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Proper holomorphic embeddings of finitely connected planar domains into $\mathbb{C}^n$

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Abstract. In this paper we consider proper holomorphic embeddings of finitely connected planar domains into $\mathbb{C}^n$ that approximate given proper embeddings on smooth curves. As a side result we obtain a tool for approximating a $\mathcal{C}^\infty$ diffeomorphism on a polynomially convex set in $\mathbb{C}^n$ by an automorphism of $\mathbb{C}^n$ that satisfies some additional properties along a real embedded curve.

1. Introduction

Let D be an open Riemann surface. Recall that a map $f\colon D\to\mathbb{C}^2$ is said to be a *proper holomorphic embedding* if it is a one-to-one holomorphic immersion such that the preimage of every compact set is compact. It is an open question whether every open Riemann surface embeds properly into $\mathbb{C}^2$.

In 1995 Globevnik and Stensønes [8] proved that any bounded finitely connected planar domain without isolated boundary points embeds properly into $\mathbb{C}^2$. Wold [16] improved their result to all finitely connected domains as well as to some infinitely connected domains in $\mathbb{C}$. (For further results see also [17], [7] and [11].)

In this paper we consider proper holomorphic embeddings of finitely connected planar domains into $\mathbb{C}^n$ satisfying additional requirements. This problem is related to the following extension of the Carleman theorem, proven by Buzzard and Forstnerič in [3]: *Let $n>1$ and $r\geq 0$ be integers. Given a proper $\mathcal{C}^r$ embedding λ of $\mathbb{R}$ into $\mathbb{C}^n$ and a continuous positive function $\eta\colon\mathbb{R}\to(0,\infty)$, there exists a proper holomorphic embedding $f\colon\mathbb{C}\to\mathbb{C}^n$ such that*

$$|f^{(s)}(t)-\lambda^{(s)}(t)|<\eta(t)\quad \textit{for all } t\in\mathbb{R} \textit{ and } 0\leq s\leq r.$$

We generalize their result by replacing $\mathbb{R}$ by a union of curves and also allowing $\mathbb{C}$ to be replaced with a finitely connected planar domain D.

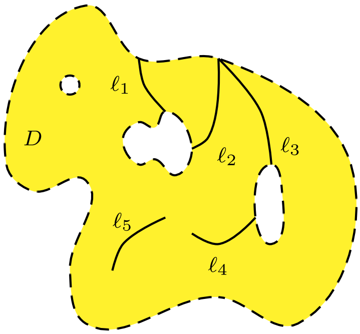

Figure 1. Planar domain D and smooth curves ℓ_i as in Theorem 1.1.

Theorem 1.1. *Let $N\geq 2$ and $r\geq 1$ be integers and D be a finitely connected planar domain. Let*

$$\ell_i=\{\gamma_i(t)\colon t\in[0,1]\},\quad i=1,2,...,k,$$

be smooth embedded curves in $\mathbb{C}$ such that $\gamma_i((0,1))\subset D$ and

$$\ell_i\cap\ell_j\cap D=\varnothing\quad \textit{for all } 1\leq i<j\leq k.$$

Let ℓ denote the curves $\bigcup_{i=1}^k \ell_i\backslash\partial D$. Given a proper $\mathcal{C}^r$ embedding $\lambda\colon \ell\hookrightarrow\mathbb{C}^N$ and a positive continuous function $\eta\colon \ell\to(0,\infty)$ there exists a proper holomorphic embedding $f\colon D\hookrightarrow\mathbb{C}^N$ such that

$$|f\circ\gamma_i^{(s)}(t)-\lambda\circ\gamma_i^{(s)}(t)|<\eta\circ\gamma_i(t)\quad \textit{for all } t\in(0,1),\ 1\leq i\leq k \textit{ and } 0\leq s\leq r.$$

While proving this theorem we formulate an additional tool related to holomorphic automorphisms of $\mathbb{C}^n$ that might be of independent interest. A holomorphic map $\phi\colon \mathbb{C}^n\to\mathbb{C}^n$ with a holomorphic inverse $\phi^{-1}\colon \mathbb{C}^n\to\mathbb{C}^n$ is called a *holomorphic automorphism* of $\mathbb{C}^n$. We denote the group of all holomorphic automorphisms of $\mathbb{C}^n$ by $\mathrm{Aut}(\mathbb{C}^n)$.

Definition 1.2. (Definition 2.1 in [9]) A family ℓ of finitely many disjoint embedded smooth real curves

$$\ell_i=\{\gamma_i(t)\colon t\in[0,\infty) \text{ or } t\in(-\infty,\infty)\},\quad i=1,2,...,m,$$

in $\mathbb{C}^n$ has a *nice projection property* if there is a holomorphic automorphism $\alpha\in\mathrm{Aut}(\mathbb{C}^n)$ such that, if π_1 is the projection to the first coordinate, $\widetilde{\gamma}_i(t)=\alpha(\gamma_i(t))$ and $\widetilde{\ell}=\alpha(\ell)$, then the following hold:

(1) $\lim_{|t|\to\infty} |\pi_1(\widetilde{\gamma}_i(t))|=\infty$ for $i=1,2,...,m$,

(2) there is a number $s\in\mathbb{R}$ such that $\mathbb{C}\backslash(\pi_1(\widetilde{\ell})\cup\overline{\mathbb{D}}_r)$ does not contain any relatively compact connected components for all $r\geq s$.

We call any number $r\geq s$ as in (2) *a nice projection radius.* If $\alpha=\mathrm{id}$, we say that ℓ has *an immediate nice projection property.*

The results we also obtain in this paper are the following.

