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An improved Riemann mapping theorem and complexity in potential theory

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Abstract. We discuss applications of an improvement on the Riemann mapping theorem which replaces the unit disc by another "double quadrature domain," i.e., a domain that is a quadrature domain with respect to both area and boundary arc length measure. Unlike the classic Riemann mapping theorem, the improved theorem allows the original domain to be finitely connected, and if the original domain has nice boundary, the biholomorphic map can be taken to be close to the identity, and consequently, the double quadrature domain is close to the original domain. We explore some of the parallels between this new theorem and the classic theorem, and some of the similarities between the unit disc and the double quadrature domains that arise here. The new results shed light on the complexity of many of the objects of potential theory in multiply connected domains.

1. Introduction

The unit disc is the most famous example of a "double quadrature domain." The averages of an analytic function on the disc with respect to both area measure and with respect to boundary arc length measure yield the value of the function at the origin when these averages make sense. The Riemann mapping theorem states that when Ω is a simply connected domain in the plane that is not equal to the whole complex plane, a biholomorphic map of Ω to this famous double quadrature domain exists.

We proved a variant of the Riemann mapping theorem in [16] that allows the domain $\Omega\neq\mathbb{C}$ to be simply or finitely connected. The theorem states that there is a biholomorphic mapping of such regions onto a one-point double quadrature domain, i.e., a bounded domain such that the average of a holomorphic function with respect to area measure is a fixed finite linear combination of the function

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and its derivatives evaluated at a single point, and such that the same is true of the average with respect to arc length measure (with different constants, of course) when these averages make sense. If the domain is a bounded domain bounded by finitely many C^∞ smooth non-intersecting curves, the double quadrature domain can be taken to be as C^∞ close to the original one as desired, and the biholomorphic map C^∞ close to the identity. (For definitions and a precise statement of the result, see [16].) If the boundary curves are Jordan curves, then a standard argument in conformal mapping theory shows that we may conformally map the domain to a nearby domain bounded by real-analytic curves via a conformal map that is close to the identity in the sup norm on the closure of the domain. If we now apply the C^∞ theorem to the domain with real-analytic boundary and compose, we find that the double quadrature domain can be taken to be close to the original domain, and the biholomorphic map close to the identity in the sup norm on the closure of the domain.

Double quadrature domains satisfy a long list of desirable properties, many of which are spelled out in [16]. The purpose of this paper is to add to that list and derive some consequences from it. We will show that double quadrature domains have many properties in common with the unit disc that allow the objects of potential theory and complex analysis to be expressed in rather simple terms. In particular, the action of many classical operators on rational functions will be seen to be particularly simple.

In Section 3, we show that the solution to the Dirichlet problem with rational data on an area quadrature domain is real algebraic modulo an explicit finite-dimensional subspace. Since derivatives of Green's function with respect to the second variable are solutions to a Dirichlet problem with rational data, this yields information about the complexity of Green's function and the closely related Poisson kernel. The results yield a method to solve the Dirichlet problem using only algebra and finite mathematics in place of the usual analytical methods.

In Section 4, we explain how biholomorphic and proper holomorphic mappings between double quadrature domains are particularly simple, and in Section 5, we show how to pull back the results of Section 3 to more general classes of domains that arise naturally. We define two classes, which we name *algebraic* and *Briot–Bouquet*, that are interesting from this point of view.

In Section 6, we show that many of the classical operators attached to a double quadrature domain map rational functions to rational or algebraic functions. The Szegő projection, for example, maps rational functions to rational functions as an operator from L^2 of the boundary to itself, and maps rational functions to algebraic functions as an operator from L^2 of the boundary to the space of holomorphic functions on the domain. The Kerzman–Stein operator maps rational functions to rational functions. Furthermore, the Dirichlet-to-Neumann map sends rational functions on the boundary to rational functions on the boundary. These results demonstrate that double quadrature domains are very much like the unit disc in this regard.

In the last Section 7, we explain an analogy between the techniques of this paper and classical Fourier analysis. The analogy reveals a way to view the main results of the paper on multiply connected domains as a way of doing Fourier analysis on multiple curves.

For the history of the study of quadrature domains and the many applications they have found, see the book [19] and the article [23] therein, and Shapiro's classic text [27]. Darren Crowdy [17] has shown that double quadrature domains arise in certain problems in fluid dynamics, so the present work could find applications in that area. Harold Shapiro has pioneered another approach to the subject matter of this paper via the Friedrichs operator (see [26] and Chapter 8 of [27]). Many of the results and techniques of this paper can no doubt be reworked from that perspective. We leave this interesting pursuit for the future.

2. Basic properties of quadrature domains

Suppose that Ω is a bounded finitely connected domain in the plane. Björn Gustafsson [21] proved that if Ω is an area quadrature domain, then Ω must have piecewise real-analytic boundary and the Schwarz function $S(z)$ associated to Ω extends meromorphically to the domain. (Aharonov and Shapiro [1] first proved extendability of the Schwarz function in the simply connected case.) Consequently, since $S(z)=\bar{z}$ on the boundary, z extends meromorphically to the double given values $\overline{S(z)}$ on the "backside," and $S(z)$ extends given values $\bar{z}$ on the backside. Gustafsson showed that the field of meromorphic functions on the double is generated by the extensions of z and $S(z)$, i.e., the extensions form a primitive pair. This implies that a meromorphic function on the domain that extends meromorphically to the double must be a rational combination of z and the Schwarz function. Since z and $S(z)$ both extend to the double, they are algebraically dependent. Therefore $S(z)$ is an algebraic function (as noted by Gustafsson, and by Aharonov and Shapiro in the simply connected case). If Ω has no cusps in the boundary, then Gustafsson [21] showed that the boundary is given by finitely many non-intersecting real-analytic real-algebraic curves of a special form.

If Ω is a boundary arc length quadrature domain, then Gustafsson [22] showed that the boundary is real-analytic and the complex unit tangent vector function $T(z)$ must extend as a meromorphic function on the double. (Shapiro and Ullemar [28] first showed this in the simply connected case.)

If Ω is a double quadrature domain, then all of the properties above hold and it follows that $T(z)$ must be a rational combination of z and $S(z)$, and consequently, since $S(z)=\bar{z}$ on the boundary, $T(z)$ is a rational function of z and $\bar{z}$. It is proved in [16] that the Bergman kernel $K(z,w)$ and the Szegő kernel $S(z,w)$ are both rational combinations of z, $S(z)$, $\bar{w}$, and $\overline{S(w)}$, and consequently are rational functions of z, $\bar{z}$, w, and $\bar{w}$ when restricted to the boundary cross the boundary. Furthermore, the complex polynomials belong to both the Bergman span and the Szegő span associated to the domain. The Kerzman–Stein kernel $A(z,w)$ is also a rational function of z, $\bar{z}$, w, and $\bar{w}$. (We will show in Section 6 that the Kerzman–Stein operator sends rational functions to rational functions on double quadrature domains. Many of the ideas used in the work leading up to the main results of this paper trace back to the papers of Kerzman and Stein [24] and Kerzman and Trummer [25].)

These are the basic properties of quadrature domains that we shall need in order to proceed.

3. The Dirichlet problem, Green's function, and Poisson kernel on quadrature domains

Peter Ebenfelt [18] proved that if Ω is a bounded simply connected area quadrature domain and $R(z,\bar{z})$ is a real-valued rational function of z and $\bar{z}$ with no singularities on $b\Omega\times b\Omega$, then the Poisson extension of $R(z,\bar{z})$ to Ω is the real part of a rational function of z and the Schwarz function $S(z)$ for Ω. Consequently, the Poisson extension is the real part of an algebraic function. Ebenfelt's proof is short and uses a very appealing reflection argument. In our desperation to generalize Ebenfelt's theorem to bounded multiply connected quadrature domains, we found two alternative proofs that we could generalize more readily than the reflection argument. Each proof gives new insight into the extension problem, and we will need both approaches later, so we now present both alternative proofs of Ebenfelt's theorem in the simply connected case as a way to launch into the generalizations. (Besides, you can never have too many proofs of a good theorem.)

