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Asymptotics for the size of the largest component scaled to "$\log n$" in inhomogeneous random graphs

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Abstract. We study inhomogeneous random graphs in the subcritical case. Among other results, we derive an exact formula for the size of the largest connected component scaled by $\log n$, with n being the size of the graph. This generalizes a result for the "rank-1 case". We also investigate branching processes associated with these graphs. In particular, we discover that the same well-known equation for the survival probability, whose positive solution determines the asymptotics of the size of the largest component in the supercritical case, also plays a crucial role in the subcritical case. However, now it is the *negative* solutions that come into play. We disclose their relationship to the distribution of the progeny of the branching process.

1. Introduction

1.1. Inhomogeneous random graphs

A general inhomogeneous random graph model that includes a number of previous models was introduced and studied in great detail by Bollobás, Janson and Riordan [3]. Let us recall the basic definition of the inhomogeneous random graph $G^{\mathcal{V}}(n,\varkappa)$, $n\geq 1$, with vertex space

$$\mathcal{V}=(S,\mu,(x_1^{(n)},...,x_n^{(n)})_{n\geq 1}).$$

Here S is a separable metric space, μ is a Borel probability measure on S, for each $n\geq 1$, $(x_1^{(n)},...,x_n^{(n)})$ is a sequence of points in S, and $\varkappa$ is a kernel defined on $S\times S$. No relationship is assumed between $x_i^{(n)}$ and $x_i^{(n')}$. To simplify notation we shall write $(x_1,...,x_n)=(x_1^{(n)},...,x_n^{(n)})$. For each $n\geq 1$ let $(x_1,...,x_n)$ be a random

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sequence of points in S, such that for any μ-continuity set $A\subseteq S$,

$$\frac{\#\{i : x_i \in A\}}{n} \xrightarrow{P} \mu(A), \quad \text{as } n\to\infty. \tag{1.1}$$

Given a sequence $x_1, ..., x_n$, we let $G^{\mathcal{V}}(n,\varkappa)$ be the random graph on $\{1, ..., n\}$, such that any two vertices i and j are connected by an edge independently of the others and with a probability

$$p_{ij}(n) = \min\left\{\frac{\varkappa(x_i,x_j)}{n}, 1\right\}, \tag{1.2}$$

where the kernel $\varkappa$ is a symmetric non-negative measurable function on $S\times S$. We shall also assume that the kernel $\varkappa$ is *graphical* on $\mathcal{V}$, which means that

(i) $\varkappa$ is continuous *a.e.* on $S\times S$;

(ii) $\varkappa\in L^1(S\times S, \mu\times\mu)$;

(iii)
$$\frac{1}{n}\mathbf{E}e(G^{\mathcal{V}}(n,\varkappa)) \longrightarrow \frac{1}{2}\int_{S^2}\varkappa(x,y)\,d\mu(x)\,d\mu(y),$$

where $e(G)$ denotes the number of edges in the graph G.

Let $C_1(G)$ denote the size of the largest connected component in the graph G. This is the most studied characteristic of a random graph. For instance, a striking phenomena of phase transition is seen in the abrupt change of the value $C_1(G)$ depending on the parameters of the model. There is a close connection between $C_1(G)$ and the survival probability of a certain multi-type Galton–Watson process $B_\varkappa(x)$ defined below.

Definition 1.1. The type space of $B_\varkappa(x)$ is S, and initially there is a single particle of type $x\in S$. At any step a particle of type $x\in S$ is replaced in the next generation by a set of particles, where the number of particles of type y has a Poisson distribution with intensity $\varkappa(x,y)\,d\mu(y)$.

Let $\rho_\varkappa(x)$ denote the survival probability of $B_\varkappa(x)$. Then Theorem 3.1 in [3] states that

$$\frac{C_1(G^{\mathcal{V}}(n,\varkappa))}{n} \xrightarrow{P} \rho_\varkappa := \int_S \rho_\varkappa(x)\,d\mu(x), \tag{1.3}$$

where, as it was proved in [3], $\rho_\varkappa(x)$ is the maximum solution to

$$f(x) = 1 - e^{-T_\varkappa[f](x)}, \tag{1.4}$$

and $T_\varkappa$ is the integral operator on S defined by

$$(T_\varkappa f)(x) = \int_S \varkappa(x,y) f(y)\, d\mu(y). \tag{1.5}$$

Whether $\rho_\varkappa$ is zero or strictly positive depends only on the norm

$$\|T_\varkappa\| = \sup\{\|T_\varkappa f\|_2 : f \geq 0 \text{ and } \|f\|_2 \leq 1\}.$$

More precisely, Theorem 3.1 in [3] says

$$\rho_\varkappa \begin{cases} >0, & \text{if } \|T_\varkappa\|>1, \\ =0, & \text{if } \|T_\varkappa\|\leq 1. \end{cases} \tag{1.6}$$

This together with (1.3) tells us that in the subcritical case, i.e., when $\|T_\varkappa\|\leq 1$, we have $C_1(G^{\mathcal{V}}(n,\varkappa))=o_P(n)$.

Under the additional assumption

$$\sup_{x,y} \varkappa(x,y) < \infty,$$

Theorem 3.12 in [3] establishes that if $\|T_\varkappa\|<1$ then $C_1(G^{\mathcal{V}}(n,\varkappa))=O(\log n)$ with probability tending to one as $n\to\infty$.

On the other hand, it was pointed out in [3] that whenever the kernel is unbounded, the condition $\|T_\varkappa\|<1$ is not sufficient for the size of the largest component to be of order $\log n$. For an example see the random growth model in [2]. Recently Janson showed in [6] that a subcritical inhomogeneous random graph (in the rank-1 case) can also have largest component of order $n^{1/\gamma}$ under the assumption of a power-law degree distribution with exponent $\gamma+1$, $\gamma>2$.

We shall obtain sufficient conditions under which $C_1(G^{\mathcal{V}}(n,\varkappa))/\log n$ converges in probability to a finite constant, even for unbounded kernels. The exact value of this constant was until recently only known for the Erdős–Rényi random graph [5]. The first result related to ours for the inhomogeneous model, but only in the rank-1 case, that is, when

$$\varkappa(x,y) = \Phi(x)\Phi(y), \tag{1.7}$$

was derived in [10]. However, in [10] the formula for the asymptotic value of $C_1(G^{\mathcal{V}}(n,\varkappa))/\log n$ is given in terms of the function Φ and thus is not applicable for a general kernel. Here we derive the first exact asymptotic formula for the scaled size of the largest component in a setup that includes, but is not restricted to, the rank-1 case (i.e., condition (1.7)).

1.2. Results

Let $\mathcal{X}(x)$ denote the size of the total progeny of $B_\varkappa(x)$, and set

$$r_\varkappa = \sup\left\{ z \geq 1 : \int_S \mathbf{E}(z^{\mathcal{X}(x)})\, d\mu(x) < \infty \right\}. \tag{1.8}$$

We shall show that $r_\varkappa$ is the determining parameter for the size $C_1(G^{\mathcal{V}}(n,\varkappa))$ in the subcritical case. In particular, we need to know whether $r_\varkappa=1$ or $r_\varkappa>1$. Therefore we shall first study $r_\varkappa$. We shall see that there is a direct relation between $r_\varkappa$ and the tail of the distribution of the total progeny $\mathcal{X}(x)$. For example, if the tail of the distribution of $\mathcal{X}(x)$ decays exponentially, then $r_\varkappa$ defines the constant in the exponent. In the case of a single-type branching process the exact result on the relation between $r_\varkappa$ and the distribution of the total progeny was proved in [8].

Whenever the assumption

$$\int_S \varkappa(x,y)\, d\mu(y) < \infty \quad \text{for all } x \in S, \tag{1.9}$$

is satisfied, Lemma 5.16 in [3] states the following: if $\|T_\varkappa\|>1$ then $\rho_\varkappa>0$ on a set of positive measure. This means that $\mathcal{X}=\infty$ on a set of positive measure, which immediately implies

$$r_\varkappa = 1, \quad \text{if } \|T_\varkappa\| > 1. \tag{1.10}$$

We shall assume from now on that

$$\inf_{x,y\in S} \varkappa(x,y) > 0. \tag{1.11}$$

Theorem 1.2. *$r_\varkappa$ is the supremum of all $z\geq 1$ for which equation*

$$g(x) = z e^{T_\varkappa[g-1](x)} \tag{1.12}$$

has a.s. (i.e., μ-a.s.) a finite solution $g\geq 1$.

Theorem 1.2 yields immediately the following criteria.

Corollary 1.3. *$r_\varkappa>1$ if and only if at least for some $z>1$, (1.12) has an a.s. finite solution $g>1$. Otherwise, $r_\varkappa=1$.*

It turns out that $r_\varkappa=1$ holds as well in the case $\|T_\varkappa\|=1$.

Corollary 1.4. *Let $\varkappa$ satisfy (1.9). Then*

$$r_\varkappa = 1, \quad \text{if } \|T_\varkappa\| \geq 1. \tag{1.13}$$

Theorem 1.2 and Corollary 1.3 will permit us to derive some sufficient conditions for $r_{\varkappa}>1$. Assume that $T_{\varkappa}$ has finite Hilbert–Schmidt norm, i.e.,

$$\|T_{\varkappa}\|_{\mathrm{HS}} := \|\varkappa\|_{L^2(S\times S)} = \left(\int_S\int_S \varkappa^2(x,y)\,d\mu(x)\,d\mu(y)\right)^{1/2} < \infty. \tag{1.14}$$

Define

$$\psi(x) = \left(\int_S \varkappa^2(x,y)\,d\mu(y)\right)^{1/2}, \tag{1.15}$$

and assume that for some positive constant $a>0$,

$$\int_S e^{a\psi(x)}\,d\mu(x) < \infty. \tag{1.16}$$

Theorem 1.5. *Let $\varkappa$ satisfy (1.16). Then*

$$r_{\varkappa} > 1, \quad \text{if } \|T_{\varkappa}\| < 1 \tag{1.17}$$

and at least one of the following conditions is satisfied: $\|T_{\varkappa}\|_{\mathrm{HS}}<1$, *and*

$$\varkappa(x,y) \le c_1 T_{\varkappa}[1](x) T_{\varkappa}[1](y) \tag{1.18}$$

for some constant $c_1>0$ and for all $x,y\in S$.