Theorem 1.3. *Let K be a compact, polynomially convex set in $\mathbb{C}^n$, $n\geq 2$, and let $\ell=\{\gamma(t)\colon t\in[0,\infty)\}$ be a $\mathcal{C}^\infty$ embedded real curve in $\mathbb{C}^n$ with a nice projection property. Set $C=\gamma([0,1])$ and assume that $K\cap\ell=\{\gamma(0)\}$. Given a $\mathcal{C}^\infty$ diffeomorphism $F\colon C\to C'\subset\mathbb{C}^n$ such that $F|_{C\cap U}=\mathrm{id}$ for some neighborhood U of K, a compact, polynomially convex set $A\subset\mathbb{C}^n$ with $F(\gamma(1))\notin A$, a number $\varepsilon>0$ and a non-negative integer r, there exists a neighborhood V of K and an automorphism $\Phi\in\mathrm{Aut}(\mathbb{C}^n)$ satisfying*

$$\|\Phi-\mathrm{id}\|_{\mathcal{C}^r(V)}<\varepsilon, \quad \|\Phi-F\|_{\mathcal{C}^r(C)}<\varepsilon \quad \textit{and} \quad \Phi(\ell\backslash C)\cap A=\varnothing.$$

Corollary 1.4. *Let K be a compact, polynomially convex set in $\mathbb{C}^n$, $n\geq 2$, and let $\ell=\{\gamma(t)\colon t\in[0,\infty)$ or $t\in(-\infty,\infty)\}$ be a $\mathcal{C}^\infty$ embedded real curve in $\mathbb{C}^n$ with a nice projection property. Assume that $K\cap\ell=\varnothing$. Given a compact set $A\subset\mathbb{C}^n$, a number $\varepsilon>0$ and a non-negative integer r, there exists a neighborhood V of K and an automorphism $\Phi\in\mathrm{Aut}(\mathbb{C}^n)$ satisfying*

$$\|\Phi-\mathrm{id}\|_{\mathcal{C}^r(V)}<\varepsilon \quad \textit{and} \quad \Phi(\ell)\cap A=\varnothing.$$

These results hold also if $C=\bigcup_{i=1}^k C_i$, resp. $\ell=\bigcup_{i=1}^k \ell_i$, is a union of pairwise disjoint arcs, resp. curves, with the same properties.

2. Preliminaries

In the proof of Theorem 1.1 we achieve properness by using a sequence of holomorphic automorphisms with certain properties. Determining convergence of such a sequence has been examined by Forstnerič in [4]. In our setting Proposition 5.1 in [4] can be stated as follows.

Proposition 2.1. *Suppose that for each $j=1,2,3,...$, ϕ_j is a holomorphic automorphism of $\mathbb{C}^n$ satisfying*

$$|\phi_j(z)-z|<\frac{1}{2^j}, \quad z\in\overline{\mathbb{B}}_j.$$

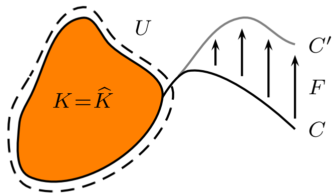

Figure 2. Moving an arc as in Proposition 2.2.

Set $\Phi_m = \phi_m \circ \phi_{m-1} \circ \ldots \circ \phi_1$. *Then there is an open set* $\Omega \subset \mathbb{C}^n$ *such that* $\lim_{m\to\infty} \Phi_m = \Phi$ *exists on* Ω *(uniformly on compact sets), and* Φ *is a biholomorphic map of* Ω *onto* $\mathbb{C}^n$. *In fact,* $\Omega = \bigcup_{j=1}^{\infty} \Phi_j^{-1}(\overline{\mathbb{B}}_j)$.

A powerful tool for obtaining automorphisms of $\mathbb{C}^n$ is Theorem 2.1 in [6]. In relations with curves, this result has been developed even further, for example in [5].

Proposition 2.2. *Let* $K \subset \mathbb{C}^n$, $n \geq 2$, *be a compact, polynomially convex set, and let* $C \subset \mathbb{C}^n$ *be an embedded arc of class* $\mathcal{C}^\infty$ *which is attached to* K *in a single point of* K. *Given a* $\mathcal{C}^\infty$ *diffeomorphism* $F\colon C \to C' \subset \mathbb{C}^n$ *such that* F *is the identity on* $C \cap U$ *for some open neighborhood* U *of* K *(see Figure 2), and given numbers* $r \geq 0$ *and* $\varepsilon > 0$, *there exists a neighborhood* W *of* K *and an automorphism* $\Phi \in \mathrm{Aut}(\mathbb{C}^n)$ *satisfying*

$$\|\Phi - \mathrm{id}\|_{\mathcal{C}^r(W)} < \varepsilon \quad \text{and} \quad \|\Phi - F\|_{\mathcal{C}^r(C)} < \varepsilon.$$

Note that the same holds with any finite number of disjoint arcs attached to K. To avoid lengthier formulations we only state the results for a single arc C or curve ℓ while keeping in mind that the same is true if $C = \bigcup_{i=1}^{k} C_i$, resp. $\ell = \bigcup_{i=1}^{k} \ell_i$, is a finite union of pairwise disjoint arcs resp. curves with the same properties. The proofs are the same in both cases, to get a proof for the latter just add indices to the existing proofs.

Let K and C be as in the proposition and assume that $\ell = \{\gamma(t) : t \in [0, \infty)\}$ is an embedded smooth curve in $\mathbb{C}^n$ with $C = \gamma([0,1])$ and $K \cap \ell = \gamma(0)$. Using the proposition we get automorphisms of $\mathbb{C}^n$ approximating movements of C while remaining close to the identity on K (see Figure 3). The obtained automorphism might map some part of $\ell \setminus C$ really close to K, which is what we want to avoid in proving Theorem 1.1.