Suppose Ω is a bounded simply connected area quadrature domain with no cusps in the boundary, and suppose $\psi(z)=R(z,\bar{z})$ is a real-valued rational function of z and $\bar{z}$ with no singularities on $b\Omega\times b\Omega$. We know that Ω has C^∞ smooth real-analytic boundary and that the Schwarz function $S(z)$ for Ω extends meromorphically to Ω. Furthermore, the field of meromorphic functions on the double of Ω is generated by z and $S(z)$. Since $S(z)=\bar{z}$ on the boundary, by writing $R(z,\bar{z})=R(z,S(z))$ for z in the boundary, we can see that the only possible type of singularity for a rational function of z and $\bar{z}$ on the boundary would have pole-like behavior at isolated points. Note that writing $R(z,\bar{z})=R(z,S(z))$

yields an extension of ψ from $b\Omega$ to Ω as a meromorphic function and that writing $R(z,\bar{z})=R(\overline{S(z)},\bar{z})$ yields an extension of ψ from $b\Omega$ to Ω as an antimeromorphic function. The Poisson extension of ψ to Ω is given as the real part of a holomorphic function h that extends C^∞-smoothly up to the boundary.

Note that a meromorphic function that extends smoothly to the boundary extends meromorphically to the double if and only if there is an anti-meromorphic function on the domain that extends smoothly to the boundary with the same boundary values.

Since $h+\bar{h}=2\psi$ on the boundary, we may write

$$h=-\bar{h}+2R(\overline{S(z)},\bar{z})$$

on the boundary to see that h extends meromorphically to the double. Consequently, h is a rational functions of z and $S(z)$. Since, as remarked above, $S(z)$ is algebraic, it follows that h is algebraic. Hence, the Poisson extension is real algebraic.

When we extend this argument to the multiply connected setting, we will need to use a generalization of the fact proved in [3] (see also [4], p. 35), that the Poisson extension of ψ to Ω, when Ω is C^∞ smooth and simply connected, is given by $h+\overline{H}$, where

$$h=\frac{P(S_a\psi)}{S_a} \quad \text{and} \quad H=\frac{P(L_a\bar{\psi})}{L_a}.$$

Here, P denotes the Szegő projection associated to Ω, $S_a(z)=S(z,a)$ is the Szegő kernel, $L_a(z)=L(z,a)$ is the Garabedian kernel, and a is a fixed point in Ω. For the basic properties of these objects, see [4], pp. 1–35. Note, in particular, that on a C^∞ smooth bounded finitely connected domain, $L(z,a)$ is C^∞ smooth on $\overline{\Omega}\times\overline{\Omega}$ minus the diagonal and $S(z,a)$ is C^∞ smooth on $\overline{\Omega}\times\overline{\Omega}$ minus the boundary diagonal. Furthermore, $L(z,a)$ has a simple pole in z at $z=a$ and is non-vanishing on $\overline{\Omega}\times\overline{\Omega}$ minus the diagonal. If Ω is simply connected, then $S(z,a)$ is non-vanishing on $\overline{\Omega}\times\overline{\Omega}$ minus the boundary diagonal. Since the Szegő projection maps $C^\infty(b\Omega)$ into itself (see [4], p. 13), it follows that h and H are holomorphic functions in $C^\infty(\overline{\Omega})$. In the simply connected case, it is easy to show that h and H extend meromorphically to the double as above, and that is a major part of what we will do in the multiply connected setting.

The second proof of Ebenfelt's theorem uses Green's function. Let $f\colon\Omega\to D_1(0)$ be a Riemann map associated to our simply connected bounded area quadrature domain. Aharonov and Shapiro [1] showed that f is algebraic. Green's function associated to Ω is given by

$$G(z,w)=-\log\left|\frac{f(z)-f(w)}{1-f(z)\,\overline{f(w)}}\right|.$$

Since

$$\frac{\partial}{\partial w}\log\left|\frac{z-w}{1-z\,\overline{w}}\right|=\frac{1-|z|^2}{2(w-z)(1-w\bar{z})},$$

it is easy to use the complex chain rule to verify that derivatives of the form

$$G^{(m)}(z,w):=\frac{\partial^m}{\partial w^m}G(z,w)$$

are rational combinations of $f(z)$ and $\overline{f(z)}$ in the z variable for fixed w. Note that $G^{(m)}(z,w)$ is harmonic in z, vanishes for z in the boundary, and has a singularity of the form $c/(z-w)^m$ in z with $c\neq 0$ at $z=w$. If $R(z,\bar{z})$ is rational in z and $\bar{z}$ without singularities on $b\Omega$, then $R(z,S(z))$ extends meromorphically to Ω and is pole free on the boundary. It is therefore possible to find constants c_{jk} and positive integers m_j associated to the poles w_j of the meromorphic function so that

$$R(z,S(z))-\sum_{j=1}^{N}\sum_{k=1}^{m_j}c_{jk}G^{(k)}(z,w_j)$$

has removable singularities at the poles w_j. This, therefore, is the Poisson extension of $R(z,\bar{z})$ to the interior. Since $S(z)$ and $f(z)$ are both algebraic and extend meromorphically to the double of Ω, Ebenfelt's result follows.

We will now generalize these arguments to the multiply connected setting. The ideas generalize nicely, but the end results are considerably more complicated to state. Although the next theorem seems long and complicated, the bottom line is that the solution to the Dirichlet problem with rational boundary data on a double quadrature domain is real algebraic modulo an explicit finite-dimensional subspace. (Note that double quadrature domains are bounded area quadrature domains without cusps on the boundary because bounded boundary arc length quadrature domains have real-analytic boundaries.)

Theorem 3.1. *Suppose that Ω is a bounded n-connected area quadrature domain with no cusps in the boundary. Suppose further that $\psi(z)=R(z,\bar{z})$ is a rational function of z and $\bar{z}$ with no singularities on $b\Omega\times b\Omega$. The solution to the Dirichlet problem with boundary data ψ is equal to*

$$h_0(z)+\overline{H_0(z)}+\sum_{j=1}^{n-1}c_j\big(h_j(z)+\overline{H_j(z)}+\log|z-b_j|\big),$$

where h_0 and H_0 are algebraic meromorphic functions on Ω that are rational combinations of z and the Schwarz function $S(z)$ for Ω. The c_j are complex constants, and the points b_j are fixed points in the complement of Ω, one in the interior of *each bounded component of* $\mathbb{C}\setminus\Omega$. *The functions* h_j *and* H_j, $j=1,...,n-1$, *are meromorphic functions on* Ω *whose derivatives are algebraic functions that extend meromorphically to the double of* Ω, *and hence* h_j' *and* H_j' *are rational combinations of* z *and* $S(z)$. *The functions* H_j, $j=0,1,...,n-1$, *are holomorphic on* Ω, *but the* h_j *may have poles at* $n-1$ *points in* Ω *specified in the proof below. The constants* c_j *are determined via the condition that the poles on* Ω *should cancel so that*

$$h_0+\sum_{j=1}^{n-1} c_j h_j$$

is holomorphic on Ω.

It should also be noted that all the functions h_j and H_j, $j=0,1,...,n-1$, extend C^∞ smoothly up to the boundary of Ω in Theorem 3.1. The functions h_0 and H_0 and the constants c_j depend on the boundary data, but the h_j and H_j for $j=1,2,...,n-1$ do not. Since z and $S(z)$ extend to form a primitive pair for the double, the functions that extend to the double in Theorem 3.1 can be expressed as

$$\sum_{k=0}^{N-1} P_k(z)S(z)^k,$$

where P_k are rational functions in z and N is the number of poles of $S(z)$ in Ω counted with multiplicities (see Farkas and Kra [20], p. 249). It is interesting to note that h_0 and H_0 are rational functions of z and $\bar{z}$, when restricted to the boundary, that extend algebraically to the interior of Ω.