Clearly, condition (1.18) holds in the rank-1 case (1.7). Here are some other examples:

(i) condition $\sup_{x,y}\varkappa(x,y)<\infty$ implies (1.18);

(ii) under assumption (1.16) the kernel $\varkappa(x,y)=\max\{x,y\}$ satisfies (1.18) as well. (Models with kernels of this type were considered, e.g., in [3] and [9].)

Observe that for all kernels we have $\|T_{\varkappa}\|\le\|T_{\varkappa}\|_{\mathrm{HS}}$, with equality holding only in the rank-1 case (1.7). Hence, under assumption (1.16) in the rank-1 case condition $\|T_{\varkappa}\|<1$ is necessary and sufficient for $r_{\varkappa}>1$.

Recall the model $G^{\mathcal{V}}(n,\varkappa)$ for which (1.1) and (1.2) hold. To be able to approximate a component in $G^{\mathcal{V}}(n,\varkappa)$ by a branching process we need some additional conditions on the distribution of the types of vertices $x_1,\dots,x_n$.

Assumption 1.6. *Let $S\subseteq\{1,2,\dots\}$, and suppose that for any $\varepsilon>0$ and $q>0$,*

$$\mathbf{P}\left\{\frac{\#\{1\le i\le n : x_i = x\}}{n} - \mu(x) \le \varepsilon e^{qT_{\varkappa}[1](x)}\mu(x) \text{ for all } x\in S\right\} \to 1, \tag{1.19}$$

as $n\to\infty$.

Notice, that in the case when S is finite, convergence (1.19) follows readily by assumption (1.1). In [11] one can find examples of models with countable S that satisfy condition (1.19).

Theorem 1.7. *Let $\varkappa$ satisfy* (1.16) *and* (1.18). *Then under Assumption* 1.6,

$$\frac{C_1(G^{\mathcal{V}}(n,\varkappa))}{\log n} \xrightarrow{P} \frac{1}{\log r_\varkappa}, \tag{1.20}$$

where

$$r_\varkappa \begin{cases} >1, & \text{if } \|T_\varkappa\|<1, \\ =1, & \text{if } \|T_\varkappa\|\geq 1. \end{cases} \tag{1.21}$$

Theorem 1.7 provides sufficient conditions when the convergence (1.20) takes place, even for unbounded kernels. Observe, however, that condition (1.16) seems to be necessary as well. In particular, in the "rank-1" case (1.7), condition (1.16) excludes the possibility of a power-law degree distribution. Such a distribution is proved in [6] to yield order $n^{1/\gamma}$ ($\gamma>2$) for the largest component in a subcritical graph.

Clearly, Theorem 1.7 complements statement (1.3) together with (1.6). There is even a direct relation between the values $r_\varkappa$ and $\rho_\varkappa$ as we shall see now. Setting $f(x)=-(g(x)-1)$ in (1.12), we get from Corollary 1.3 that $r_\varkappa>1$ *if and only if at least for some* $z>1$ *the equation*

$$f(x)=1-ze^{-T_\varkappa[f](x)} \tag{1.22}$$

has an a.s. finite solution $f<0$. Notice that when $z=1$, (1.22) coincides with (1.4). This observation leads to an unexpected direct relation to the supercritical case.

Proposition 1.8. *Let $\varkappa$ satisfy* (1.11). *Then $r_\varkappa>1$ if* (1.4) *has an a.s. finite solution f such that* $\sup_{x\in S} f(x)<0$.

In the case of a homogeneous Erdős–Rényi graph $G_{n,p}$ (consult, e.g., [1]), where the probability of any edge is $p=c/n$, the relation between the supercritical and subcritical cases is most transparent. Placing $G_{n,p}$ into the general definition of an inhomogeneous random graph model gives us $|S|=1$ and $\varkappa\equiv c$. In the corresponding branching process $B_\varkappa$ (see Definition 1.1), the offspring (of a single type) has the Po(c) distribution. The survival probability ρ_c of this process is again the maximum solution to (1.4), which in this case has the simple form

$$f=1-e^{-cf}. \tag{1.23}$$

By Corollary 1.3 we have here $r_c>1$ if and only if (1.12) holds, which in this case implies that

$$g = ze^{c(g-1)} \tag{1.24}$$

has a finite solution $g>1$ for some $z>1$. It is not difficult to compute, for all $0<c<1$,

$$r_c = \sup\{z \geq 1 : g = ze^{c(g-1)} \text{ for some } g > 1\} = \frac{1}{ce^{1-c}},$$

which by Theorem 1.7 recovers the result known already in [5], namely for all $0<c<1$,

$$\frac{C_1(G_{n,p})}{\log n} \xrightarrow{P} \frac{1}{c-1-\log c}.$$

It is straightforward to check that (1.24) has a finite positive solution for some $z>1$ if and only if (1.23) has a strictly negative solution (or equivalently if and only if $c<1$). Hence, in the case of Erdős–Rényi graphs the condition in Proposition 1.8 is necessary as well. However, whether this statement holds for general kernels remains an open question. One may start with the rank-1 case, but we do not consider this question here.

Observe, that while all the non-negative solutions to (1.4) are bounded by 1, the non-positive ones can be unbounded. This certainly makes a great difference in our analysis. It was to ease our difficulties in the unbounded case that we introduced condition (1.18), which resembles the rank-1 case. One may surmise, however, that the results of Theorems 1.5 and 1.7 should hold in a much more general setup than we are able to treat here. In particular, condition (1.18) is far from optimal.

In the special "rank-1 case" (1.7) (which implies condition (1.18)) the convergence (1.20) was established previously in [10] under additional conditions on the function Φ. In fact, in the rank-1 case one can derive an explicit formula for $r_{\varkappa}$, for the details see [10].

In the general situation $r_{\varkappa}$ can be found at least numerically with the help of the presented results.

An example with an unbounded kernel is a graph constructed on vertices with independent identically distributed types $x_1, ..., x_n \in \{1, 2, ...\}$, such that $\mathbf{E}e^{\alpha x_1} < \infty$ for some $\alpha>0$, and kernel $\varkappa(x, y)=c(x\vee y)$. For this model Theorem 1.7 is applicable for all positive c. We refer to [3], Section 16.6, for a discussion on the norm of $T_{\varkappa}$ in this case.

2. The generating function for the progeny of branching process

Recall that we denote by $\mathcal{X}(x)$ the size of the total progeny of $B_\varkappa(x)$ (see Definition 1.1 in the introduction). We shall study the function

$$h_z(x) = \mathbf{E} z^{\mathcal{X}(x)}, \quad x \in S,$$

for $z \geq 1$. It is standard to derive (consult, e.g., [7], Chapter 6) that $h_z(k)$ as a generating function for a branching process satisfies the equation

$$h_z(k) = z \exp \int_S \varkappa(k,x)(h_z(x)-1)\, d\mu(x), \quad k \in S,$$

or in a functional form

$$h_z = z e^{T_\varkappa[h_z - 1]} =: \Phi_{z,\varkappa} h_z. \tag{2.1}$$

Theorem 2.1. *For any $z \geq 1$, the function h_z is the minimal solution $f \geq 1$ to the equation*

$$f = \Phi_{z,\varkappa} f, \tag{2.2}$$

and moreover

$$h_z = \lim_{k\to\infty} \Phi^k_{z,\varkappa}[1]. \tag{2.3}$$

Proof. Let us denote by $\mathcal{X}_k(x)$, $k \geq 0$, the number of offspring of the process $B_\varkappa(x)$ in the first k generations. In particular, $\mathcal{X}_0(x) = 1$ and

$$\mathcal{X}_1(x) =_d 1 + P_x,$$

where P_x is the number of offspring of a particle of type x, among which the number of particles of each type $y \in S$ has $\mathrm{Po}(\varkappa(x,y)\, d\mu(y))$-distribution. Set

$$h_{k,z}(x) = \mathbf{E} z^{\mathcal{X}_k(x)}$$

for $k \geq 0$ and $x \in S$. It is straightforward to derive that

$$h_{1,z}(x) = \mathbf{E} z^{\mathcal{X}_1(x)} = \Phi_{z,\varkappa}[z](x) = \Phi_{z,\varkappa}[h_{0,z}](x),$$

and similarly, for any $k \geq 1$,

$$h_{k+1,z}(x) = \Phi_{z,\varkappa}[h_{k,z}](x).$$

Noticing that $h_{0,z}(x) = z = \Phi_{z,\varkappa}[1](x)$ for all $x \in S$, we derive from here that

$$h_{k,z}(x) = \Phi^{k+1}_{z,\varkappa}[1](x). \tag{2.4}$$

Obviously, $h_{k,z}(x)\uparrow h_z(x)$, i.e.,

$$h_z(x)=\lim_{k\to\infty}\Phi^k_{z,\varkappa}[1](x)$$

for all $x\in S$. By the monotone convergence

$$T_\varkappa[h_z](x)=\int_S\varkappa(x,y)\lim_{k\to\infty}\Phi^k_{z,\varkappa}[1](y)\,d\mu(y)=\lim_{k\to\infty}T_\varkappa[\Phi^k_{z,\varkappa}[1]](x),$$

and therefore

$$\Phi_{z,\varkappa}[h_z](x)=e^{\lim_{k\to\infty}T_\varkappa[\Phi^k_{z,\varkappa}[1]-1](x)}=\lim_{k\to\infty}\Phi_{z,\varkappa}[\Phi^k_{z,\varkappa}[1]](x)=h_z(x). \tag{2.5}$$

Hence, $h_z=\lim_{k\to\infty}\Phi^k_{z,\varkappa}[1]$ is a solution to (2.2).

Since $\Phi_{z,\varkappa}$ is monotone and $\Phi_{z,\varkappa}[1](x)=z\ge 1$, it follows by induction that

$$h_z(x)=\lim_{k\to\infty}\Phi^k_{z,\varkappa}[1](x)\ge 1 \tag{2.6}$$

for all $x\in S$.