Tackling a problem of this kind, Buzzard and Forstnerič in [3] found an elegant solution. The automorphism of $\mathbb{C}^n$ that approximately maps C to C' while staying close to identity on K, and does not map any part of $\ell \setminus C$ close to K, was in their case an automorphism of $\mathbb{C}^n$, provided by Proposition 2.2, precomposed with an

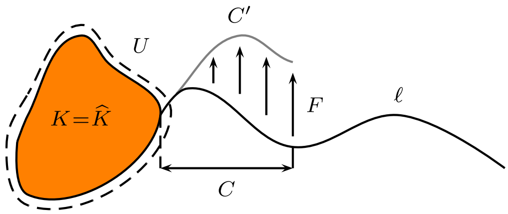

Figure 3. Moving an arc which is a part of a curve ℓ.

automorphism (called shear) of the form

$$z \longmapsto z+f(z_1)\vec{e}_2, \quad \text{where } z_1=\pi_1(z) \text{ and } \vec{e}_2=(0,1,0,...,0).$$

Although they worked with ℓ being a real line $\mathbb{R}\times\{0\}^{n-1}$, their idea can be carried over to a more general setting. For example, in [16] Wold used it for curves ℓ having a nice projection property.

An important property of sets on which we want to approximate by holomorphic automorphisms of $\mathbb{C}^n$ is polynomial convexity. By Theorem C in [15] a union $K\cup C$ of a polynomially convex set K and pairwise disjoint embedded curves $C=\bigcup_{i=1}^k C_i$ is polynomially convex, if C is simply connected and each C_i meets K in at most one point. When dealing with a union of slightly more general sets, we will use the following proposition, which is essentially the content of the proof of Proposition 1 in [17].

Proposition 2.3. *Let M be a bordered complex, one-dimensional submanifold of $\mathbb{C}^n$, $n\geq 2$, whose boundary is a set of smooth curves that are all unbounded. Assume that K is a polynomially convex compact set in $\mathbb{C}^n$, K_i is a holomorphically convex compact set in M, and $K\cap M\subset K_i$. Then $K\cup K_i$ is polynomially convex.*

3. Controlling unbounded arcs

Proving Theorem 1.3 we will follow the idea of Buzzard and Forstnerič [3], namely applying Proposition 2.2 and precomposing with a shear. We first explain the construction of the shear (see also the related results [3, Lemma 3.2] and [16, Lemma 1]).

Lemma 3.1. *Let A and K be compact, polynomially convex sets in $\mathbb{C}^n$, and*

$$\ell=\{\gamma(t): t\in[0,\infty)\}$$

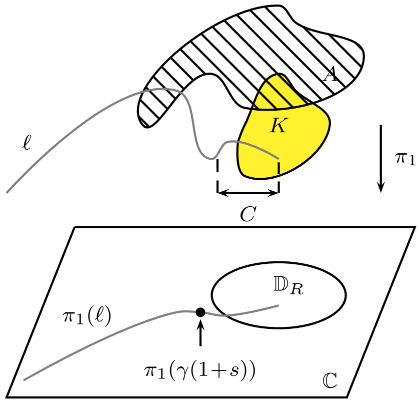

Figure 4. Properties of A, K, ℓ and C in Lemma 3.1.

be an embedded $\mathcal{C}^\infty$ real curve in $\mathbb{C}^n$ that has an immediate nice projection property. Take a nice projection radius R with $\pi_1(K)\subset\mathbb{D}_R$, set $C=\gamma([0,1])$, and assume that $\pi_1(\ell)\cap\overline{\mathbb{D}}_R=\pi_1(C)$ and $\gamma(1)\notin A$ (see Figure 4). Given $\varepsilon>0$ and an integer $r\geq 0$, there exists an automorphism $\varphi(z)=z+g(z_1)\vec{e}_2$ of $\mathbb{C}^n$ such that

$$\|\varphi-\mathrm{id}\|_{\mathcal{C}^r(\pi_1^{-1}(\overline{\mathbb{D}}_R))}<\varepsilon,\quad \|\varphi-\mathrm{id}\|_{\mathcal{C}^r(C)}<\varepsilon, \tag{1}$$

and

$$\varphi(\ell\backslash C)\subset\mathbb{C}^n\backslash A. \tag{2}$$

Proof. Since $\gamma(1)\notin A$, there is $s>0$ such that $\gamma(t)\notin A$ for all $t\in[1,1+s]$. Set $R'=|\pi_1(\gamma(1+s))|$. Choose $S>R'$ such that $A\subset\mathbb{B}_S$. Let

$$P=\{\gamma(1+s)+\zeta\,\vec{e}_2:\zeta\in\mathbb{C}\}$$

and $E=A\cap P$. Since A is polynomially convex, so is E, which implies that $P\backslash E$ is connected. As $\gamma(1+s)\notin A$, we may choose a smooth curve $\lambda\colon[0,1]\to\mathbb{C}$ with $\lambda(0)=0$,

$$\gamma(1+s)+\lambda(t)\vec{e}_2\in P\backslash E \text{ for } t\in[0,1] \quad\text{and}\quad |\lambda(1)+\pi_2(\gamma(1+s))|>S+1$$

(see Figure 5). Define $\tilde{\ell}_S=\overline{\mathbb{D}}_S\cap\pi_1(\ell)$ and let $S-R'>\delta>0$ be such that

$$\gamma(1+s)+\lambda([0,1])\vec{e}_2+\overline{\mathbb{B}}_{3\delta}\subset\mathbb{C}^n\backslash A \quad\text{and}\quad \pi_1(\gamma([1+s,1+s+\delta]))\subset\mathbb{D}_S.$$