Theorem 3.1 can be interpreted to mean that the solution to the Dirichlet problem with rational boundary data on a bounded area quadrature domain without cusps in the boundary is a real algebraic function $h_0+\overline{H}_0$ modulo an $(n-1)$-dimensional subspace spanned by the functions

$$h_j+\overline{H}_j+\log|z-b_j|,$$

which also have a rather simple structure. In particular, since the derivatives of h_j and H_j are rational functions of z and $S(z)$, where $S(z)$ is algebraic, these functions are abelian integrals in the classical sense. We will show that there are other interesting domain functions that can serve as the basis for the $(n-1)$-dimensional subspace in Theorem 3.1 as we proceed.

The starting point for proving Theorem 3.1 is the following general theorem. Note that, on an area quadrature domain, rational functions of z and $\bar{z}$ without singularities on the boundary are precisely the functions on the boundary that are restrictions to the boundary of meromorphic functions on the double without poles on the boundary. Indeed, as we have seen, since $S(z)=\bar{z}$ on the boundary, we have $R(z,\bar{z})=R(z,S(z))$ on the boundary, which extends meromorphically to the double if R is rational, and which cannot have any poles on the boundary curves if $R(z,\bar{z})$ has no singularities. Conversely, any meromorphic function on the double can be expressed as a rational combination of z and $S(z)$, and hence, when restricted to the boundary, as a rational function of z and $\bar{z}$.

Theorem 3.2. *Suppose that Ω is a bounded n-connected domain bounded by n non-intersecting C^∞ smooth curves, and suppose ψ is a C^∞ function on the boundary of Ω that is the restriction to the boundary of a meromorphic function on the double of Ω. Then the solution to the Dirichlet problem on Ω with boundary data ψ is given by*

$$h_0(z)+\overline{H_0(z)}+\sum_{j=1}^{n-1} c_j(h_j(z)+\overline{H_j(z)}+\log|z-b_j|),$$

where h_0 and H_0 are meromorphic functions on Ω that extend meromorphically to the double of Ω. The functions h_j, $j=1,...,n-1$, are meromorphic functions on Ω such that

$$\left(h_j'-\frac{1}{2}\frac{1}{z-b_j}\right)dz$$

extends to the double as a meromorphic 1-form. The functions H_j, $j=1,...,n-1$, are holomorphic functions on Ω such that

$$\left(H_j'-\frac{1}{2}\frac{1}{z-b_j}\right)dz$$

extends to the double as a meromorphic 1-form. The H_j, $j=0,1,...,n-1$, are holomorphic on Ω, but the h_j may have poles at $n-1$ points in Ω specified in the proof below. The c_j are complex constants, and the points b_j are fixed points in the complement of Ω, one in the interior of each bounded component of $\mathbb{C}\setminus\Omega$. The constants c_j are determined via the condition that the poles on Ω should cancel so that

$$h_0+\sum_{j=1}^{n-1} c_j h_j$$

is holomorphic on Ω.

We will explain how Theorem 3.1 follows from Theorem 3.2 after we prove Theorem 3.2.

Proof. To prove Theorem 3.2, let a be a point in Ω such that the $n-1$ zeroes of $S(z,a)$ are distinct and simple. (That such an a exists is proved in [3], or see [4], p. 106.) Let $b_1, b_2, ..., b_{n-1}$ be points, one from the interior of each of the bounded components of $\mathbb{C}\backslash\Omega$. We will solve the Dirichlet problem with boundary data ψ by the method described on p. 1367 of [2] (or better, see pp. 53–55 of [4]). There, it is shown that there are constants $c_1, c_2, ..., c_{n-1}$ such that the Poisson extension of ψ is given by

$$h(z)+\overline{H(z)}+\sum_{j=1}^{n-1} c_j \log|z-b_j|,$$

where, writing $\phi=\psi-\sum_{j=1}^{n-1} c_j \log|z-b_j|$, we have

$$h=\frac{P(S_a\phi)}{S_a} \quad \text{and} \quad H=\frac{P(L_a\bar{\phi})}{L_a}.$$

Here, both h and H are holomorphic on Ω. (The function h looks like it might possibly have simple poles at the $n-1$ simple zeroes of S_a, but the method ensures that the numerator vanishes at the zeroes of $S_a(z)$.) Both h and H extend C^∞ smoothly to the boundary. Let $\mathcal{L}_j(z)=\log|z-b_j|$. We now consider h as a linear combination of

$$h_0=\frac{P(S_a\psi)}{S_a}$$

and the functions

$$h_j=-\frac{P(S_a\mathcal{L}_j)}{S_a}.$$

We next apply Theorem 6.1 of [2] (see also Theorem 14.1 of [4] on p. 53) which states that a C^∞ smooth function u on the boundary can be expressed as $f+\overline{F}$ on the boundary, where

$$f=\frac{P(S_a u)}{S_a}$$

is a meromorphic function that extends C^∞-smoothly to the boundary and

$$F=\frac{P(L_a\bar{u})}{L_a}$$

is a holomorphic function that extends C^∞-smoothly to the boundary. Hence,

$$h_0=\frac{P(S_a\psi)}{S_a}=\psi-\frac{\overline{P(L_a\bar{\psi})}}{\bar{L}_a}.$$

Since ψ extends to the double as a meromorphic function without poles on the boundary curves of Ω, it follows that $\psi(z)=\overline{G(z)}$ on the boundary, where G is a meromorphic function on Ω that extends smoothly to the boundary. Hence, it follows that h_0 has the same boundary values as a function that extends to Ω as an antimeromorphic function. Thus h_0 extends to the double.

We will now show that the functions h_j have special extension properties, too. We may reason as above to see that

$$h_j = -\frac{P(S_a \mathcal{L}_j)}{S_a} = -\mathcal{L}_j + \frac{\overline{P(L_a \bar{\mathcal{L}}_j)}}{\bar{L}_a}.$$

Hence,

$$h_j(z) = -\log|z - b_j| + \overline{g(z)}$$

on the boundary, where g is a holomorphic function on Ω that extends smoothly to the boundary. Notice that if u is a function defined on a neighborhood of the boundary of Ω and $z(t)$ parametrizes the boundary in the standard sense, then

$$\frac{d}{dt} u(z(t)) = \frac{\partial u}{\partial z} z'(t) + \frac{\partial u}{\partial \bar{z}} \overline{z'(t)}.$$

Apply this idea to both sides of the last formula for h_j, divide by $|z'(t)|$, and note that $z'(t)/|z'(t)|$ is equal to the complex unit tangent vector function $T(z)$ at $z = z(t)$ to obtain

$$h_j'(z)T(z) = -\frac{1}{2}\frac{T(z)}{z - b_j} - \frac{1}{2}\frac{\overline{T(z)}}{\bar{z} - \bar{b}_j} + \overline{g'(z)}\,\overline{T(z)}$$

for z in the boundary. This shows that

$$\left(h_j'(z) + \frac{1}{2}\frac{1}{z - b_j}\right)T(z) = \left(\overline{g'(z)} - \frac{1}{2}\frac{1}{\bar{z} - \bar{b}_j}\right)\overline{T(z)} \tag{1}$$

on the boundary. It follows that

$$\left(h_j'(z) + \frac{1}{2}\frac{1}{z - b_j}\right) dz$$

extends to the double of Ω as a meromorphic 1-form.