Finally, we show that h_z is the minimal solution $f\ge 1$ to (2.2). Assume, that there is a solution $f\ge 1$ such that $1\le f(x)<h_z(x)$ for some x. Then due to the monotonicity of $\Phi_{z,\varkappa}$ we have also

$$\Phi^k_{z,\varkappa}[1](x)\le\Phi^k_{z,\varkappa}[f](x)=f(x)<h_z(x)=\lim_{N\to\infty}\Phi^N_{z,\varkappa}[1](x)$$

for all $k\ge 1$. Letting $k\to\infty$ in the last formula we come to the contradiction with the strict inequality in the middle. Therefore h_z is the minimal solution $f\ge 1$ to (2.2). □

Remark 2.2. If $f\ge 1$ satisfies (2.2) and $f(x)<\infty$ at least for some x, then it follows straight from the definition of $\Phi_{z,\varkappa}$ that $\int_S\varkappa(x,y)f(y)\,d\mu(y)<\infty$. Hence, under assumption (1.11), if $f\ge 1$ satisfies (2.2), then either $f=\infty$ a.s. or $f<\infty$ a.s., in which case also $\int_S\varkappa(x,y)f(y)\,d\mu(y)<\infty$ a.s. The latter together with the assumption (1.11) yields $f\in L_1(S,\mu)$ as well.

Remark 2.3. Theorem 1.2 and Corollary 1.3 follow directly from Theorem 2.1 and Remark 2.2.

Next we describe a sufficient condition when the minimal solution $f\ge 1$ to (2.2) is finite.

Lemma 2.4. *If $\Phi_{z,\varkappa}f\le f$ for some $f\ge 1$, then there exists a function $1\le g\le f$ which is a solution to (2.2), i.e., $\Phi_{z,\varkappa}g=g$.*

Proof. (The proof almost repeats the one of Lemma 5.12 from [3].) The monotonicity of $\Phi_{z,\varkappa}$ and the assumption $\Phi_{z,\varkappa}f \le f$ yield by induction

$$f \ge \Phi_{z,\varkappa}f \ge \Phi^2_{z,\varkappa}f \ge \dots .$$

Since $f \ge 1$ we have for all x,

$$\Phi_{z,\varkappa}[f](x) = ze^{T_\varkappa[f-1](x)} \ge z \ge 1,$$

which implies by induction that also $\Phi^k_{z,\varkappa}f \ge 1$ for all $k \ge 1$. Hence the limit

$$f(x) \ge g(x) = \lim_{k\to\infty} \Phi^k_{z,\varkappa}[f](x) \ge 1$$

exists for every x. By the monotone convergence (repeat the argument from (2.5)) g is a solution to (2.2). □

Theorem 2.5. *Let $\varkappa$ satisfy condition* (1.16).

(i) *If $\|T_\varkappa\|_{\mathrm{HS}} < 1$ then at least for some $z > 1$ there is a finite function $f \ge 1$ which satisfies* (2.2).

(ii) *If $\|T_\varkappa\| < 1$ and the kernel $\varkappa$ satisfies condition* (1.18) *from Theorem* 1.5, *then at least for some $z > 1$ there is a finite function $f \ge 1$ which satisfies* (2.2).

Proof. To prove (i) we shall construct a function $f \ge 1$ which satisfies the conditions of Lemma 2.4. Let $\lambda = \|T_\varkappa\|_{\mathrm{HS}} < 1$. Then

$$\|\psi\|_2 = \lambda \tag{2.7}$$

(see the definition of ψ in (1.15)). For any $\varepsilon \ge 0$ let us define

$$g(x,\varepsilon) = T_\varkappa[e^{\varepsilon\psi} - 1](x) = \int_S \varkappa(x,y)(e^{\varepsilon\psi(y)} - 1)\, d\mu(y). \tag{2.8}$$

By the Cauchy–Bunyakovskii inequality

$$g(x,\varepsilon) \le \left(\int_S \varkappa^2(x,y)\, d\mu(y)\right)^{1/2} \left(\int_S (e^{\varepsilon\psi(y)} - 1)^2\, d\mu(y)\right)^{1/2} = \psi(x)A(\varepsilon), \tag{2.9}$$

where the function

$$A(\varepsilon) := \left(\int_S (e^{\varepsilon\psi(y)} - 1)^2\, d\mu(y)\right)^{1/2}$$

is increasing and, by assumption (1.16) and dominated convergence, continuous on $[0, a/4]$. Furthermore, for $0 < \varepsilon < a/4$ we can compute

$$A'(\varepsilon) = \frac{\int_S \psi(y) e^{\varepsilon\psi(y)} (e^{\varepsilon\psi(y)} - 1)\, d\mu(y)}{(\int_S (e^{\varepsilon\psi(y)} - 1)^2\, d\mu(y))^{1/2}}.$$

Using again the Cauchy–Bunyakovskii inequality and condition (1.16) we derive from here that for any small positive ε,

$$A'(\varepsilon) \le \left(\int_S \psi^2(y) e^{2\varepsilon\psi(y)}\, d\mu(y) \right)^{1/2} \le \left(\int_S M e^{3\varepsilon\psi(y)}\, d\mu(y) \right)^{1/2} < \infty,$$

where M is some absolute positive constant. Hence, taking into account (2.7) we have

$$\limsup_{\varepsilon \downarrow 0} A'(\varepsilon) \le \|\psi\|_2 = \lambda < \lambda + \frac{1-\lambda}{2} =: \lambda_1 < 1.$$

This bound together with $A(0)=0$ and the mean-value theorem allows us to conclude that there exists some positive value $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$,

$$A(\varepsilon) < \lambda_1 \varepsilon. \tag{2.10}$$

Therefore for all $0 < \varepsilon < \varepsilon_0$ we get by (2.9) that

$$g(x,\varepsilon) \le \psi(x) A(\varepsilon) < \lambda_1 \varepsilon \psi(x). \tag{2.11}$$

Now fix $z > 1$ arbitrarily and set $\tilde{\psi} = z\psi$. Define also the function

$$\tilde{g}(x,\varepsilon) = z T_{\varkappa}[e^{\varepsilon\tilde{\psi}} - 1](x) = z T_{\varkappa}[e^{\varepsilon z \psi} - 1](x) = z g(x, z\varepsilon). \tag{2.12}$$

According to (2.11) we have

$$\tilde{g}(x,\varepsilon) \le z\varepsilon\lambda_1 \tilde{\psi}(x) \tag{2.13}$$

for all $0 < \varepsilon < \varepsilon_0$. Let us set

$$f_z = z(e^{\varepsilon z \psi} - 1) + 1 = z(e^{\varepsilon\tilde{\psi}} - 1) + 1. \tag{2.14}$$

We claim, that for some $z > 1$,

$$\Phi_z[f_z] \le f_z. \tag{2.15}$$

Indeed, consider

$$\Phi_z[f_z] := \Phi_{z,\varkappa}[f_z] = z e^{T_{\varkappa}[f_z - 1]} = z e^{z T_{\varkappa}[e^{\varepsilon\tilde{\psi}} - 1]}. \tag{2.16}$$

Using definition (2.12) and bound (2.13) we obtain from here that

$$\Phi_z[f_z](x) = z e^{\tilde{g}(x,\varepsilon)} \le z e^{z\varepsilon\lambda_1 \tilde{\psi}(x)}. \tag{2.17}$$

Let us assume now that $1 < z < \delta/\lambda_1$ for some $\lambda_1 < \delta < 1$. Then we have

$$e^{z\varepsilon\lambda_1\tilde{\psi}(x)} \le e^{\varepsilon\delta\tilde{\psi}(x)}. \tag{2.18}$$

Under assumption (1.11) we have $\psi(x)>b>0$ for some positive b, which implies that $\tilde{\psi}(x)>b$ as well. Therefore we can find $1<z<\delta/\lambda_1$ such that for all $x\in S$,

$$e^{\varepsilon\delta\tilde{\psi}(x)} \le e^{\varepsilon\tilde{\psi}(x)} - \frac{z-1}{z},$$

which together with (2.18) gives us

$$ze^{z\varepsilon\lambda_1\tilde{\psi}(x)} \le ze^{\varepsilon\delta\tilde{\psi}(x)} \le z\left(e^{\varepsilon\tilde{\psi}(x)} - \frac{z-1}{z}\right) = z(e^{\varepsilon\tilde{\psi}(x)}-1)+1 = f_z(x).$$

Substituting this bound into (2.17) we finally get (2.15). Hence, f_z satisfies the conditions of Lemma 2.4, by which (i) of Theorem 2.5 follows.

The proof of (ii) is very similar to the previous one. Let $\|T_\varkappa\|=\lambda<1$. Recall that by Lemma 5.15 in [3] an operator $T_\varkappa$ with a finite Hilbert–Schmidt norm (assumption (1.14)) has a non-negative eigenfunction $\phi\in L^2(S,\mu)$ such that $T_\varkappa\phi = \|T_\varkappa\|\phi$. Hence, there is a function ϕ such that $\|\phi\|_2=1$ and

$$\phi(x) = \frac{1}{\lambda}\int_S \varkappa(x,y)\phi(y)\,d\mu(y). \tag{2.19}$$

This together with (1.11) yields

$$\phi(x) \ge c_0 > 0 \tag{2.20}$$

for all $x\in S$ and some $c_0>0$, and therefore

$$\phi(x) = \frac{1}{\lambda}\int_S \varkappa(x,y)\phi(y)\,d\mu(y) \ge \frac{c_0}{\lambda}T_\varkappa[1](x). \tag{2.21}$$

Also (2.19) together with the Cauchy–Bunyakovskii inequality implies

$$\phi(x) \le \frac{1}{\lambda}\psi(x), \tag{2.22}$$

where due to condition (1.18),

$$\begin{aligned}\psi^2(x) &= \int_S \varkappa^2(x,y)\,d\mu(y) \le c_1^2(T_\varkappa[1](x))^2\int_S (T_\varkappa[1](y))^2\,d\mu(y)\\ &\le c_1^2(T_\varkappa[1](x))^2\|T_\varkappa\|_{\mathrm{HS}}^2.\end{aligned} \tag{2.23}$$

Combining now (2.21)–(2.23) we get

$$\frac{c_0}{\lambda}T_\varkappa[1](x) \le \phi(x) \le \frac{1}{\lambda}\psi(x) \le \frac{1}{\lambda}c_1T_\varkappa[1](x)\|T_\varkappa\|_{\mathrm{HS}},$$

which immediately yields

$$\frac{\psi(x)}{\phi(x)} \le \frac{c_1 \lambda \|T_\varkappa\|_{\mathrm{HS}}}{c_0} \tag{2.24}$$

for all $x \in S$.