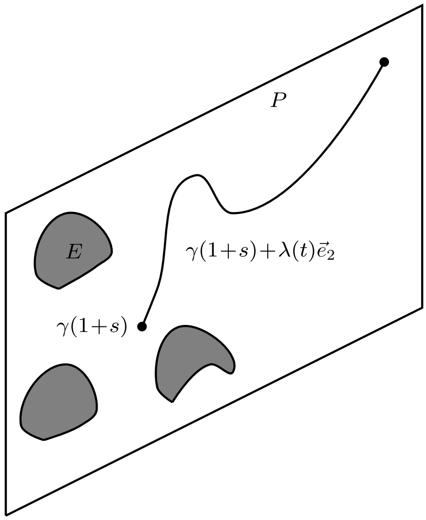

Figure 5. Choosing the curve λ as in the proof of Lemma 3.1.

Define a function $h\colon \overline{\mathbb{D}}_{R'}\cup\tilde{\ell}_S\to\mathbb{C}$ by

$$h(\zeta)=\begin{cases}0, & \zeta\in\overline{\mathbb{D}}_{R'},\\ \lambda\left(\dfrac{t-1-s}{\delta}\right), & \text{if } \zeta=\pi_1(\gamma(t)) \text{ for } t\in(1+s,1+s+\delta),\\ \lambda(1), & \text{if } \zeta=\pi_1(\gamma(t)) \text{ for } t\geq 1+s+\delta.\end{cases}$$

Since $\lambda(0)=0$, h is continuous. Clearly h is holomorphic in $\mathbb{D}_{R'}$, and thus for any given $\eta>0$ Mergelyan's theorem provides an entire function g such that

$$|g(\zeta)-h(\zeta)|<\eta \quad \text{for } \zeta\in\overline{\mathbb{D}}_{R'}\cup\tilde{\ell}_S.$$

A shear $\varphi(z)=z+g(z_1)\vec{e}_2$ satisfies conditions (1) by Cauchy's estimates if we take $\eta<\frac{1}{2}\min\{\varepsilon, R'\varepsilon, ..., (R')^r\varepsilon/r!\}$.

For $z\in\ell$ with $\pi_1(z)\notin\tilde{\ell}_S$, $|\pi_1(\varphi(z))|>S$ holds. Since $A\subset\mathbb{B}_S$, this implies that $\varphi(z)\notin A$. If $z\in\ell\setminus C$ is such that $\pi_1(z)\in\tilde{\ell}_S$, then $\varphi(z)\notin A$ by the choice of λ and h. This implies (2). □

Proof of Theorem 1.3. The automorphism Φ satisfying the conclusions of the theorem will be a composition of three automorphisms, $\Phi=\phi_2\circ\phi_1\circ\phi_0$. First, we get ϕ_1 by applying Proposition 2.2. The aim of ϕ_0 is gaining control over $\ell\setminus C$, for which Lemma 3.1 is used. The automorphism ϕ_2 is constructed before ϕ_0 (using Proposition 2.2 again) to meet the conditions of Lemma 3.1.

Proposition 2.2 provides an automorphism ϕ_1 and a neighborhood V of K such that

$$\|\phi_1 - \mathrm{id}\|_{\mathcal{C}^r(V)} < \frac{\varepsilon}{2} \quad \text{and} \quad \|\phi_1 - F\|_{\mathcal{C}^r(C)} < \frac{\varepsilon}{2}.$$

We may assume ε to be small enough so that the second condition also implies $\phi_1(\gamma(1)) \notin A$.

Let $\alpha \in \mathrm{Aut}(\mathbb{C}^n)$ be as in the definition of the nice projection property of ℓ. Choose R to be a nice projection property radius with $\pi_1(\alpha(K) \cup \alpha(C)) \subset \mathbb{D}_R$. Let $c = \ell \cap \alpha^{-1}(\pi_1^{-1}(\mathbb{D}_R))$. Let G be a diffeomorphism defined on $\phi_1(c)$ which is an identity on $\phi_1(C)$ and such that $G(\phi_1(c) \backslash \phi_1(C)) \cap A = \varnothing$. Applying Proposition 2.2 (for the polynomially convex set $\phi_1(K)$, arc $\phi_1(c)$ and the diffeomorphism G) we get for any $\delta > 0$ an automorphism ϕ_2 and a neighborhood W of $\phi_1(K)$ such that

$$\|\phi_2 - \mathrm{id}\|_{\mathcal{C}^r(W)} < \delta \quad \text{and} \quad \|\phi_2 - G\|_{\mathcal{C}^r(\phi_1(c))} < \delta.$$

If necessary, we take δ even smaller so that the last condition also implies that

$$\phi_2 \circ \phi_1(c \backslash C) \cap A = \varnothing.$$

By the product and chain rules we get for all δ small enough that

$$\|\phi_2 \circ \phi_1 - \mathrm{id}\|_{\mathcal{C}^r(V)} < \frac{\varepsilon}{2} \quad \text{and} \quad \|\phi_2 \circ \phi_1 - F\|_{\mathcal{C}^r(C)} < \frac{\varepsilon}{2}.$$