Very similar reasoning can be used to handle H. Indeed, consider H as a linear combination of

$$H_0 = \frac{P(L_a \bar{\psi})}{L_a}$$

and the functions

$$H_j = -\frac{P(L_a \mathcal{L}_j)}{L_a}.$$

Since L_a has a simple pole at a and no zeroes on $\overline{\Omega}\backslash\{a\}$, these functions are holomorphic on Ω. Now, we may apply Theorem 6.1 of [2] as we did with h_0 to see that

$$H_0=\frac{P(L_a\bar{\psi})}{L_a}=\bar{\psi}-\frac{\overline{P(S_a\psi)}}{\bar{S}_a}.$$

Since ψ extends to the double as a meromorphic function without poles on the boundary curves of Ω, it also follows that $\psi(z)=G(z)$ on the boundary, where G is a meromorphic function on Ω that extends smoothly to the boundary. Hence, it follows that H_0 extends meromorphically to the double. Next, we apply the same idea to H_j to see that

$$H_j=-\frac{P(L_a\mathcal{L}_j)}{L_a}=-\mathcal{L}_j+\frac{\overline{P(S_a\mathcal{L}_j)}}{\bar{S}_a}.$$

Thus,

$$H_j(z)=-\log|z-b_j|+\overline{g(z)},$$

where g is a meromorphic function on Ω that extends smoothly to the boundary. It follows that

$$H_j'(z)T(z)=-\frac{1}{2}\frac{T(z)}{z-b_j}-\frac{1}{2}\frac{\overline{T(z)}}{\bar{z}-\bar{b}_j}+\overline{g'(z)}\,\overline{T(z)}$$

for z in the boundary. This shows that

$$\left(H_j'(z)+\frac{1}{2}\frac{1}{z-b_j}\right)T(z)=\left(\overline{g'(z)}-\frac{1}{2}\frac{1}{\bar{z}-\bar{b}_j}\right)\overline{T(z)} \tag{2}$$

on the boundary. It follows that

$$\left(H_j'(z)+\frac{1}{2}\frac{1}{z-b_j}\right)dz$$

extends to the double of Ω as a meromorphic 1-form. Finally, the last sentence in the theorem about the cancellation of the poles follows from the proof of the formula for the solution of the Dirichlet problem given in [4]. □

Conditions like equations (1) and (2) that appear in the proof of Theorem 3.2 yield that

$$h_j'(z)+\frac{1}{2}\frac{1}{z-b_j}\quad\text{and}\quad H_j'(z)+\frac{1}{2}\frac{1}{z-b_j}$$

belong to the Class $\mathcal{A}$ of [9], and are therefore given by finite linear combinations of functions from a list of simple domain functions, as shown in [9].

We now turn to the proof of Theorem 3.1.

Proof of Theorem 3.1. Since Ω is a bounded area quadrature domain, we know that the Schwarz function $S(z)$ for Ω extends meromorphically to Ω and $S(z)=\bar{z}$ on the boundary. We also know that the field of meromorphic functions on the double is generated by the extensions of z and $S(z)$. It was shown in [11] (Theorem 1.5) that $T(z)^2$ extends to the double as a meromorphic function on such domains. (If Ω is also a boundary arc length quadrature domain, then $T(z)$ itself extends to the double as a meromorphic function.) These simple facts are all that is needed to deduce Theorem 3.1 from Theorem 3.2. The conclusions about h_0 and H_0 follow immediately. Since $T(z)^2$ extends meromorphically to the double, it is given by the boundary values of a meromorphic function τ on Ω that extends smoothly to the boundary. Now, noting that $1/T(z)=\overline{T(z)}$, equations (1) and (2) can be further manipulated to yield the conclusions in Theorem 3.1 about h_j and H_j. Indeed, equation (1) yields that

$$h_j'(z) = -\frac{1}{2}\frac{1}{\overline{S(z)}-b_j} + \left(\overline{g'(z)} - \frac{1}{2}\frac{1}{\bar{z}-\bar{b}_j}\right)\overline{\tau(z)}$$

on the boundary, and this shows that h_j' extends meromorphically to the double. Consequently, h_j' is a rational combination of z and $S(z)$. Similarly, H_j' can be shown to satisfy the same conditions. □

Another nice consequence of Theorem 3.1 is the following result about the solution to the Dirichlet problem with rational data on an area quadrature domain without cusps on the boundary. The Bergman span associated to a domain is the complex linear span of functions of z of the form $K^{(m)}(z,w)$ where w are points in the domain, $K^{(0)}(z,w)=K(z,w)$, and $K^{(m)}(z,w)=(\partial/\partial\bar{w})^m K(z,w)$ are derivatives of the Bergman kernel in the second variable.

Theorem 3.3. *Suppose that Ω is a bounded n-connected area quadrature domain without cusps on the boundary, and suppose $\psi(z)=R(z,\bar{z})$ is a rational function of z and $\bar{z}$ with no singularities on $b\Omega\times b\Omega$. The solution to the Dirichlet problem with boundary data ψ is equal to*

$$u = h + \overline{H} + \sum_{j=1}^{n-1} c_j \log|z-b_j|,$$

where h and H are holomorphic functions on Ω such that all their derivatives are algebraic functions that are rational combinations of z and the Schwarz function $S(z)$ for Ω. Furthermore, h'' and H'' and all higher-order derivatives are in the

Bergman span. The functions $\partial u/\partial z$ and $\partial u/\partial \bar{z}$ are holomorphic and antiholomorphic functions on Ω, respectively, that extend meromorphically and antimeromorphically to the double. All the real partial derivatives of u are real algebraic functions. The points b_j are fixed points in the complement of Ω, one in the interior of each bounded component of $\mathbb{C}\setminus\Omega$.

Proof. To prove Theorem 3.3, we need Theorem 1.4 in [11], which states that, if g is a holomorphic function that extends meromorphically to the double of a bounded area quadrature domain with no cusps on the boundary, then g' extends meromorphically to the double. Hence, if we express the solution u to the Dirichlet problem as we did in the proof of Theorem 3.2 via

$$u = h + \overline{H} + \sum_{j=1}^{n-1} c_j \log|z - b_j|,$$

then

$$\frac{\partial u}{\partial z} = h' + \sum_{j=1}^{n-1} c_j \frac{1}{2(z-b_j)},$$

which extends to the double because Theorem 3.2 plus Theorem 1.4 in [11] yield that

$$h' = h_0' + \sum_{j=1}^{n-1} c_j h_j'$$

extends to the double (and $1/(z-b_j)$ extends to the double because z does). Similarly, H' extends meromorphically to the double, and

$$\frac{\partial u}{\partial \bar{z}} = \overline{H}' + \sum_{j=1}^{n-1} c_j \frac{1}{2(\bar{z}-\bar{b}_j)}$$

extends to the double as an antimeromorphic function.

Note that once it is known that a function f extends meromorphically to the double, it is seen to be a rational function of z and $S(z)$. Since $S(z)$ extends to the double, so does $S'(z)$. Consequently, $S'(z)$ is a rational function of z and $S(z)$, and it follows inductively that all the derivatives of f extend to the double as rational combinations of z and $S(z)$.