We will show now that the function

$$F_z(x) = z(e^{\varepsilon z \phi(x)} - 1) - 1, \tag{2.25}$$

defined similarly to (2.14), satisfies the conditions of Lemma 2.4 for some positive ε.

Consider similarly to (2.8) the function

$$G(x,\varepsilon) = T_\varkappa[e^{\varepsilon\phi} - 1](x) = \int_S \varkappa(x,y)(e^{\varepsilon\phi(y)} - 1) d\mu(y).$$

From here we derive using assumption (1.16) and bound (2.22), that at least for all $\varepsilon < a\lambda/4$ the derivatives

$$\frac{\partial}{\partial\varepsilon} G(x,\varepsilon) = \int_S \varkappa(x,y)\phi(y) e^{\varepsilon\phi(y)}\, d\mu(y) \tag{2.26}$$

and

$$\frac{\partial^2}{\partial\varepsilon^2} G(x,\varepsilon) = \int_S \varkappa(x,y)\phi(y)^2 e^{\varepsilon\phi(y)}\, d\mu(y) \tag{2.27}$$

are finite and non-negative for any $x \in S$. Note that for all $x \in S$,

$$G(x,0) = 0$$

and

$$\frac{\partial}{\partial\varepsilon} G(x,\varepsilon)\Big|_{\varepsilon=0} = \int_S \varkappa(x,y)\phi(y) d\mu(y) = \lambda\phi(x). \tag{2.28}$$

Therefore for all $x \in S$ and $0 \le \varepsilon < a\lambda/4$ we have

$$G(x,\varepsilon) \le \varepsilon\left(\lambda\phi(x) + \varepsilon \frac{\partial^2}{\partial\varepsilon^2} G(x,\varepsilon)\right). \tag{2.29}$$

Under the assumption (1.16) and (2.22) we get from (2.27) using the Cauchy–Bunyakovskii inequality that for all $0 \le \varepsilon < a\lambda/4$,

$$\frac{\partial^2}{\partial\varepsilon^2} G(x,\varepsilon) \le \psi(x)\left(\int_S \phi(y)^4 e^{2\varepsilon\phi(y)}\, d\mu(y)\right)^{1/2} \le \psi(x) c_2, \tag{2.30}$$

where c_2 is some positive constant. Taking also into account bound (2.24), we derive from (2.30),

$$\frac{\partial^2}{\partial\varepsilon^2}G(x,\varepsilon)\le c_3\phi(x)$$

for some positive constant c_3. Substituting this into (2.29), we get

$$G(x,\varepsilon)\le\varepsilon\phi(x)(\lambda+\varepsilon c_3). \tag{2.31}$$

For all small $\varepsilon>0$ we have

$$\lambda+\varepsilon c_3<\lambda+\frac{1-\lambda}{2}=:\lambda_1<1.$$

This together with (2.31) immediately yields

$$G(x,\varepsilon)\le\lambda_1\varepsilon\phi(x) \tag{2.32}$$

for all small $\varepsilon>0$.

Repeating now almost exactly the same argument which led from (2.11) to (2.15), one can derive from (2.32) that for F_z defined by (2.25),

$$\Phi_z[F_z]\le F_z.$$

Hence, F_z satisfies the conditions of Lemma 2.4, which yields (ii). □

3. Proofs of the main results

3.1. Proof of Theorem 1.2

Recall that

$$r_{\varkappa}=\sup\left\{z\ge1:\int_S h_z(x)\,d\mu(x)<\infty\right\}.$$

Hence, the statement of Theorem 1.2 follows immediately from Theorem 2.1 and Remark 2.2.

3.2. Proof of Corollary 1.4

Lemma 3.1. *If* $\|T_{\varkappa}\|=1$ *one has*

$$\lim_{c\uparrow1}r_{c\varkappa}=1. \tag{3.1}$$

Proof. Let $\mathcal{X}^c$ denote the total progeny of $B_{c\varkappa}$ (see Definition 1.1). It is clear that if $0<c<c'$ then $\mathcal{X}^{c'}$ stochastically dominates $\mathcal{X}^c$, and therefore it is obvious that $r_{c\varkappa}$ is a monotone non-increasing function in $c>0$. Also, it follows from the definition of $r_{c\varkappa}$, that $r_{c\varkappa}\geq 1$. Hence, the limit $\lim_{c\uparrow 1} r_{c\varkappa}\geq 1$ exists. Assume that

$$\lim_{c\uparrow 1} r_{c\varkappa} =: r > 1. \tag{3.2}$$

Define

$$1 < z := \frac{1}{2} + \frac{r}{2} < \lim_{c\uparrow 1} r_{c\varkappa}.$$

Then by Theorem 1.2 for any fixed $c<1$ there exists a minimal solution $1\leq f<\infty$ to (2.2), i.e.

$$f = z e^{T_{c\varkappa}[f-1]}.$$

Notice that also

$$f(x) \geq z$$

for all $x\in S$. Let $c'=\sqrt{z}>1$ and set

$$g := \frac{f}{c'} \geq \sqrt{z} > 1. \tag{3.3}$$

It is straightforward to derive

$$\Phi_{\sqrt{z},c'c\varkappa}[g] = \sqrt{z} e^{c' T_{c\varkappa}[g-1]} = \frac{1}{\sqrt{z}}\, z e^{T_{c\varkappa}[c'g-1]-(c'-1)T_{c\varkappa}[1]} = \frac{1}{c'} f e^{-(c'-1)T_{c\varkappa}[1]} \leq g.$$

Hence, by Lemma 2.4 there exists a function $1<h<\infty$ such that

$$h = \Phi_{\sqrt{z},c'c\varkappa}[h] \equiv \sqrt{z} e^{T_{\sqrt{z}c\varkappa}[h-1]}. \tag{3.4}$$

Choose now $1/\sqrt{z}<c<1$. Then existence of an a.s. finite solution $h>1$ to (3.4) with $\sqrt{z}>1$ implies by Theorem 1.2 that $r_{\sqrt{z}c\varkappa}>1$ even when $\|T_{\sqrt{z}c\varkappa}\|=\sqrt{z}c>1$. We get a contradiction with (1.10), which finishes the proof of the lemma. □

By Lemma 3.1 when $\|T_\varkappa\|=1$ we have

$$1 = \lim_{c\uparrow 1} r_{c\varkappa} \geq r_\varkappa \geq 1,$$

which yields

$$r_\varkappa = 1, \quad \text{if } \|T_\varkappa\| = 1. \tag{3.5}$$

This together with (1.10) completes the proof of Corollary 1.4.

3.3. Proof of Theorem 1.5

The statement follows immediately from Theorems 1.2 and 2.5.

3.4. Proof of Proposition 1.8

Assume that (1.4) has an a.s. finite solution f such that $\sup_{x\in S} f(x)<0$. We shall show that in this case there is $z>1$ such that (1.12) has an a.s. finite solution $g\geq 1$. This by Theorem 1.2 will imply that $r_{\mathscr{K}}>1$.

By our assumption,

$$f=1-e^{-T_{\mathscr{K}}[f]}<0.$$

Then for $h:=-(f-1)>1$ we have

$$h=e^{T_{\mathscr{K}}[h-1]}. \tag{3.6}$$

Claim. *There are $0<\varepsilon<1$ and $z>1$ such that the function*

$$H=\varepsilon+(1-\varepsilon)h \tag{3.7}$$

satisfies the inequality

$$\Phi_z[H]:=ze^{T_{\mathscr{K}}[H-1]}\leq H. \tag{3.8}$$

Proof. By (3.6) we have for any $z>1$,

$$\Phi_z[H]=ze^{T_{\mathscr{K}}[\varepsilon+(1-\varepsilon)h-1]}=z(e^{T_{\mathscr{K}}[h-1]})^{1-\varepsilon}=zh^{1-\varepsilon}\equiv z\frac{h^{1-\varepsilon}}{\varepsilon+(1-\varepsilon)h}H. \tag{3.9}$$

Define for all numbers $q\geq 1$,

$$Q(\varepsilon,q):=\frac{q^{1-\varepsilon}}{\varepsilon+(1-\varepsilon)q}.$$

It is straightforward to compute that for any $0<\varepsilon<1$ and for any $q>1$,

$$\frac{\partial}{\partial q}Q(\varepsilon,q)=\frac{(1-\varepsilon)q^{-\varepsilon}(\varepsilon+(1-\varepsilon)q)-q^{1-\varepsilon}(1-\varepsilon)}{(\varepsilon+(1-\varepsilon)q)^2}<0. \tag{3.10}$$

Recall that by the assumption,

$$h_*=\inf_{x\in S}h(x)=1-\sup_{x\in S}f(x)>1.$$

Hence, by (3.10) for all $x\in S$,

$$Q(\varepsilon,h(x))\leq Q(\varepsilon,h_*)<Q(\varepsilon,1)=1.$$

Setting now $z=1/Q(\varepsilon,h_*)>1$ we derive from (3.9),

$$\Phi_z[H]\le H,$$

which proves our claim. □

Notice, that by the definition

$$H=\varepsilon+(1-\varepsilon)h=1+(1-\varepsilon)(h-1)\ge 1,$$

since $h>1$. This and (3.8) confirm that the conditions on Lemma 2.4 are fulfilled by the function H. Therefore by Lemma 2.4 there exists an a.s. finite solution $g\ge 1$ to (1.12) with some $z>1$. This completes the proof of Proposition 1.8.

3.5. Proof of Theorem 1.7

3.5.1. The upper bound

Theorem 3.2. *If $\|T\|<1$ then under the conditions of Theorem 1.7 one has $r_\varkappa>1$ and*

$$\lim_{n\to\infty}\mathbf{P}\left\{C_1(G^{\mathcal{V}}(n,\varkappa))>\left(\frac{1}{\log r_\varkappa}+\delta\right)\log n\right\}=0 \tag{3.11}$$

for any $\delta>0$.