The final diffeomorphism ϕ_0 will be of the form $\phi_0 = \alpha^{-1} \circ \varphi \circ \alpha$. Let $\tilde{A}$ denote the set $\alpha \circ (\phi_2 \circ \phi_1)^{-1}(A)$. We have that $\alpha(\ell)$ is a curve with an immediate projection property and $\alpha(c)$ is an initial part of this curve with one endpoint attached to $\alpha(K)$ and the other endpoint outside $\tilde{A}$. By the choice of R and c it follows that the hypotheses of Lemma 3.1 are satisfied with $\alpha(K)$ in place of K, $\tilde{A}$ in place of A, and $\alpha(\ell)$ and $\alpha(c)$ in place of ℓ and C. For any $\mu > 0$ the lemma gives a shear $\varphi(z) = z + g(z_1)\vec{e}_2$ such that

$$\|\varphi - \mathrm{id}\|_{\mathcal{C}^r(\pi_1^{-1}(\mathbb{D}_R))} < \mu, \quad \|\varphi - \mathrm{id}\|_{\mathcal{C}^r(\alpha(c))} < \mu \quad \text{and} \quad \varphi(\alpha(\ell) \backslash \alpha(c)) \subset \mathbb{C}^n \backslash \tilde{A}.$$

The last condition gives $\phi_2 \circ \phi_1 \circ \phi_0(\ell \backslash c) \cap A = \varnothing$. Taking μ small enough the chain and product rules imply that

$$\|\phi_2 \circ \phi_1 \circ \phi_0 - \mathrm{id}\|_{\mathcal{C}^r(V)} < \varepsilon, \quad \|\phi_2 \circ \phi_1 \circ \phi_0 - F\|_{\mathcal{C}^r(C)} < \varepsilon$$

and

$$|\phi_2 \circ \phi_1 \circ \phi_0(t) - G \circ \phi_1(t)| < \varepsilon \quad \text{for } t \in c.$$

This gives the required estimates for $\Phi = \phi_2 \circ \phi_1 \circ \phi_0$ and yields $\Phi(\ell \backslash C) \cap A = \varnothing$. $\square$

Proof of Corollary 1.4. Since A is compact and ℓ has a nice projection property, there exist positive numbers s and $t_1 < t_2$, such that $A \subset \mathbb{B}_s$ and $\gamma([t_1, t_2]) \cap \mathbb{B}_s = \varnothing$. By Theorem C in [15] the set $L := K \cup \gamma([t_1, t_2])$ is polynomially convex. Choose $0 < t_0 < t_1$ and $t_3 > t_2$ such that the arcs $C_1 := \gamma([t_0, t_1])$ and $C_2 := \gamma([t_2, t_3])$ do not intersect $\overline{\mathbb{B}}_s$. Let

$$F\colon C_1 \cup C_2 \longrightarrow C_1 \cup C_2$$

be the identity map. Set

$$\ell_1 = \{\gamma(t) : t \in (-\infty, t_1] \text{ or } t \in [0, t_1]\} \quad \text{and} \quad \ell_2 = \{\gamma(t) : t \in [t_2, \infty)\}.$$

If ℓ_1 is a finite arc, extend F to a smooth diffeomorphism

$$\widetilde{F}\colon \ell_1 \cup C_2 \longrightarrow \ell_1' \cup C_2 \subset \mathbb{C}^n,$$

such that $\ell_1' = \widetilde{F}(\ell_1)$ does not intersect $\overline{\mathbb{B}}_s$. Now apply Proposition 2.2 (with $L \cup C_2$ in place of K, and ℓ_1 and ℓ_1' in place of C and C') to approximate $\widetilde{F}$ by an automorphism Φ_1. Then use Theorem 1.3 (with $C = \Phi_1(C_2)$, $\ell = \Phi_1(\ell_2)$ and $\Phi_1(L \cup \ell_1)$ in place of K, $\overline{\mathbb{B}}_s$ in place of A, and the identity map in place of F) to get Φ_2. Take $\Phi = \Phi_2 \circ \Phi_1$.

If ℓ_1 is unbounded, the result immediately follows from Theorem 1.3 with L in place of K, $\overline{\mathbb{B}}_s$ in place of A and the two pairs of curves (C_1, ℓ_1) and (C_2, ℓ_2). ☐

4. Proper embeddings with approximation

In this section we prove Theorem 1.1. We first embed D into $\mathbb{C}^N$ using a rational shear map g in such a way that $g(D)$ has only unbounded boundary components, and that $g|_\ell$ is proper. We show that under a suitable choice of g, the boundary $\partial g(D)$ has a nice projection property. Automorphisms of $\mathbb{C}^N$ are then used to push the rest of the boundary to infinity. The approximation of λ is achieved with the help of Theorem 1.3.

Proof of Theorem 1.1. Every finitely connected domain on the Riemann sphere is conformally equivalent to a domain bounded by smooth Jordan curves and isolated points. Thus we may assume ∂D to be smooth.

By approximation we may assume that $\lambda\colon \Gamma \to \mathbb{C}^N$ is a proper $\mathcal{C}^\infty$ embedding and η is small enough so that if $\mu\colon \Gamma \to \mathbb{C}^N$ satisfies the inequalities $|\mu(t) - \lambda(t)| < \eta(t)$ and $|\mu'(t) - \lambda'(t)| < \eta(t)$, then μ is a proper embedding (see [13, Proposition 2.15.4]).

Let K_1 be a holomorphically convex, compact set in D with

$$\bigcup_{i=1}^{k} (\gamma_i(0) \cup \gamma_i(1)) \cap D \subset K_1,$$

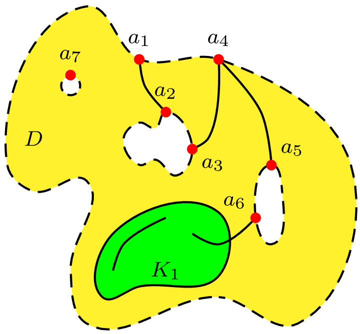

Figure 6. Choosing K_1 and $a_1, a_2, ..., a_M$.

which also satisfies the following property: if $\ell_i \subset D$ for some i, then $\ell_i \subset K_1$ (see Figure 6).