We now combine Theorem 3.3 with Theorem 1.4 in [11] and Lemma 4.1 in [10] to complete the proof. Lemma 4.1 in [10] states that, if g is a holomorphic function that extends meromorphically to the double of a bounded smoothly bounded finitely connected domain with no poles on the boundary, then g' is in the Bergman span of the domain. Hence, if we differentiate a second time, we see that h'' and H''

belong to the Bergman span, and we may repeat this process to see that all higher order derivatives also belong to the Bergman span. □

The arguments leading to the conclusions of Theorems 3.2 and 3.1 can be repeated using the harmonic measure functions $\omega_j(z)$ in place of $\log|z-b_j|$ to obtain similar results. If γ_j is one of the $n-1$ boundary curves of Ω bounding one of the bounded components of $\mathbb{C}\setminus\Omega$, then ω_j is the solution to the Dirichlet problem with boundary values equal to one on γ_j and equal to zero on the other boundary curves. The Dirichlet problem can be solved in a manner very similar to the Szegő projection method used above using the ω_j in place of $\log|z-b_j|$. Indeed, this method was developed in [2] and [3] and is described on pp. 89–91 of [4]. The proofs of Theorems 3.2 and 3.1 can be repeated line for line and at the point where equations (1) and (2) come into play, the following argument is invoked. The classical functions F_j' are defined via $F_j'(z)=2(\partial/\partial z)\omega_j(z)$. (They are holomorphic on Ω, but they are not the derivative of a holomorphic function on Ω in spite of the prime in the notation. They are locally the derivative of a holomorphic function with real part equal to ω_j.) In the proof, where the formula

$$h_j(z)=-\omega_j(z)+\overline{g(z)}$$

is obtained, differentiate along the boundary as before to obtain

$$h_j'(z)T(z)=-\tfrac{1}{2}F_j'(z)T(z)-\tfrac{1}{2}\overline{F_j'(z)T(z)}+\overline{g'(z)}\,\overline{T(z)}.$$

This shows that $\left(h_j'+\frac{1}{2}F_j'\right)dz$ extends as a meromorphic 1-form to the double. If Ω is an area quadrature domain, then the functions F_j' extend meromorphically to the double of Ω (see Theorem 1.1 of [11]). The rest of the proof follows smoothly after this point. It is worth stating the resulting theorem for area quadrature domains here.

Theorem 3.4. *Suppose that Ω is a bounded n-connected area quadrature domain without cusps in the boundary, and suppose $\psi(z)=R(z,\bar z)$ is a rational function of z and $\bar z$ with no singularities on $b\Omega\times b\Omega$. The solution to the Dirichlet problem with boundary data ψ is equal to*

$$h_0+\overline{H}_0+\sum_{j=1}^{n-1}c_j(h_j+\overline{H}_j+\omega_j),$$

where h_0 and H_0 are algebraic meromorphic functions on Ω that are rational combinations of z and the Schwarz function $S(z)$ for Ω. The c_j are complex constants. The functions h_j and H_j, $j=1,\dots,n-1$, are meromorphic functions on Ω whose *derivatives are algebraic functions that extend meromorphically to the double of* Ω, *and hence* h_j' *and* H_j' *are rational combinations of* z *and* $S(z)$. *The functions* H_j, $j=0,1,...,n-1$, *are holomorphic on* Ω, *but the* h_j *may have poles at* $n-1$ *points in* Ω *specified in the proof above. The constants* c_j *are determined via the condition that the poles on* Ω *should cancel so that*

$$h_0+\sum_{j=1}^{n-1} c_j h_j$$

is holomorphic on Ω.

The theorem analogous to Theorem 3.4 corresponding to Theorem 3.2 with ω_j in place of $\log|z-b_j|$ also holds.

We have generalized the first alternative proof of Ebenfelt's theorem to the multiply connected setting. Next, we generalize the second alternative proof. In [6], [8] and [9], the derivative $G^{(1)}(z,w)=(\partial/\partial w)G(z,w)$ is expressed in terms of simpler domain functions of one variable in various ways. The virtue of this second proof allows us to apply these results to obtain information about the form of the solution to the Dirichlet problem with rational data in double quadrature domains.

Suppose that Ω is a bounded n-connected double quadrature domain. Theorem 1.1 of [8] states that there are points A_1, A_2, A_3 and w_k, $k=1,2,...,n-1$ in Ω such that

$$G^{(1)}(z,w)=r_0(z,w)+\sum_{k=1}^{n-1} \rho_k(z) r_k(w), \tag{3}$$

where $r_0(z,w)$ is a rational combination of $S(w,A_j)$, $S(z,A_j)$, and $\overline{S(z,A_j)}$ for $j=1,2,3$, $r_k(w)$ is a rational combination of $S(w,A_j)$ for $j=1,2,3$, and ρ_k is given by

$$\rho_k(z)=G^{(1)}(z,w_k)-\frac{S(z,w_k)L(z,w_k)}{S(w_k,w_k)}.$$

On a double quadrature domain, the Szegő kernel $S(z,b)$ extends meromorphically to the double in z for each fixed $b\in\Omega$. The same holds for the Garabedian kernel $L(z,b)$. The functions $G^{(1)}(z,w_k)$ are solutions to the Dirichlet problem with boundary data $\frac{1}{2}(z-w_k)^{-1}$. Consequently, we may apply Theorem 3.1 to express $G^{(1)}(z,w_k)$ in elementary terms. When we collect everything together, we find that

$$G^{(1)}(z,w)=R_0(z,w)+\sum_{k=1}^{n-1} \sigma_k(z) R_k(w), \tag{4}$$

where $R_0(z,w)$ is a rational combination of w, $S(w)$, z, $\bar{z}$, $S(z)$ and $\overline{S(z)}$, $R_k(w)$ is a rational combination of w and $S(w)$, and $\sigma_k(z)$ is equal to the function $(h_k(z)+\overline{H_k(z)}+\log|z-b_k|)$ appearing in Theorem 3.1. Higher order derivatives $G^{(m)}(z,w)$ have the same form since $S'(w)$ also extends to the double, and are therefore rational functions of w and $S(w)$. Hence, we could use the idea of the second proof of Ebenfelt's theorem to first extend $R(z,\bar{z})$ as a meromorphic function, and then to subtract off a linear combination of terms $G^{(m)}(z,w_j)$ to remove the poles to obtain the solution of the Dirichlet problem with boundary data $R(z,\bar{z})$. Although formula (4) is new, the form of the solution to the Dirichlet problem obtained is very similar to that given by Theorem 3.1.

We remark here that if we use Theorem 8.1 of [6] to express the derivatives of Green's function instead, we obtain the following theorem, which does seem new.

Theorem 3.5. *If Ω is a bounded double quadrature domain, then the solution to the Dirichlet problem with rational boundary data is equal to*

$$\Phi+\sum_{k=1}^{n-1} c_k(\lambda_k-\omega_k),$$

where Φ is a function in $C^\infty(\overline{\Omega})$ that is a rational combination of z, the Schwarz function for Ω, and the conjugates of these two functions. Here,

$$\lambda_k(z)=\frac{1}{S(z,z)}\int_{w\in\gamma_k}|S(z,w)|^2\,ds$$

are the non-harmonic measure functions studied in [6].

The Poisson kernel $p(z,w)$ associated to a bounded smoothly bounded domain Ω is related to the classical Green's function via

$$p(z,w)=-\frac{i}{\pi}\frac{\partial G}{\partial w}(z,w)T(w),$$

where $z\in\Omega$ and $w\in b\Omega$. On a double quadrature domain, since $T(w)$ extends meromorphically to the double, $T(w)$ is therefore a rational function of w and $\bar{w}=S(w)$, and it follows from equation (4) that the Poisson kernel associated to the domain is equal to an expression analogous to the right-hand side of equation (4) as follows.

Theorem 3.6. *The Poisson kernel associated to a double quadrature domain Ω is given by*

$$p(z,w)=Q_0(z,w)+\sum_{k=1}^{n-1}\sigma_k(z)Q_k(w)$$

for $z \in \Omega$ *and* $w \in b\Omega$, *where* $Q_0(z,w)$ *is a rational combination of* w, $\bar{w}$, z, $\bar{z}$, $S(z)$, *and* $\overline{S(z)}$, $Q_k(w)$ *is a rational combination of* w *and* $\bar{w}$, *and* $\sigma_k(z)$ *is equal to the function* $h_k(z)+\overline{H_k(z)}+\log|z-b_k|$ *from Theorem 3.1*, h_k' *and* H_k' *being rational functions of* z *and* $S(z)$.