Proof. Notice that here $r_\varkappa>1$ simply by Theorem 1.5.

Recall the usual algorithm of finding a connected component in a random graph. Given the sequence $(x_1,\dots,x_n)$ and a corresponding graph $G^{\mathcal{V}}(n,\varkappa)$, take any vertex $1\le i\le n$ to be the root. Find all vertices connected to this vertex i in the graph $G^{\mathcal{V}}(n,\varkappa)$, and then mark i as "saturated". Then for each non-saturated revealed vertex, we find all vertices connected to it but which have not been used previously. We continue this process until we end up with a tree of saturated vertices.

Denote by τ_n^i the set of vertices in the tree constructed according to the above algorithm with the root at vertex i. Then for any $\omega>0$,

$$\mathbf{P}\{C_1(G^{\mathcal{V}}(n,\varkappa))>\omega\}=\mathbf{P}\Big\{\max_{1\le i\le n}|\tau_n^i|>\omega\Big\}. \tag{3.12}$$

Let the constant a be the one from condition (1.16). Then for any

$$0\le q<\tfrac{1}{2}a \tag{3.13}$$

define the auxiliary probability measure on S,

$$\mu_q(x) = m_q e^{qT_\varkappa[1](x)}\mu(x) \tag{3.14}$$

with normalizing constant

$$m_q := \left(\sum_S e^{qT_\varkappa[1](x)}\mu(x)\right)^{-1},$$

which is strictly positive due to assumption (1.16). Notice that $\mu_0(x)=\mu(x)$ for all $x\in S$, and m_q is continuous in q on $[0, a/2]$ with $m_0=1$. This implies in particular that, for any $\varepsilon'>0$, one can choose a positive q so that

$$\mu(x) \le (1+\varepsilon')\mu_q(x) \tag{3.15}$$

for all x. Fix $\varepsilon>0$ and $0<q<a/2$ arbitrarily and define the event

$$\mathcal{B}_n = \left\{ \frac{\#\{1\le i\le n : x_i = x\}}{n} - \mu(x) \le \varepsilon\mu_q(x) \text{ for all } x\in S \right\}. \tag{3.16}$$

By assumption (1.19) we have

$$\mathbf{P}\{\mathcal{B}_n\} = 1-o(1). \tag{3.17}$$

Then we derive from (3.12) that

$$\mathbf{P}\{C_1(G^{\mathcal{V}}(n,\varkappa)) > \omega\} \le \mathbf{P}\left\{ \max_{1\le i\le n} |\tau_n^i| > \omega \,\middle|\, \mathcal{B}_n \right\} + o(1). \tag{3.18}$$

Notice that the distribution of the size $|\tau_n^i|$ depends only on the type x_i of vertex i. Then setting

$$|\tau_n(x)| =_d |\tau_n^i|\,\big|_{x_i=x} \tag{3.19}$$

for each $x\in S$, we derive from (3.18),

$$\mathbf{P}\{C_1(G^{\mathcal{V}}(n,\varkappa)) > \omega\} \le n\sum_{x\in S}(\mu(x)+\varepsilon\mu_q(x))\mathbf{P}\{|\tau_n(x)| > \omega \,|\, \mathcal{B}_n\} + o(1), \tag{3.20}$$

as $n\to\infty$.

To approximate the distribution of $|\tau_n(x)|$ we shall use the following branching processes. For any $c\ge 1$ and $q\ge 0$ let $B_{c,q}$ denote the process defined similar to $B_\varkappa$ in Definition 1.1, but with the distribution of the offspring being

$$\mathrm{Po}(c\varkappa(x,y)\mu_q(y))$$

instead of $\mathrm{Po}(\varkappa(x,y)\mu(y))$. Notice, that $B_{1,0}$ is defined exactly as $B_\varkappa$. Let further $\mathcal{X}^{c,q}(x)$ denote the total number of particles (including the initial one) produced by the branching process $B_{c,q}$ with the initial single particle of type x.

Proposition 3.3. *Under the conditions of Theorem 1.7 one can find $q>0$ and $c>1$ arbitrarily close to 0 and 1, respectively, such that for some $\varepsilon>0$ in the definition of $\mathcal{B}_n$,*

$$\mathbf{P}\{|\tau_n(x)|>\omega \mid \mathcal{B}_n\} \le \mathbf{P}\{\mathcal{X}^{c,q}(x)>\omega\} \tag{3.21}$$

for all $x\in S$, $\omega>0$, and for all large n.

Proof. Observe that at each step of the exploration algorithm which defines τ_n^i, the number of type-y offsprings of a particle of type x has the binomial distribution $\mathrm{Bin}(N_y', p_{xy}(n))$, where N_y' is the number of the remaining vertices of type y.

We shall use the well-known fact that the binomial $\mathrm{Bin}(n,p)$ distribution is dominated by the Poisson distribution $\mathrm{Po}(-n\log(1-p))$. First we shall derive an upper bound for N_y'. Notice that conditionally on $\mathcal{B}_n$ we have

$$N_y' \le \#\{1\le i\le n : x_i = y\} \le n(\mu(y)+\varepsilon\mu_q(y)) \tag{3.22}$$

for each $y\in S$. The last inequality implies that for any y such that

$$\#\{1\le i\le n : x_i = y\} > 0$$

we have

$$n(\mu(y)+\varepsilon\mu_q(y)) \ge 1. \tag{3.23}$$

By the Cauchy–Bunyakovskii inequality and by assumption (1.16) we have

$$\sum_S e^{qT_\varkappa[1](x)}\mu(x) \le \sum_S e^{q\psi(x)}\mu(x) < \infty$$

for all $q\le a$. Hence, for all $0<q<a/2$ and for all $y\in S$,

$$\mu(y)+\varepsilon\mu_q(y) \le b_3(e^{-aT_\varkappa[1](y)}+\varepsilon m_q e^{(q-a)T_\varkappa[1](y)}) \le b_2 e^{-aT_\varkappa[1](y)/2}$$

for all $0<\varepsilon<1$, where b_2 and b_3 are some positive constants. Combining this with (3.23) we obtain for all y such that $\#\{1\le i\le n : x_i = y\}>0$,

$$\frac{1}{n} \le \mu(y)+\varepsilon\mu_q(y) \le b_2 e^{-aT_\varkappa[1](y)/2}.$$

This implies that conditionally on $\mathcal{B}_n$,

$$\max_{x\in\{x_1,\dots,x_n\}} T_\varkappa[1](x) \le A_1 \log n$$

for some constant A_1. Taking into account assumption (1.18), we derive from here that for all large n conditionally on $\mathcal{B}_n$,

$$p_{x_i x_j}(n) = \frac{\varkappa(x_i, x_j)}{n} \le \frac{c_1 (A_1 \log n)^2}{n}. \tag{3.24}$$

The last bound and (3.22) together with (3.15) allow us for any fixed positive ε_1 to choose ε and q so that conditionally on $\mathcal{B}_n$ we get

$$\begin{aligned} -N'_y \log(1-p_{xy}(n)) &\le (\mu(y)+\varepsilon\mu_q(y))n|\log(1-p_{xy}(n))| \\ &\le (1+\varepsilon_1)\mu_q(y)\varkappa(x,y) =: c\mu_q(y)\varkappa(x,y) \end{aligned} \tag{3.25}$$

for all large n and all $x, y \in \{x_1, ..., x_n\}$. Hence, (3.25) holds for any $q>0$ and $c>1$ arbitrarily close to 0 and 1, respectively. It follows by (3.25) that the binomial distribution $\mathrm{Bin}(N'_y, p_{xy}(n))$ is dominated stochastically by the Poisson distribution $\mathrm{Po}(c\mu_q(y)\varkappa(x,y))$. Therefore if conditionally on $\mathcal{B}_n$ at each step of the exploration algorithm which reveals τ_n^i, we replace the $\mathrm{Bin}(N'_y, p_{xy}(n))$ variable with the $\mathrm{Po}(c\mu_q(y)\varkappa(x,y))$ one, we arrive at the statement (3.21) of the proposition. □

Substituting (3.21) into (3.20) we derive that for any $q>0$ and $c>1$, one has

$$\mathbf{P}\{C_1(G^{\mathcal{V}}(n,\varkappa)) > \omega\} \le bn \sum_{x\in S} \mu_q(x) \mathbf{P}\{\mathcal{X}^{c,q}(x) > \omega\} + o(1),$$

as $n\to\infty$, where b is some positive constant. This bound together with the Markov inequality imply for all $z \ge 1$,

$$\mathbf{P}\{C_1(G^{\mathcal{V}}(n,\varkappa)) > \omega\} \le bnz^{-\omega} \sum_{x\in S} \mu_q(x) \mathbf{E} z^{\mathcal{X}^{c,q}(x)} + o(1). \tag{3.26}$$

Let $T_{c\varkappa,\mu_q}$ denote the following integral operator associated with the branching process $B_{c,q}$,

$$T_{c\varkappa,\mu_q}[f](x) = \int_S c\varkappa(x,y) f(y)\, d\mu_q(y) = \sum_S c\varkappa(x,y) f(y) \mu_q(y). \tag{3.27}$$

Assume from now on that $q>0$ and $c>1$ are such that

$$cm_q \ge 1. \tag{3.28}$$

We shall now extend the result from Lemma 7.2 in [3] on the approximation of kernels to our special case of unbounded kernels. First, taking into account conditions (1.16) and (1.18) we derive that for any fixed $q<a/4$ and $c>1$,

$$\|T_{c\varkappa,\mu_q}\|^2_{\mathrm{HS}} = \int_S \int_S (cm_q)^2 \varkappa^2(x,y) e^{qT_\varkappa[1](x)} e^{qT_\varkappa[1](y)}\, d\mu(x)\, d\mu(y) < \infty.$$