Choose finitely many points $a_1, a_2, ..., a_M$ in ∂D, at least one in each component of ∂D, so that $\bar{\ell}$ meets ∂D in a subset of $\{a_1, a_2, ..., a_M\}$ (see Figure 6). Set

$$\Gamma := (\partial D \backslash \{a_1, ..., a_M\}) \cup (\ell \backslash K_1) .$$

Let $g\colon \mathbb{C} \backslash \{a_1, ..., a_M\} \to \mathbb{C}^N$ be defined by

$$g(z) = \left(z, \sum_{i=1}^{M} \frac{c_i}{z - a_i}, 0, ..., 0 \right).$$

Clearly g embeds D into $\mathbb{C}^N$.

Claim. *The numbers $c_1, ..., c_M$ can be chosen in such a way that the family of curves $g(\Gamma)$ has a nice projection property.*

Proof. Let $\alpha(t)$ and $\beta(t)$ for $t \in [0, 1]$ be the parameterizations of two curves in $\overline{\Gamma}$ with $\alpha(0) = a_i$ and $\beta(0) = a_j$. If $i \neq j$, it follows by [16, Proposition 3.2] that for some positive number R and all positive t close to 0, the estimates

$$\left| \pi_2 \circ g \circ \alpha(t) - \frac{c_i}{(\pi_2 \circ g \circ \alpha)'(0) t} \right| < R \quad \text{and} \quad \left| \pi_2 \circ g \circ \beta(t) - \frac{c_j}{(\pi_2 \circ g \circ \beta)'(0) t} \right| < R$$

hold. If c_i and c_j are non-zero and

$$\frac{c_i}{(\pi_2 \circ g \circ \alpha)'(0)} \neq \frac{c_j}{(\pi_2 \circ g \circ \beta)'(0)}, \tag{3}$$

the estimates imply that $\pi_2 \circ g \circ \alpha(t)$ and $\pi_2 \circ g \circ \beta(s)$ do not intersect for positive t and s close to 0.

Choose numbers $c_1, c_2, ..., c_M$ in such a way that (3) holds for all pairs (i,j), $1 \le i < j \le k$, and all possible pairs of curves (α, β) in $\overline{\Gamma}$ with endpoints $\alpha(0) = a_i$ and $\beta(0) = a_j$.

It remains to consider pairs of curves in $\overline{\Gamma}$ with a common endpoint. Denote the parameterizations again by $\alpha(t)$ and $\beta(t)$ for $t \in [0,1]$ and assume that $\alpha(0) = a_i = \beta(0)$. Choose a number $C > 0$ such that for positive t and s close to 0 we have

$$\left| \sum_{\substack{j=1 \\ j \neq i}}^{M} \frac{c_j}{(\alpha(t) - a_j)(\beta(s) - a_j)} \right| < C.$$

Clearly, if the positive numbers t and s are small enough, the estimate

$$|(\alpha(t) - a_i)(\beta(s) - a_i)| < \frac{|c_i|}{2C}$$

holds. For such t and s this gives that

$$\begin{aligned} &|\pi_2 \circ g \circ \alpha(t) - \pi_2 \circ g \circ \beta(s)| \\ &\qquad = \left| \sum_{j=1}^{M} c_j \left(\frac{1}{\alpha(t) - a_j} - \frac{1}{\beta(s) - a_j} \right) \right| \\ &\qquad = |\alpha(t) - \beta(s)| \left| \frac{c_i}{(\alpha(t) - a_i)(\beta(s) - a_i)} + \sum_{j \neq i} \frac{c_j}{(\alpha(t) - a_j)(\beta(s) - a_j)} \right| \\ &\qquad > |\alpha(t) - \beta(s)|(2C - C) \\ &\qquad = |\alpha(t) - \beta(s)| C. \end{aligned}$$

The number $|\alpha(t) - \beta(s)|C$ is positive, since α and β do not intersect in D. This means that for small positive t and s, the curves $\pi_2 \circ g \circ \alpha$ and $\pi_2 \circ g \circ \beta$ do not intersect, which yields a nice projection property of $g(\Gamma)$. □

We inductively construct a sequence Φ_n of holomorphic automorphisms of $\mathbb{C}^n$, converging to a map $\Phi\colon \Omega \to \mathbb{C}^n$. Besides establishing convergence and insuring $g(D) \subset \Omega$ and $g(\partial D \setminus \{a_1, ..., a_M\}) \subset \partial \Omega$, additional requirements arise from demanding that $f = \Phi \circ g$ satisfies

$$|f \circ \gamma_i^{(s)}(t) - \lambda \circ \gamma_i^{(s)}(t)| < \eta \circ \gamma_i(t) \quad \text{for all } t \in (0,1),\ 1 \le i \le k \text{ and } 0 \le s \le r.$$

It will be clear from the inductive construction that there is no loss of generality in assuming that ℓ consists of a single curve $\gamma([0,1))$ (that means that $\gamma(0) \in D$ and

$\gamma(1)\in\partial D$). Choose constants $0<c_1<c_2<...$ converging to 1, such that $\gamma([0,c_1])$ contains $\ell\cap K_1$ and

$$\lambda(\gamma([c_j,1]))\cap\overline{\mathbb{B}}_{j+2}=\varnothing.$$

Set

$$L_j=\left\{z\in D:\ \operatorname{dist}(z,\partial D\cup\gamma([c_j,1]))\geq\frac{1}{2^j}\right\}.$$

Clearly $\bigcup_{j=1}^{\infty}L_j=D$.

