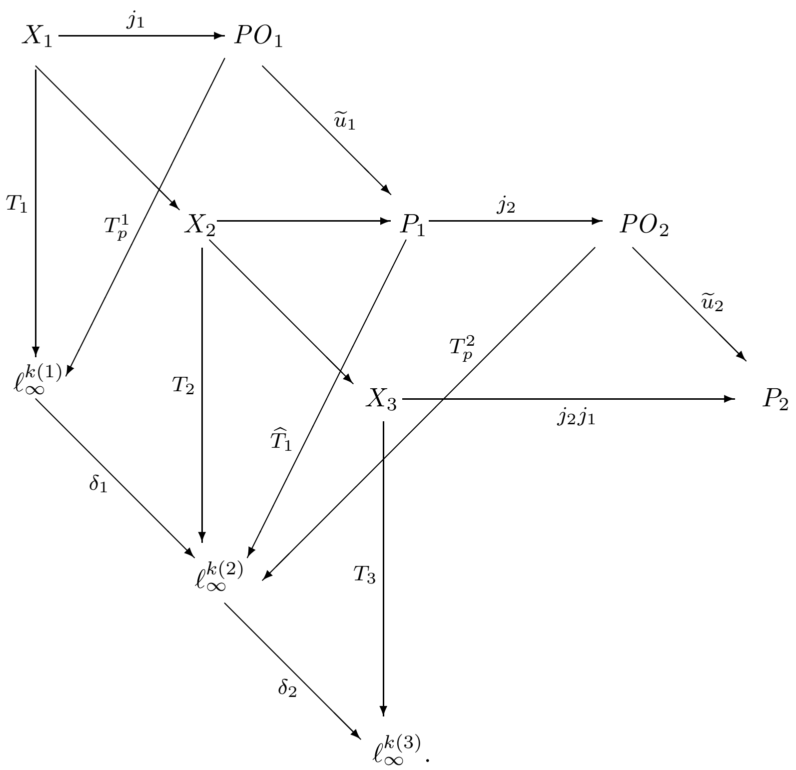
The push-out universal property provides us with an appropriate operator $T_p^2\colon PO_2\to\mathcal{L}_{\infty,\lambda}$. And once more, the universal property gives us a commuting $\widehat{T}_2\colon P_2\to\mathcal{L}_{\infty,\lambda}$ with norm bounded by $(1+\varepsilon_1)(1+\varepsilon_2)$. In this way we may produce commuting operators $\widehat{T}_n\colon P_n\to\mathcal{L}_{\infty,\lambda}$ with

$$\|\widehat{T}_n\|\le\prod_{k=1}^{n}(1+\varepsilon_k).$$

Therefore, the extension $\widehat{T}$ of T is given locally by

$$\widehat{T}(x)=\widehat{T}_n(x)\quad\text{if } x\in P_n.$$

Needless to say, one has $\|\widehat{T}\|\le 1+\varepsilon$. A diagram of the process can be useful:



And clearly the proof is complete. □

Concerning isometric extension, we have a second variant.

Theorem 2.2. *Let X be a separable Banach space and fix $1 \le \mu < \infty$. Then for every $\lambda > \mu$ one may construct a Bourgain–Pisier space associated with X, namely $\mathcal{L}_{\infty,\lambda}(X)$, with the following extension property: every operator $T\colon X \to \mathcal{L}_{\infty,\mu}$ extends to an operator $\widehat{T}\colon \mathcal{L}_{\infty,\lambda}(X) \to \mathcal{L}_{\infty,\mu}$ with $\|\widehat{T}\| = \|T\|$.*

Proof. Let us observe that for $\mu < \lambda$ one may easily find δ such that $\mu/\lambda < \delta < 1$. Thus take $\eta := \delta/\mu$. It is trivial to check that this choice of η still satisfies $\lambda^{-1} < \eta < 1$. We consider the resulting Bourgain–Pisier space for this specific η. The extension procedure works as in the previous proof and our choice of η will make the trick. □

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