Then by the dominated convergence theorem

$$\|T_{c\varkappa,\mu_q}-T_\varkappa\|^2_{\mathrm{HS}}\to 0, \tag{3.29}$$

as $c\to 1$ and $q\to 0$. Furthermore, since

$$\|T_\varkappa\|\le\|T_{c\varkappa,\mu_q}\|\le\|T_\varkappa\|+\|T_{c\varkappa,\mu_q}-T_\varkappa\|_{\mathrm{HS}},$$

the convergence (3.29) implies as well that

$$\|T_{c\varkappa,\mu_q}\|\to\|T_\varkappa\|,$$

as $c\to 1$ and $q\to 0$. Hence, if $\|T_\varkappa\|<1$ then we can choose $0<q<a/4$ and $c>1$ so that (3.28) holds together with

$$\|T_{c\varkappa,\mu_q}\|<1. \tag{3.30}$$

Now for all values c and q for which (3.30) holds we have by Theorem 2.5(ii) that

$$r(q,c):=\sup\left\{z\ge 1:\sum_{x\in S}\mathbf{E}z^{\mathcal{X}^{c,q}(x)}\mu_q(x)<\infty\right\}>1, \tag{3.31}$$

and therefore for all $1<z<r(q,c)$,

$$\sum_{x\in S}\mu_q(x)\mathbf{E}z^{\mathcal{X}^{c,q}(x)}<\infty. \tag{3.32}$$

Notice, that condition (3.28) implies that $\mathcal{X}^{c,q}(x)$ is stochastically larger than $\mathcal{X}(x)$ for any $x\in S$, which clearly yields

$$r(q,c)\le r_\varkappa. \tag{3.33}$$

Lemma 3.4. *For any $z<r_\varkappa$ there are $q>0$ and $c\ge 1/m_q$ such that*

$$z\le r(q,c)\le r_\varkappa. \tag{3.34}$$

Proof. Notice that when $z\le 1$ statement (3.34) follows from (3.31) and (3.33).

Let us fix $1<z<r_\varkappa$ arbitrarily. We shall show that for some $q>0$ and $c\ge 1/m_q$ the equation

$$f=ze^{T_{c\varkappa,\mu_q}[f-1]} \tag{3.35}$$

has a finite solution $f\ge 1$. This by Theorem 1.2 will imply that $z\le r(q,c)$. The latter together with (3.33) immediately yield (3.34).

First we rewrite (3.35). Let $q>0$ and $c\geq 1/m_q$ be such that (3.30) holds. Set

$$c_q := cm_q \geq 1 \quad \text{and} \quad \widetilde{\varkappa}(x,y) := c_q \varkappa(x,y) e^{qT_\varkappa[1](y)}.$$

Then (3.35) becomes

$$f = \Phi_{z,\tilde{\varkappa}}[f], \tag{3.36}$$

where

$$\Phi_{z,\tilde{\varkappa}}[f] = z \exp(T_\varkappa[c_q e^{qT_\varkappa[1]} f] - T_{\tilde{\varkappa}}[1]).$$

Setting $g = c_q e^{qT_\varkappa[1]} f$ we derive from (3.36),

$$g = c_q z \exp(T_\varkappa[g-1] + T_\varkappa[1] + qT_\varkappa[1] - T_{\tilde{\varkappa}}[1]) = \Phi_{c_q z,\varkappa}[g] e^{(1+q)T_\varkappa[1] - T_{\tilde{\varkappa}}[1]}.$$

Hence, (3.36) has a finite solution $f\geq 1$ if and only if the equation

$$g = \Phi_{c_q z,\varkappa}[g] e^{(1+q)T_\varkappa[1] - T_{\tilde{\varkappa}}[1]} =: G[g] \tag{3.37}$$

has a finite solution $g \geq c_q e^{qT_\varkappa[1]}$. Observe that G is a monotone operator, i.e., if $g \geq g_1$ then $G[g] \geq G[g_1]$. Since

$$G[g] = c_q e^{qT_\varkappa[1]} \Phi_{z,\tilde{\varkappa}}[c_q^{-1} e^{-qT_\varkappa[1]} g],$$

for any

$$g \geq c_q e^{qT_\varkappa[1]}$$

we have

$$G[g] \geq c_q e^{qT_\varkappa[1]}.$$

If we find a function g_0 such that

$$g_0 \geq c_q e^{qT_\varkappa[1]} \tag{3.38}$$

and

$$G[g_0] \leq g_0, \tag{3.39}$$

then we can derive (using an argument similar to the proof of Lemma 2.1) that

$$g := \lim_{n\to\infty} G^n[g_0] \geq c_q e^{qT_\varkappa[1]} \tag{3.40}$$

is the finite solution to (3.37).

Let g_0 be the minimal positive solution to

$$g_0 = \Phi_{c_q z,\varkappa}[g_0], \tag{3.41}$$

where we assume that

$$c_q z < r_{\varkappa}. \tag{3.42}$$

By Theorem 1.2 the minimal positive solution to (3.41) is finite. Furthermore, by formula (2.6) we have

$$g_0 \geq \Phi^2_{c_q z,\varkappa}[1] = c_q z e^{(c_q z-1)T_{\varkappa}[1]} \geq c_q e^{(c_q-1)T_{\varkappa}[1]}, \tag{3.43}$$

where we used the fact that $z>1$. Now for the fixed previously $1<z<r_{\varkappa}$ we can choose

$$0 < q < \frac{r_{\varkappa}}{z} - 1, \tag{3.44}$$

and then set

$$c_q = 1+q. \tag{3.45}$$

With this choice of constants condition (3.42) is satisfied, and moreover from (3.43) we derive

$$g_0 \geq c_q e^{(c_q-1)T_{\varkappa}[1]} = c_q e^{qT_{\varkappa}[1]}, \tag{3.46}$$

which means that condition (3.38) is satisfied as well. Notice also that by (3.44) and (3.45),

$$\begin{aligned}(1+q)T_{\varkappa}[1](x) - T_{\tilde{\varkappa}}[1](x) &= \int_S (1+q-c_q e^{qT_{\varkappa}[1](x)})\varkappa(x,y)\,d\mu(y) \\ &\leq \int_S (1+q-c_q)\varkappa(x,y)d\mu(y) = 0.\end{aligned} \tag{3.47}$$

Therefore with constants (3.44) and (3.45) we derive from (3.37), (3.47) and (3.41) that

$$G[g_0] = \Phi_{c_q z,\varkappa}[g_0] e^{(1+q)T_{\varkappa}[1]-T_{\tilde{\varkappa}}[1]} \leq \Phi_{c_q z,\varkappa}[g_0] = g_0.$$

Hence, both conditions (3.38) and (3.39) are fulfilled. Then by (3.40) equation (3.37) has a desired finite solution. In turn, this implies that (3.36), and thus also (3.35) has a finite solution $f \geq 1$, which yields statement (3.34). □

By Lemma 3.4 for any $\delta>0$ we can choose a small $\delta'>0$ and (q,c) close to $(0,1)$ so that (3.32) holds with

$$z = r(q,c) - \delta' > 1,$$

and moreover

$$\left(\frac{1}{\log r_\varkappa}+\delta\right)\log(r(q,c)-\delta')>1. \tag{3.48}$$

Now setting $\omega=(1/\log r_\varkappa+\delta)\log n$ and $z=r(q,c)-\delta'$ in (3.26) we derive with help of (3.32),

$$\mathbf{P}\left\{C_1(G^{\mathcal{V}}(n,\varkappa))>\left(\frac{1}{\log r_\varkappa}+\delta\right)\log n\right\}$$
$$\le b_1 n z^{-\omega}+o(1)=b_1 n\exp\left(-\log(r(q,c)-\delta')\left(\frac{1}{\log r_\varkappa}+\delta\right)\log n\right)+o(1),$$

where b_1 is some finite positive constant. This together with (3.48) yields statement (3.11). □

3.5.2. The lower bound

Theorem 3.5. *If $\|T\|<1$ then under the conditions of Theorem 1.7 one has $r_\varkappa>1$ and*

$$\lim_{n\to\infty}\mathbf{P}\left\{C_1(G^{\mathcal{V}}(n,\varkappa))<\left(\frac{1}{\log r_\varkappa}-\delta\right)\log n\right\}=0 \tag{3.49}$$

for any $\delta>0$.

Proof. Fix any small positive $\delta<1/\log r_\varkappa$ and set

$$\omega=\left(\frac{1}{\log r_\varkappa}-\delta\right)\log n, \tag{3.50}$$

$$N=N(n)=\frac{n}{\omega^2}. \tag{3.51}$$

Introduce also for arbitrarily fixed finite $D\in S$ and $\varepsilon_1>0$ the event

$$\mathcal{A}_n=\left\{\frac{\#\{x_i:x_i=y\}}{n}-\mu(y)\ge-\varepsilon_1\mu(y)\ \text{ for all } 0\le y\le D\right\}\cap\mathcal{B}_n$$

with $\mathcal{B}_n$ defined by (3.16). Observe that by assumption (1.1) and by (3.17),

$$\mathbf{P}\{\mathcal{A}_n\}\to 1, \tag{3.52}$$

as $n\to\infty$. Let

$$\mathbf{P}_{\mathcal{A}_n}(\,\cdot\,)=\mathbf{P}\{\,\cdot\,|\,\mathcal{A}_n\}$$

denote the conditional probability.

Given the graph $G^{\mathcal{V}}(n,\varkappa)$ we shall reveal recursively its connected components in the following way:

Fix $x^0<D$ with $\mu(x^0)>0$ arbitrarily. Note that conditionally on $\mathcal{A}_n$ we have

$$N_{x^0}:=\#\{1\le i\le n: x_i=x^0\}\ge(1-\varepsilon_1)\mu(x^0)n. \tag{3.53}$$

Let V_1 be a random vertex uniformly distributed on $\{1\le i\le n: x_i=x^0\}$, and let $L_1=\tau_n^{V_1}$ be the set of vertices in the tree with the root at vertex V_1 (see the definition of the algorithm in Section 3.5.1).

For any $U\subset\{1,\dots,n\}$ let $\tau_n^{i,U}$ denote the set of vertices of the tree constructed in the same way as τ_n^i but on the set of vertices $\{1,\dots,n\}\setminus U$ instead of $\{1,\dots,n\}$. In particular, with this notation $\tau_n^{i,\varnothing}=\tau_n^i$.