Choose smooth functions $\chi_j:\ell\to\mathbb{R}$ with $\chi_j(t)=1$ if $t\in\gamma([0,c_j])$ and $\chi_j(t)=0$ if $t\in\gamma([c_{j+1},1))$. Let C_j be constants such that

$$\|\chi_j h\|_{\mathcal{C}^r(\ell)}\leq C_j\|h\|_{\mathcal{C}^r(\ell)}\quad\text{for all } h\in\mathcal{C}^r(\ell).$$

We proceed by induction. The assumptions for the nth step are that we have

– holomorphically convex sets $K_1\Subset K_2\Subset...\Subset K_n$ in D, such that $L_j\subset K_j$ for $j=1,...,n$;

– a number $0<\delta_n<1/C_n\inf\{\eta(t): t\in\gamma([0,c_{n+1}])\}$;

– automorphisms $\Phi_n=\phi_n\circ\phi_{n-1}\circ...\circ\phi_1$ of $\mathbb{C}^N$;

such that

(1_n) $|\phi_n(x)-x|<1/2^{n+1}$ for $x\in\overline{\mathbb{B}}_{n-1}\cup\Phi_{n-1}\circ g(K_n)$;

(2_n) $|\Phi_n\circ g(x)|>n$ for $x\in\partial D$;

(3_n) $|\Phi_n\circ g\circ\gamma^{(s)}(t)-\lambda\circ\gamma^{(s)}(t)|<\frac{1}{2}\eta\circ\gamma(t)$ for $t\in[0,c_n]$ and $0\leq s\leq r$;

(4_n) $|\Phi_n\circ g\circ\gamma^{(s)}(t)-\lambda\circ\gamma^{(s)}(t)|<\delta_n$ for $t\in[c_n,c_{n+1}]$ and $0\leq s\leq r$.

We will show how to obtain these hypotheses at step $n+1$. Step 1 is achieved using the same construction.

Condition (2_n) implies that $\Phi_n\circ g(\partial D)\cap\overline{\mathbb{B}}_n=\varnothing$, and thus $L:=\Phi_n\circ g(D)\cap\overline{\mathbb{B}}_n$ is a compact set in D. Let K_{n+1} be a holomorphically convex compact set in D such that

$$K_n\cup L\cup L_{n+1}\subset K_{n+1}\quad\text{and}\quad \ell\cap K_{n+1}\subset\gamma([0,c_{n+1}]).$$

For $t\in\ell$ define

$$\lambda_n(t):=\Phi_n\circ g(t)\chi^n(t)+\lambda(t)(1-\chi^n(t)).$$

If $t\in[0,c_n]$, (3_n) yields

$$|\lambda_n\circ\gamma^{(s)}(t)-\lambda\circ\gamma^{(s)}(t)|=|\Phi_n\circ g\circ\gamma^{(s)}(t)-\lambda\circ\gamma^{(s)}(t)|<\tfrac{1}{2}\eta\circ\gamma(t),\quad 0\leq s\leq r.$$

If $t\in[c_{n+1},1)$, then $\lambda_n(\gamma(t))-\lambda(\gamma(t))=0$. It follows from (4_n) and the choice of δ_n that

$$\begin{aligned}\|\lambda_n-\lambda\|_{\mathcal{C}^r(\gamma([c_n,c_{n+1}]))} &= \|(\Phi_n\circ g-\lambda)\chi_n(t)\|_{\mathcal{C}^r(\gamma([c_n,c_{n+1}]))}\\ &\le C_n\|\Phi_n\circ g-\lambda\|_{\mathcal{C}^r(\gamma([c_n,c_{n+1}]))}\\ &< \inf\{\eta(t): t\in\gamma([0,c_{n-1}])\},\end{aligned}$$

which gives that

$$|\lambda_n\circ\gamma^{(s)}(t)-\lambda\circ\gamma^{(s)}(t)|<\eta(\gamma(t)) \quad \text{for all } t\in[c_n,c_{n+1}] \text{ and } 0\le s\le r.$$

By the assumptions we posed on λ and η, these estimates imply that λ_n is an embedding.

Set

$$K=\Phi_n\circ g(K_{n+1})\cup\overline{\mathbb{B}}_n.$$

Since $\overline{\mathbb{B}}_n\cap\Phi_n\circ g(D)\subset\Phi_n\circ g(K_{n+1})$, it follows by Proposition 2.3 that K is polynomially convex. Set

$$C=\Phi_n\circ g(\gamma([0,c_{n+1}]))\setminus\Phi_n\circ g(\mathring{K}_{n+1}), \qquad \ell'=\Phi_n\circ g(\ell)\setminus\Phi_n\circ g(\mathring{K}_{n+1})$$

and $A=\overline{\mathbb{B}}_{n+1}$. Define $F\colon C\to C'$ by $F(z)=\lambda_n\circ(\Phi_n\circ g)^{-1}(z)$. Let

$$\delta_{n+1}=\frac{1}{2C_{n+1}}\inf\{\eta(t): t\in\gamma([0,c_{n+2}])\} \quad \text{and} \quad \varepsilon=\min\left\{\frac{1}{2^{n+1}},\delta_{n+1},C_{n+1}\delta_{n+1}\right\}.$$

Applying Theorem 1.3 we get an automorphism ϕ_{n+1} and a neighborhood V of K such that

$$\|\phi_{n+1}-\mathrm{id}\|_{\mathcal{C}^r(V)}<\varepsilon, \quad \|\phi_{n+1}-F\|_{\mathcal{C}^r(C)}<\varepsilon \quad \text{and} \quad \phi_{n+1}(\ell'\setminus C)\cap A=\varnothing.$$

In particular these three conditions imply (1_{n+1}), (2_{n+1}), (3_{n+1}) and (4_{n+1}), and this concludes induction step $n+1$.