If we use formula (7.5) from [6] to express the Poisson kernel associated to a double quadrature domain instead, we may deduce that $p(z,w)$ is a rational combination of z, $S(z)$, $\bar{z}$, $\overline{S(z)}$, w, and $\bar{w}$, plus a sum of the form

$$\sum_{k=1}^{n-1}(\lambda_k(z)-\omega_k(z))\mu_k(w),$$

where the functions μ_k are rational functions of w and $\bar{w}$.

We have derived formulas for derivatives of Green's function in quadrature domains. It is also possible to deduce formulas for Green's function itself. Indeed, if Ω is a bounded area quadrature domain without cusps in the boundary, then Green's function associated to Ω for a point $b_0 \in \Omega$ is given by

$$G(z,b_0) = -\log|z-b_0|+u,$$

where u solves the Dirichlet problem on Ω with boundary data $\log|z-b_0|$. We may express u as we did in the proof of Theorem 3.1 to write

$$u = h(z)+\overline{H(z)}+\sum_{j=1}^{n-1} c_j \log|z-b_j|,$$

where, writing $\phi = \log|z-b_0| - \sum_{j=1}^{n-1} c_j \log|z-b_j|$, we have

$$h = \frac{P(S_a\phi)}{S_a} \quad \text{and} \quad H = \frac{P(L_a\bar{\phi})}{L_a}.$$

Let $\mathcal{L}_j(z) = \log|z-b_j|$ for $j=0,1,2,...,n-1$. We now consider h as a linear combination of

$$h_0 = \frac{P(S_a\mathcal{L}_0)}{S_a}$$

and the functions

$$h_j = -\frac{P(S_a\mathcal{L}_j)}{S_a}.$$

The same argument used above in the proof of Theorem 3.1 yields that

$$h_0 = \frac{P(S_a\mathcal{L}_0)}{S_a} = \mathcal{L}_0 - \frac{\overline{P(L_a\bar{\mathcal{L}}_0)}}{\bar{L}_a}.$$

Hence,

$$h_0 = \mathcal{L}_0 + \bar{g},$$

where g is holomorphic on Ω. We may now differentiate this formula along the boundary to obtain

$$h_0'(z)T(z) = \frac{1}{2}\frac{T(z)}{z-b_0} + \frac{1}{2}\frac{\overline{T(z)}}{\bar{z}-\bar{b}_0} + \overline{g'(z)}\,\overline{T(z)}$$

for z in the boundary. This shows that

$$\left(h_0'(z) - \frac{1}{2}\frac{1}{z-b_0}\right)T(z) = \left(\overline{g'(z)} + \frac{1}{2}\frac{1}{\bar{z}-\bar{b}_0}\right)\overline{T(z)}$$

on the boundary. We can now use the argument that we used in the proof of Theorem 3.1 to see that h_0' extends to the double as a meromorphic function. Similar reasoning can be applied to H. The bottom line is that

$$G(z,b_0) = -\log|z-b_0| + h_0(z) + \overline{H_0(z)} + \sum_{j=1}^{n-1} c_j\left(h_j(z) + \overline{H_j(z)} + \log|z-b_j|\right),$$

where the h_j and H_j are such that h_j' and H_j' are algebraic functions of z which are rational combinations of z and $S(z)$ for $j=0,1,2,\dots,n-1$. It would be interesting to figure out the dependence of the functions on the right-hand side of this formula in b_0, but the results of [13] lead one to believe that it might be rather messy.

We close this section by remarking that the harmonic measure functions ω_k associated to a bounded area quadrature domain Ω without cusps in the boundary can be expressed in terms of simpler functions. There exist real constants c_j so that

$$u := \omega_k - \sum_{j=1}^{n-1} c_j \log|z-b_j|$$

is equal to the real part of a holomorphic function G on Ω (i.e., so that the periods vanish). Note that $G' = 2\partial u/\partial z$ is equal to F_k' plus a linear combination of $1/(z-b_j)$. Since F_k' and all the $1/(z-b_j)$ extend meromorphically to the double on area quadrature domains, it follows that G' extends meromorphically to the double. Hence, we may state the following theorem.

Theorem 3.7. *The harmonic measure functions associated to a bounded area quadrature domain without cusps in the boundary can be expressed via*

$$\omega_k = \operatorname{Re} G_k + \sum_{j=1}^{n-1} c_{kj} \log|z-b_j|,$$

where G_k are holomorphic functions on Ω such that G_k' extends meromorphically to the double. Consequently, G_k' are algebraic functions that are rational combinations of z and the Schwarz function. The points b_j are points in the complement of Ω, one in the interior of each bounded component of the complement, and (c_{kj}) is a non-singular matrix of real constants.

4. Mappings between quadrature domains

Gustafsson [21] proved that if Ω_1 is a bounded finitely connected domain bounded by Jordan curves and $f\colon \Omega_1 \to \Omega_2$ is a biholomorphic mapping, then Ω_2 is an area quadrature domain if and only if f extends meromorphically to the double of Ω_1. If f is a biholomorphic mapping between two area quadrature domains, this result can be applied in both directions. We also know from [21] that finitely connected area quadrature domains have Schwarz functions that extend meromorphically to the domain, and the field of meromorphic functions on the double is generated by the extensions of z and the Schwarz function. Hence, it follows that f is a rational combination of z and the Schwarz function $S_1(z)$ for Ω_1. Since $S_1(z)=\bar{z}$ on the boundary, it follows that f is a rational function of z and $\bar{z}$ when restricted to the boundary. As $S_1(z)$ is algebraic, f is algebraic. It is shown in [11] (Theorem 1.4) that, on an area quadrature domain, if H extends to the double, then so does H'. Thus, f' is also a rational function of z and $S_1(z)$ on Ω, and a rational function of z and $\bar{z}$ when restricted to the boundary. In fact, all the derivatives of f have this property.

We may conclude from this discussion that biholomorphic mappings between area quadrature domains are rather like automorphisms of the unit disc (where $S(z)=1/z$).

Gustafsson's result about biholomorphic maps was generalized to proper holomorphic mappings in [14] (Theorem 1.3 and p. 168). Hence, exactly the same conclusions can be drawn about proper holomorphic mappings between bounded area quadrature domains. Proper holomorphic mappings between area quadrature domains are rather like finite Blaschke products.

We next consider a biholomorphic mapping $f\colon \Omega_1 \to \Omega_2$ between bounded double quadrature domains. Such a map would have all of the properties mentioned above, plus the property proved in [16] that $\sqrt{f'}$ belongs to the Szegő span of Ω_1. Since the Szegő kernel $S(z,a)$ and its derivatives $(\partial/\partial\bar{a})^m S(z,a)$ extend meromorphically to the double in z for each fixed a on double quadrature domains (see [16]), it follows that $\sqrt{f'}$ is a rational function of z and $S_1(z)$. Consequently, f' is the square of such a function and f' is the square of a rational function of z and $\bar{z}$ when restricted to the boundary.

5. Classes of domains with simple kernels

The results of Section 3 can be pulled back via conformal mappings. When the conformal mappings have additional properties, more detailed information can be gleaned.

Call a bounded finitely connected domain Ω an *algebraic* domain if there is a complex algebraic proper holomorphic mapping of Ω onto the unit disc. This class of domains was studied in [6]. There, it is shown that these domains are characterized by having algebraic Bergman kernels, or Szegő kernels, or Ahlfors maps. A list of equivalent ways to characterize the domains is given in [6] (see also [9]).