Given constructed components $L_1,\dots,L_k$ for $1\le k\le[N]$, let V_{k+1} be a vertex uniformly distributed on $\{1\le i\le n: x_i=x^0\}\setminus\bigcup_{i=1}^k L_i$ (whenever this set is non-empty) and set $L_{k+1}=\tau_n^{V_k,\bigcup_{i=1}^k L_i}(V_{k+1})$. If $\{1\le i\le n: x_i=x^0\}\setminus\bigcup_{i=1}^k L_i=\varnothing$, we simply set $L_{k+1}=\varnothing$. Then according to (3.52) we have

$$\mathbf{P}\left\{C_1(G^{\mathcal{V}}(n,\varkappa))<\left(\frac{1}{\log r_\varkappa}-\delta\right)\log n\right\}\le\mathbf{P}_{\mathcal{A}_n}\left\{\max_{1\le i\le[N]+1}|L_i|<\omega\right\}+o(1), \tag{3.54}$$

as $n\to\infty$.

Consider now

$$\begin{aligned}\mathbf{P}_{\mathcal{A}_n}&\left\{\max_{1\le i\le[N]+1}|L_i|<\omega\right\}\\&=\mathbf{P}_{\mathcal{A}_n}\{|L_1|<\omega\}\prod_{i=1}^{[N]}\mathbf{P}_{\mathcal{A}_n}\left\{|L_{i+1}|<\omega\,\middle|\,|L_1|<\omega,\dots,|L_i|<\omega\right\}.\end{aligned} \tag{3.55}$$

Observe that by (3.53) for all large n,

$$\omega(N+1)=o(n)<(1-\varepsilon_1)\mu(x^0)n\le N_{x^0}.$$

Hence, conditionally on $\{|L_1|<\omega,\dots,|L_k|<\omega\}$ the set $\{1\le i\le n:\ x_i=x^0\}\setminus\bigcup_{i=1}^k L_i$ is non-empty for any $k\le N$. Notice also that if $U\subset U'$ then $|\tau_n^{i,U'}|$ is stochastically dominated by $|\tau_n^{i,U}|$ for any i. This allows us to derive from (3.55) that

$$\mathbf{P}_{\mathcal{A}_n}\left\{\max_{1\le i\le[N]+1}|L_i|<\omega\right\}\le\prod_{i=1}^{N}\max_{\substack{U\subset\{1,\dots,n\}\\|U|\le N\omega}}\mathbf{P}_{\mathcal{A}_n}\{|\tau_n^{V_i,U}|<\omega\}. \tag{3.56}$$

To approximate the distribution of $|\tau_n^{i,U}|$ we introduce another branching process which will be stochastically dominated by $B_\varkappa$. First define for any value $D\in S$ the probability measure

$$\hat\mu_D(y)=\begin{cases} M_D^{-1}\mu(y), & \text{if } y\le D,\\ 0, & \text{otherwise,}\end{cases} \tag{3.57}$$

where $M_D:=\sum_{y\le D}\mu(y)$ is the normalizing constant. Then for any positive c and D let $\widehat B_{c,D}$ be the process defined similar to $B_\varkappa$, but with the distribution of offspring

$$\mathrm{Po}(c\varkappa(x,y)\hat\mu_D(y)) \tag{3.58}$$

instead of $\mathrm{Po}(\varkappa(x,y)\mu(y))$. Notice, that $\widehat B_{1,\infty}$ is defined exactly as $B_\varkappa$. Let $\widehat{\mathcal{X}}^{c,D}(x)$ denote the total number of particles (including the initial one) produced by the branching process $\widehat B_{c,D}$ with the initial particle of type x.

Lemma 3.6. *Assume that the conditions of Theorem 1.7 are fulfilled. For all large D and all small ε_1 in the definition of $\mathcal{A}_n$ one can find $c<1$, arbitrarily close to 1, so that*

$$\mathbf{P}_{\mathcal{A}_n}\{|\tau_n^{V_i,U}|<\omega\}\le\left(1+b\frac{\log^4 n}{n^2}\right)^{n\omega}\mathbf{P}\{\widehat{\mathcal{X}}^{c,D}(x^0)<\omega\}$$

for all large n, $V_i\in\{i:x_i=x^0\}$ and all $U\subset\{1,\dots,n\}$ with $|U|\le N\omega$, where b is some positive constant independent of x, c and D (ω and N are defined by (3.50) and (3.51)).

Proof. At each step of the exploration algorithm which defines $\tau_n^{V_i,U}$, the number of type-y offspring of a particle of type x has the binomial distribution $\mathrm{Bin}(N'_y,p_{xy}(n))$, where N'_y is the number of remaining vertices of type y.

Here we shall explore another relation between the binomial and the Poisson distributions. Let $Y_{n,p}\in\mathrm{Bin}(n,p)$ and $Z_\lambda\in\mathrm{Po}(\lambda)$. Then it is straightforward to derive from the formulae for the corresponding probabilities that for all $0<p<\frac14$ and $0\le k\le n$,

$$\begin{aligned}\mathbf{P}\{Y_{n,p}=k\}&=\frac{n!}{k!(n-k)!}p^k(1-p)^{n-k}\\ &=\frac{n!}{n^k(n-k)!}((1-p)e^{p/(1-p)})^n e^{-np/(1-p)}\frac{(np/(1-p))^k}{k!}\\ &\le(1+\gamma p^2)^n\mathbf{P}\{Z_{np/(1-p)}=k\},\end{aligned} \tag{3.59}$$

where γ is some positive constant (independent of n, k and p). Also notice, that (3.59) trivially holds for all $k>n$.

We shall now find a lower bound for N'_y. Conditionally on $\mathcal{A}_n$ we have

$$N_y := \#\{x_i : x_i = y\} \geq (1-\varepsilon_1)\mu(y)n \tag{3.60}$$

for all $y<D$. By deleting an arbitrary set U with $|U|\leq N\omega$ from $\{1,...,n\}$, we may delete at most $N\omega$ vertices of type y. Hence, conditionally on $\mathcal{A}_n$ at any step of the exploration algorithm which defines $\tau_n^{i,U}$ with $|\tau_n^{i,U}|<\omega$, the number N'_y of the remaining vertices of type y is bounded from below as

$$N'_y \geq N_y - \omega - N\omega,$$

and thus for all $y<D$ by (3.60),

$$N'_y \geq n(1-\varepsilon_1)\mu(y) - \omega - N\omega.$$

Taking into account definitions (3.50) and (3.51) we derive from here that for any $\varepsilon'>0$ one can choose a small $\varepsilon_1>0$ so that

$$N'_y \geq (1-\varepsilon')\mu(y)n$$

for all $y\leq D$ and large n. This implies that conditionally on $\mathcal{A}_n$ at any step of the exploration algorithm we have

$$N'_y \frac{p_{xy}(n)}{1-p_{xy}(n)} \geq \mu(y)(1-\varepsilon')\varkappa(x,y) \tag{3.61}$$

for any $y\leq D$ and large n. Now with help of (3.57) we rewrite (3.61) as

$$N'_y \frac{p_{xy}(n)}{1-p_{xy}(n)} \geq \hat{\mu}_D(y) M_D(1-\varepsilon')\varkappa(x,y) =: \hat{\mu}_D(y) c\varkappa(x,y) \tag{3.62}$$

for all $x,y\in S$, where

$$c = M_D(1-\varepsilon').$$

Recall that $M_D\uparrow 1$, as $D\to\infty$. Therefore choosing the constants D and ε_1 appropriately we can make c arbitrarily close to 1.

Now using relation (3.59) between the Poisson and the binomial distributions, and taking into account (3.62), we derive for all k and $N'_y\leq n$,

$$\begin{aligned}\mathbf{P}\{Y_{N'_y,p_{xy}(n)} = k\} &\leq (1+\gamma p_{xy}^2(n))^{N'_y}\mathbf{P}\{Z_{N'_y p_{xy}(n)/(1-p_{xy}(n))} = k\}\\ &\leq \left(1+\gamma c_1^2 A_1^4 \frac{\log^4 n}{n^2}\right)^n \mathbf{P}\{Z_{N'_y p_{xy}(n)/(1-p_{xy}(n))} = k\},\end{aligned}$$

where we used bound (3.24). Note that in the last formula $Z_{N'_y p_{xy}(n)/(1-p_{xy}(n))}$ stochastically dominates $Z_{\hat{\mu}_D(y)c\varkappa(x,y)}$ due to (3.62). This implies that if conditionally on $\mathcal{A}_n$, at each of at most ω steps of the exploration algorithm we replace the $\mathrm{Bin}(N'_y, p_{xy}(n))$ variable with the

$$\mathrm{Po}(\hat{\mu}_D(y)c\varkappa(x,y))$$

one, we arrive at the following bound using the branching process $\widehat{B}_{c,D}$,

$$\mathbf{P}_{\mathcal{A}_n}\{|\tau_n^{V_i,U}|<\omega\} \le \left(1+\gamma c_1^2 A_1^4 \frac{\log^4 n}{n^2}\right)^{n\omega} \mathbf{P}\{\widehat{\mathcal{X}}^{c,D}(x^0)<\omega\} \tag{3.63}$$

for all large n. This yields the statement of Lemma 3.6. □

Now, combining (3.54) with (3.56) and using Lemma 3.6, we derive

$$\begin{aligned}\mathbf{P}\{C_1(G^{\mathcal{V}}(n,\varkappa))<\omega\} &\le \left(\left(1+b\frac{\log^4 n}{n^2}\right)^{n\omega} \mathbf{P}\{\widehat{\mathcal{X}}^{c,D}(x^0)<\omega\}\right)^N + o(1)\\ &\le e^{b_1 \log^3 n}(1-\mathbf{P}\{\widehat{\mathcal{X}}^{c,D}(x^0)\ge\omega\})^{n/\omega^2}+o(1),\end{aligned} \tag{3.64}$$

as $n\to\infty$, where b_1 is some positive constant independent of c and D.