By (1_n) the map

$$\Phi=\lim_{n\to\infty}\Phi_n\colon\Omega\longrightarrow\mathbb{C}^2$$

converges (Theorem 2.1) and satisfies $g(D)\subset\Omega$. Condition (2_n) yields that $g(\partial D)\subset\partial\Omega$, and thus the composition $f=\Phi\circ g\colon D\to\mathbb{C}^2$ is a proper holomorphic embedding and (3_n) and (4_n) give that

$$|f\circ\gamma^{(s)}(t)-\lambda\circ\gamma^{(s)}(t)|<\eta(\gamma(t)) \quad \text{for all } t\in[0,1) \text{ and } 0\le s\le r. \qquad \square$$

5. Remark

Theorem 1.1 gives a generalization of a result by Buzzard and Forstnerič [3], which they named the Carleman-type theorem for proper holomorphic embeddings. In the complex plane, Arakelian's theorem [1] (see also [14]) generalizes Carleman's theorem. An *Arakelian set* is a closed subset E in $\mathbb{C}$ such that $\mathbb{C}\setminus E$ has no bounded components and for every closed disc $D\subset\mathbb{C}$, the union of all bounded components of $\mathbb{C}\setminus(E\cup D)$ is a bounded set. Arakelian's theorem states: *Given an Arakelian set E, $\varepsilon>0$ and a continuous function λ on E that is holomorphic in the interior of E, there is a holomorphic function $f\colon\mathbb{C}\to\mathbb{C}$ that satisfies*

$$|f(t)-\lambda(t)|<\varepsilon$$

for every $t\in E$.

Consider the following Arakelian type problem for proper holomorphic embeddings: *Assume we are given a proper embedding $\lambda\colon E\to\mathbb{C}^n$, $n>1$, which is holomorphic in $\mathring{E}$. Given $\varepsilon>0$, does there exist a proper holomorphic embedding $f\colon\mathbb{C}\to\mathbb{C}^n$, such that $|f(t)-\lambda(t)|<\varepsilon$ for all $t\in E$?*

In general, the answer is negative. By Proposition 4.5 in [2] there exists a discrete set of discs in $\mathbb{C}^2$ for which one cannot find a proper holomorphic embedding of $\mathbb{C}$ into $\mathbb{C}^2$ containing small perturbations of the given discs.

We would also like to note that there are some other recent results related to Carleman's theorem. Løw and Wold [10] gave new results regarding polynomial convexity of closed, totally real (non-compact) submanifolds M of $\mathbb{C}^n$, which enabled Manne, Øvrelid and Wold [12] to generalize the Carleman theorem to Stein manifolds. For the exact statement, see [12].

References

1. Arakelian, N. U., Uniform approximation on closed sets by entire functions, *Izv. Akad. Nauk SSSR Ser. Mat.* **28** (1964), 1187–1206 (Russian).
2. Borell, S. and Kutzschebauch, F., Embeddings through discrete sets of balls, *Ark. Mat.* **46** (2008), 251–269.
3. Buzzard, G. and Forstnerič, F., A Carleman type theorem for proper holomorphic embeddings, *Ark. Mat.* **35** (1997), 157–169.
4. Forstnerič, F., Interpolation by holomorphic automorphisms and embeddings in $\mathbb{C}^n$, *J. Geom. Anal.* **9** (1999), 93–118.
5. Forstnerič, F. and Løw, E., Global holomorphic equivalence of smooth submanifolds in $\mathbb{C}^n$, *Indiana Univ. Math. J.* **46** (1997), 133–153.
6. Forstnerič, F. and Rosay, J.-P., Approximation of biholomorphic mappings by automorphisms of $\mathbb{C}^n$, *Invent. Math.* **112** (1993), 323–349.

7. Forstnerič, F. and Wold, E. F., Bordered Riemann surfaces in $\mathbb{C}^2$, *J. Math. Pures Appl.* **91** (2009), 100–114.
8. Globevnik, J. and Stensønes, B., Holomorphic embeddings of planar domains into $\mathbb{C}^2$, *Math. Ann.* **303** (1995), 579–597.
9. Kutzschebauch, F., Løw, E. and Wold, E. F., Embedding some Riemann surfaces into $\mathbb{C}^2$ with interpolation, *Math. Z.* **262** (2009), 603–611.
10. Løw, E. and Wold, E. F., Polynomial convexity and totally real manifolds, *Complex Var. Elliptic Equ.* **54** (2009), 265–281.
11. Majcen, I., Embedding certain infinitely connected subsets of bordered Riemann surfaces properly into $\mathbb{C}^2$, *J. Geom. Anal.* **19** (2009), 695–707.
12. Manne, P. E., Øvrelid, N. and Wold, E. F., Holomorphic convexity and Carleman approximation by entire functions on Stein manifolds, *Math. Ann.* **351** (2011), 571–585.
13. Narasimhan, R., *Analysis on Real and Complex Manifolds*, North-Holland, Amsterdam, 1985.
14. Rosay, J.-P. and Rudin, W., Arakelian's approximation theorem, *Amer. Math. Monthly* **96** (1989), 432–434.
15. Stolzenberg, G., Uniform approximation on smooth curves, *Acta Math.* **115** (1969), 185–198.
16. Wold, E. F., Proper holomorphic embeddings of finitely and some infinitely connected subsets of $\mathbb{C}$ into $\mathbb{C}^2$, *Math. Z.* **252** (2006), 1–9.
17. Wold, E. F., Embedding Riemann surfaces properly into $\mathbb{C}^2$, *Internat. J. Math.* **17** (2006), 963–974.

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