Call a bounded finitely connected domain Ω a *Briot–Bouquet* domain if there is a proper holomorphic mapping f of Ω onto the unit disc such that f' and f are algebraically dependent, i.e., such that a non-zero polynomial of two complex variables $P(z,w)$ exists such that $P(f',f)\equiv 0$ on Ω. It was shown in [9] that domains in this class are characterized by having Bergman kernels or Szegő kernels generated by only *two* holomorphic functions of one complex variable. A list of equivalent ways to characterize the domains is given in [9]. One of the equivalent conditions is that Ahlfors maps associated to Ω are Briot–Bouquet functions.

The next theorem will allow us to harvest the consequences of the theorems of Section 3 for algebraic and Briot–Bouquet domains.

Theorem 5.1. *Suppose $f\colon \Omega_1\to\Omega_2$ is a biholomorphic mapping between bounded domains bounded by finitely many non-intersecting C^∞ smooth curves. Suppose further that Ω_2 is an area quadrature domain. Let $S_2(z)$ denote the Schwarz function for Ω_2.*

If Ω_1 is an algebraic domain, then f is algebraic. Consequently, so is $S_2\circ f$.

If Ω_1 is a Briot–Bouquet domain, then f is a Briot–Bouquet function, and so is $G\circ f$ whenever G is a meromorphic function on Ω_2 that extends meromorphically to the double of Ω_2. In particular, $S_2\circ f$ is a Briot–Bouquet function.

In both cases, f and $S_2\circ f$ extend meromorphically to the double of Ω_1 and form a primitive pair for the field of meromorphic functions on the double.

Proof. It is shown in [11] that area quadrature domains are algebraic domains, and it is shown in [12] that biholomorphic mappings between algebraic domains are algebraic (since biholomorphic maps in Bergman coordinates are algebraic and Bergman coordinates themselves are rational combinations of the Bergman kernel, which is algebraic in algebraic domains). Hence, the first part of the theorem is proved.

Suppose now that Ω_1 is a Briot–Bouquet domain. Since Ω_2 is an area quadrature domain, the mapping f extends to the double of Ω_1 as a meromorphic function.

Assume that G extends meromorphically to the double of Ω_2. Since the property of extending to the double is preserved under conformal changes of variables, it follows that $G\circ f$ also extends meromorphically to the double of Ω_1.

It is shown in [7] that the field of meromorphic functions on the double of a smoothly bounded finitely connected domain is generated by two Ahlfors mappings associated to the domain. Hence, there are Ahlfors maps g_1 and g_2 associated to two distinct points in Ω_1 such that $f(z)=R(g_1(z), g_2(z))$, where R is a complex rational function of two variables. Differentiating this formula reveals that f' is equal to g_1' times a rational function of g_1 and g_2 plus g_2' times a rational function of g_1 and g_2. Since Ω is a Briot–Bouquet domain, the Ahlfors maps are Briot–Bouquet functions (see [9]). Hence, there are non-zero polynomials P_1 and P_2 of two complex variables such that $P_j(g_j', g_j)\equiv 0$, $j=1,2$. Now the procedure used in [9] to construct the compact Riemann surface to which all the kernel functions extend can be applied here. Since g_1 and g_2 extend meromorphically to the double of Ω_1, we may use the identities $P_j(g_j', g_j)\equiv 0$, $j=1,2$, to extend g_1' and g_2' as meromorphic functions on a compact Riemann surface that is a finite branched cover of the double of Ω_1. Now both f and f' extend meromorphically to this compact Riemann surface, and they are therefore algebraically dependent. Hence, there is a non-zero polynomial P of two complex variables such that $P(f', f)\equiv 0$ on Ω, i.e., f is a Briot–Bouquet function. The same reasoning can be applied to $G\circ f$ to conclude that it, too, is a Briot–Bouquet function. □

If $f\colon \Omega_1\to\Omega_2$ is a biholomorphic mapping between smoothly bounded domains, then Green's functions associated to the domains transform via

$$G_1(z,w)=G_2(f(z), f(w)).$$

Consequently,

$$G_1^{(1)}(z,w)=G_2^{(1)}(f(z), f(w))f'(w),$$

and we can use the results of Section 3 to pull back results about the Poisson kernel and Green's functions on double quadrature domains to algebraic and Briot–Bouquet domains. Before we state the resulting theorem, note that the complex unit tangent function transforms via

$$T_2(f(z))=\frac{f'(z)}{|f'(z)|}T_1(z),$$

and so

$$p_1(z,w)=p_2(f(z), f(w))|f'(z)|$$

(which could also be deduced directly from considerations of arc length, but the transformation rule for the complex unit tangent vector functions is worth writing down here because it allows one to pull back complexity results about the tangent function from double quadrature domains).

Theorem 5.2. *Suppose that* Ω *is a bounded domain bounded by finitely many non-intersecting* C^∞ *smooth curves that is either an algebraic or Briot–Bouquet domain, and suppose that* $f(z)$ *is a biholomorphic mapping of* Ω *onto a double quadrature domain with Schwarz function* $S(z)$*. The Poisson kernel associated to* Ω *is given by*

$$p(z,w)=Q_0(z,w)+\sum_{k=1}^{n-1}\sigma_k(z)Q_k(w)$$

for $z\in\Omega$ *and* $w\in b\Omega$*, where* $Q_0(z,w)$ *is a rational combination of* $f(w)$*,* $S(f(w))$*,* $f(z)$*,* $S(f(z))$*,* $\overline{f(z)}$*, and* $\overline{S(f(z))}$ *times* $|f'(w)|$*,* $Q_k(w)$ *is a rational combination of* $f(w)$ *and* $S(f(w))$ *times* $|f'(w)|$*, and* $\sigma_k(z)$ *is equal to* $g_k(z)+\overline{G_k(z)}+\log|f(z)-b_k|$*, where* g_k *and* G_k *have derivatives that are rational functions of* $f(z)$ *and* $S(f(z))$ *times* $f'(z)$*. In the case when* Ω *is an algebraic domain,* $f(z)$*,* $f'(z)$*, and* $S(f(z))$ *are algebraic and* $|f'(z)|$ *is real algebraic, and* g_k *and* G_k *have algebraic derivatives. In the case when* Ω *is a Briot–Bouquet domain,* $f(z)$*,* $S(f(z))$*, and* $f'(z)$ *are Briot–Bouquet functions, and* g_k *and* G_k *have derivatives that are Briot–Bouquet functions times* $f'(z)$*.*

In Theorem 5.2, since an algebraic domain is also a Briot–Bouquet domain, the techniques of [12] can be used to show that $f'(z)$ extends meromorphically to a compact Riemann surface that is a finite cover of the double to which $f(z)$ and $S(f(z))$ also extend. Hence, if G_1 and G_2 are a primitive pair for the compact Riemann surface, all of the functions f, f', and $S\circ f$ are rational combinations of the same two functions G_1 and G_2. Thus the Poisson kernel is composed of basic building blocks that are surprisingly simple.

6. Classical boundary operators on double quadrature domains

Suppose that Ω is a bounded double quadrature domain. We will now show that the Szegő projection associated to Ω, as an operator from $L^2(b\Omega)$ to itself, maps rational functions of z and $\bar{z}$ to the same class of functions. Suppose that $u(z)=R(z,\bar{z})$ is such a function without singularities on the boundary $b\Omega$. The Szegő projection P satisfies

$$Pv=v-\overline{P(\overline{vT})}\,\overline{T}$$

on the boundary (see p. 13 of [4]). Now, since u extends antimeromorphically to Ω as $R(\overline{S(z)},\bar{z})$ and as $T(z)$ is equal to the boundary values of a meromorphic function on the double without poles on the boundary curves, we see that the holomorphic function Pu has the same boundary values as an antimeromorphic function on Ω that extends smoothly up to the boundary. Hence, Pu extends meromorphically to the double, and is therefore a rational combination of z and $S(z)$. Now, restricting to the boundary shows that Pu is a rational combination of z and $\bar{z}$, and the proof is complete. It also follows that the extension of Pu to Ω as a holomorphic function is a rational function of z and $S(z)$ without singularities on $\overline{\Omega}$. Thus, as an operator from $L^2(b\Omega)$ to the space of holomorphic functions on Ω with L^2 boundary values, P maps rational functions of z and $\bar{z}$ to algebraic functions.

It is easy to see that the Cauchy transform has the same behavior on an area quadrature domain. Indeed, we may write

$$(\mathcal{C}u)(z)=\frac{1}{2\pi i}\int_{b\Omega}\frac{u(w)}{w-z}\,dw=\frac{1}{2\pi i}\int_{b\Omega}\frac{R(w,S(w))}{w-z}\,dw,$$

and the residue theorem yields that Cu is a rational function of z with poles at the poles of the meromorphic function $R(w,S(w))$ that fall outside $\overline{\Omega}$. The L^2 adjoint of the Cauchy transform satisfies

$$\mathcal{C}^*u=u-\overline{\mathcal{C}(\,\overline{uT}\,)T}$$

(see [4], p. 12). If Ω is a double quadrature domain, then T is the restriction to the boundary of a rational function of z and $\bar{z}$. Hence, it follows that $\mathcal{C}^*$ maps the space of rational functions of z and $\bar{z}$ without singularities on the boundary into itself. It now follows that the Kerzman–Stein operator $\mathcal{A}=\mathcal{C}-\mathcal{C}^*$ enjoys the same property.

We now consider the Dirichlet-to-Neumann map on a double quadrature domain. If $\psi(z)=R(z,\bar{z})$ is a rational function without singularities on the boundary, then Theorem 3.4 states that the solution to the Dirichlet problem with boundary data ψ is given by

$$h_0+\overline{H}_0+\sum_{j=1}^{n-1}c_j(h_j+\overline{H}_j+\omega_j),$$

where the properties of the functions involved in the formula are listed in the statement of the theorem. The normal derivative of this function is

$$-ih_0'T+i\overline{H_0'T}+\sum_{j=1}^{n-1}c_j(-ih_j'T+i\overline{H_j'T}-iF_j'T)$$

(see p. 87 of [4]). All of the functions in this formula are rational in z and $\bar{z}$ on the boundary. Hence, the normal derivative of the solution to the Dirichlet problem is rational, and we have shown that the Dirichlet-to-Neumann map sends rational functions to rational functions on a double quadrature domain.

We close this section by remarking that, since $dz = T(z)\, ds$ and $d\bar{z} = \overline{T(z)}\, ds$, and since $T(z)$ extends to the double as a meromorphic function or as an antimeromorphic function on a double quadrature domain Ω, it is possible to convert an integral of the form

$$\int_{b\Omega} R(z, \bar{z})\, ds,$$

where $R(z, \bar{z})$ is a rational function of z and $\bar{z}$, to an integral

$$\int_{b\Omega} R(z, S(z)) G(z)\, dz,$$

where $G(z)$ and the Schwarz function $S(z)$ are meromorphic functions, and consequently, the integral can be computed by means of the residue theorem. The same conversion can be made for integrals involving dz or $d\bar{z}$ in place of ds. Thus, computing integrals in double quadrature domains is rather like computing integrals of rational functions of trigonometric functions on the unit circle.

7. Fourier analysis on multiply connected domains

The results of this paper are probably most interesting in the realm of double quadrature domains, however, the ideas espoused here can be viewed as a new way of thinking about Fourier analysis in multiply connected domains. Classical Fourier analysis is expanding functions in powers of $e^{i\theta}$ and $e^{-i\theta}$. If we write $e^{i\theta} = z$, then $e^{-i\theta} = \bar{z} = 1/z$, and $1/z$ is the Schwarz function for the unit disc. On the unit disc, the Poisson extension of a rational function $R(z, \bar{z})$ can be gotten by first extending $R(z, \bar{z})$ as the meromorphic function $R(z, S(z))$ and then subtracting off appropriate derivatives $(\partial/\partial w)^m G(z, w)$ of Green's function at points w where the meromorphic function has poles. Since Green's function is

$$-\log\left|\frac{z-w}{1-\bar{w}z}\right|,$$

these derivatives are rational functions of z and $\bar{z}$. The extension obtained in this way is the same as the harmonic extension of the Fourier series from the boundary obtained in classical analysis. (This gives another way to see that the harmonic extension of a rational function on the boundary of the unit disc is also rational. See [15] and [18] for more on this subject in multiply connected domains.)

In this paper, we have generalized this idea to the multiply connected setting. If Ω is a bounded finitely connected domain bounded by n non-intersecting C^∞

smooth curves, then it is proved in [7] that there are two Ahlfors maps g_1 and g_2 associated to Ω such that the field of meromorphic functions on the double of Ω is generated by g_1 and g_2. Note that g_1 and g_2 are proper holomorphic maps of Ω onto the unit disc, and that $|g_j(z)|=1$ when $z\in b\Omega$, $j=1,2$. These functions will take the place of $e^{\pm i\theta}$ in what follows.

The rational functions of z and $\bar{z}$ on the unit circle are exactly the functions that extend from the unit circle to the double of the unit disc as meromorphic functions. On Ω, this class corresponds to the class of all rational functions of $g_1(z)$, $g_2(z)$, $\overline{g_1(z)}$, and $\overline{g_2(z)}$, which is the same as the class of all rational functions of $g_1(z)$ and $g_2(z)$, since $\overline{g_j(z)}=1/g_j(z)$ when $z\in b\Omega$, $j=1,2$. It is interesting to note that this class of functions is dense in the space of C^∞ functions on the boundary, and hence also dense in the space of continuous functions on the boundary. Indeed, any C^∞ function on the boundary can be decomposed as $h+\overline{HT}$, where h and H are holomorphic functions in $C^\infty(\overline{\Omega})$ (see [4], p. 13). It was proved in [5] (see also [4], p. 29) that the complex linear span of functions of z of the form $S(z,b)$, as b ranges over Ω, is dense in the space of holomorphic functions in $C^\infty(\overline{\Omega})$. Fix a point $a\in\Omega$, and let $S_a(z)=S(z,a)$ and $L_a(z)=L(z,a)$ as always. The Szegő and Garabedian kernels satisfy the identity

$$\bar{S}_a=\frac{1}{i}L_aT \tag{5}$$

on the boundary (see [4], p. 24). Given a function φ in $C^\infty(b\Omega)$, we may decompose $S_a\varphi$ as $h+\overline{HT}$ and we may use identity (5) to obtain

$$\varphi=\frac{h}{S_a}-i\frac{\overline{H}}{\bar{L}_a}.$$

Next, if we approximate h and H by linear combinations of functions of the form S_b, we see that φ can be approximated by linear combinations of S_b/S_a and the conjugates of S_b/L_a. But such quotients extend meromorphically to the double of Ω (since identity (5) reveals that S_b/S_a is equal to the conjugate of L_b/L_a on the boundary, and L_b/S_a is equal to the conjugate of S_b/L_a on the boundary). Hence, the space of functions on the boundary that are restrictions of meromorphic functions on the double without poles on the boundary is dense in $C^\infty(b\Omega)$.

Consequently, one approach to solving the Dirichlet problem would be to first approximate a continuous function on the boundary by rational combinations of g_1 and g_2. The approximation can be extended meromorphically to the double. Next, the poles of the extended function can be subtracted off by derivatives of Green's function in the second variable to obtain a solution to the Dirichlet problem for the approximation. Various forms of the derivatives of Green's function can be used to deduce facts about the complexity of the solution. For example, Theorem 1.2 of [13] expresses Green's function in finite terms involving a single Ahlfors map, say g_1. Or Theorem 3.2 of the present work can be applied to see that the solution is a rational function of g_1 and g_2 and their conjugates modulo an explicit $(n-1)$-dimensional subspace.

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