Assume from now on that $c=M_D$. Define the following operator associated with the branching process $\widehat{B}_{c,D}$,

$$T_D[f](x) := T_{c\varkappa,\hat{\mu}_D}[f](x) = \int_0^D \varkappa(x,y)f(y)\,d\mu(y) = \sum_{y\le D} \varkappa(x,y)f(y)\mu(y).$$

Clearly, under the assumption $\|T_\varkappa\|<1$ we also have

$$\|T_D\|<1. \tag{3.65}$$

Hence, $T_{c\varkappa,\hat{\mu}_D}$ satisfies the conditions of Theorem 2.5(ii), which together with Remark 2.2 implies that (when $c=M_D$)

$$\hat{r}(D) := \sup\{z\ge 1 : \mathbf{E}z^{\widehat{\mathcal{X}}^{c,D}(x)}<\infty\} > 1 \tag{3.66}$$

for all $x\in S$. By the construction, $\mu(y)\ge\mu_D(y)c$ for all $y\in S$ (with $c=M_D$). Therefore $\mathcal{X}(x)$ is stochastically larger than $\widehat{\mathcal{X}}^{c,D}(x)$ for all $x\in S$, and hence

$$r_\varkappa \le \hat{r}(D) \tag{3.67}$$

for all $D\in S$. Furthermore, we shall prove the following result.

Lemma 3.7. *Under the conditions of Theorem 1.7,* $\lim_{D\to\infty}\hat{r}(D)=r_\varkappa$.

Proof. Note that $\hat{r}(D)$ is non-increasing in D. Therefore inequality (3.67) implies existence of the limit

$$\lim_{D\to\infty} \hat{r}(D) \geq r_{\varkappa}. \tag{3.68}$$

We shall show that if

$$z < \hat{r}(D) \quad \text{for all } D, \tag{3.69}$$

then also

$$z < r_{\varkappa}. \tag{3.70}$$

This together with (3.68) will immediately imply the statement of the lemma.

From now on we fix z which satisfies (3.69). Then for any $D \in S$ equation

$$f = ze^{T_D[f-1]} =: \Phi_{D,z}[f] \tag{3.71}$$

has the minimal solution f_D, which by (2.3) equals

$$f_D(x) := \lim_{k\to\infty} \Phi^k_{D,z}[1](x) < \infty \tag{3.72}$$

for all $x \in S$. To prove (3.70) it is sufficient to show that equation

$$f = ze^{T[f-1]} =: \Phi_z(f) \tag{3.73}$$

has a finite minimal solution as well. Therefore we shall prove that

$$f_\infty(x) := \lim_{k\to\infty} \Phi^k_{z}[1](x) < \infty \tag{3.74}$$

for all $x \in S$, which by Theorem 2.1 is the minimal solution to (3.73).

Claim. *For all $k \geq 1$ and for all $x \in S$,*

$$\lim_{D\to\infty} \Phi^k_{D,z}[1](x) = \Phi^k_z[1](x). \tag{3.75}$$

Proof. We shall use the induction argument. First, we notice that for all $x \in S$,

$$\Phi_{D,z}[1](x) = z = \Phi_z[1](x),$$

and

$$\Phi^2_{D,z}[1](x) = ze^{T_D[1](x)} \uparrow ze^{T[1](x)} = \Phi^2_z[1](x) < \infty, \tag{3.76}$$

as $D \to \infty$.

Assume now that (3.75) holds for some $k>1$. We shall show that then also

$$\lim_{D\to\infty}\Phi^{k+1}_{D,z}[1](x)=\Phi^{k+1}_{z}[1](x) \tag{3.77}$$

for all $x\in S$, which together with (3.76) will imply (3.75) for all $k\geq 1$. Set

$$g_D:=\Phi^{k}_{D,z}[1] \quad \text{and} \quad g:=\Phi^{k}_{z}[1].$$

By the assumption, $g_D\uparrow g$, as $D\to\infty$. Then with help of the monotone convergence theorem, we derive

$$\lim_{D\to\infty}\Phi^{k+1}_{D,z}[1]=\lim_{D\to\infty}\Phi_{D,z}[g_D]=\lim_{D\to\infty}ze^{T_D[g_D-1]}=ze^{T[g-1]}=\Phi^{k+1}_{z}[1],$$

which proves (3.77). □

Using (3.75) we can rewrite the function in (3.74) as

$$f_\infty(x)=\lim_{k\to\infty}\lim_{D\to\infty}\Phi^{k}_{D,z}[1](x). \tag{3.78}$$

Recall that by Theorem 2.1 and Remark 2.2 we have either $f_\infty(x)<\infty$ or $f_\infty(x)=\infty$ for all $x\in S$ (take into account that S is countable here). Our aim is to prove that $f_\infty(x)<\infty$ for all $x\in S$.

Assume that, on the contrary, $f_\infty(x)=\infty$ for all $x\in S$. Then by (3.78) for any $C>0$ and x there is $k_0=k_0(C,x)>1$ such that

$$\lim_{D\to\infty}\Phi^{k_0}_{D,z}[1](x)>C, \tag{3.79}$$

which in turn implies that there is $D_0=D_0(C,x)$ such that

$$\Phi^{k_0}_{D_0,z}[1](x)>C. \tag{3.80}$$

Due to the definition in (3.72) we have

$$f_{D_0}(x)=\lim_{k\to\infty}\Phi^{k}_{D_0,z}[1](x)=\lim_{k\to\infty}\Phi^{k}_{D_0,z}[\Phi^{k_0}_{D_0,z}[1]](x)\geq\lim_{k\to\infty}\Phi^{k}_{D_0,z}[C](x). \tag{3.81}$$

It is straightforward to derive, taking into account condition (1.11) and the definition of $\Phi_{D,z}$, that for any $D>0$, $z\geq 1$ and all large A one has $\lim_{k\to\infty}\Phi^{k}_{D,z}[A](x)=\infty$. Hence, choosing constant C large enough, we derive from (3.81) that $f_{D_0}(x)=\infty$, which contradicts inequality in (3.72). Hence, (3.74) holds, which finishes the proof of Lemma 3.7. □

By Lemma 3.7 for any given $\delta_1>0$ we can find a large constant D such that

$$\hat{r}(D)<r_{\varkappa}+\tfrac{1}{2}\delta_1. \tag{3.82}$$

It follows from definition (3.66) of $\hat{r}(D)$ that

$$\limsup_{k\to\infty} \mathbf{P}\{\widehat{\mathcal{X}}^{c,D}(x)=k\}^{1/k} = \frac{1}{\hat{r}(D)} < 1$$

(here $c=M_D$), which implies that also

$$\limsup_{k\to\infty} \mathbf{P}\{\widehat{\mathcal{X}}^{c,D}(x)\geq k\}^{1/k} = \frac{1}{\hat{r}(D)}. \tag{3.83}$$

Proposition 3.8. *Let $c=M_D$. The limit*

$$\lim_{k\to\infty} \mathbf{P}\{\widehat{\mathcal{X}}^{c,D}(x)\geq k\}^{1/k}$$

exists.

Proof. We shall use a super-multiplicativity property of $\mathbf{P}\{\widehat{\mathcal{X}}^{c,D}(x)\geq k\}$. Notice that the branching process $\widehat{B}_{c,D}$ has in the first generation at least 2 offspring of type x with probability

$$\pi_x = 1-e^{-\varkappa(x,x)\mu(x)}(1+\varkappa(x,x)\mu(x)),$$

which is positive for all x. Hence, for all $n,m\geq 0$,

$$\begin{aligned} A(n+m) := \mathbf{P}\{\widehat{\mathcal{X}}^{c,D}(x)\geq n+m\} &\geq \pi_x \mathbf{P}\{\widehat{\mathcal{X}}^{c,D}(x)\geq n\}\mathbf{P}\{\widehat{\mathcal{X}}^{c,D}(x)\geq m-1\} \\ &\geq \pi_x A(n)A(m). \end{aligned}$$

Then, by the result of Theorem 23 in [4], $\lim_{k\to\infty}\log A(k)/k$ exists, and moreover, it cannot be ∞ since $\log A(k)\leq 0$. This implies the existence of $\lim_{k\to\infty} A(k)^{1/k}$ and completes the proof of the proposition. □

Proposition 3.8 together with (3.83) gives us

$$\lim_{k\to\infty} \mathbf{P}\{\widehat{\mathcal{X}}^{c,D}(x)\geq k\}^{1/k} = \frac{1}{\hat{r}(D)} \tag{3.84}$$

for all $x\in S$. Therefore for any $\delta_1>0$ and some positive constant $A=A(\delta_1,x^0)<\infty$ we derive using (3.82) that

$$\mathbf{P}\{\widehat{\mathcal{X}}^{c,D}(x^0)>\omega\} \geq A\big(\hat{r}(D)+\tfrac{1}{2}\delta_1\big)^{-\omega} \geq A(r_\varkappa+\delta_1)^{-\omega}.$$

This allows us to derive from (3.64) that for $\omega=(1/\log r_\varkappa-\delta)\log n$, and any $\delta>0$ and $\delta_1>0$,

$$\mathbf{P}\{C_1(G^{\mathcal{V}}(n,\varkappa))<\omega\} \le e^{b_1\log^3 n}(1-A(r_\varkappa+\delta_1)^{-(1/\log r_\varkappa-\delta)\log n})^{n/(\alpha\log n)^2}+o(1), \tag{3.85}$$

where $\alpha=1/\log r_\varkappa-\delta$. Now for any $\delta>0$ we choose a positive δ_1 so that

$$\gamma_1 := \left(\frac{1}{\log r_\varkappa}-\delta\right)\log(r_\varkappa+\delta_1)<1.$$

Then (3.85) becomes

$$\mathbf{P}\left\{C_1(G^{\mathcal{V}}(n,\varkappa))<\left(\frac{1}{\log r_\varkappa}-\delta\right)\log n\right\} \le e^{b_1\log^3 n}\left(1-\frac{A_0}{n^{\gamma_1}}\right)^{n/(\alpha\log n)^2}+o(1),$$

where the right-hand side tends to zero as $n\to\infty$. This completes the proof of Theorem 3.5. □

3.5.3. Proof of Theorem 1.7

Theorems 3.5 and 3.2 yield the assertion of Theorem 1.7 when $\|T_\varkappa\|<1$.

When $\|T_\varkappa\|\ge 1$ we have that $r_\varkappa=1$ by Corollary 1.4. It is clear that for any $0<c<1/\|T_\varkappa\|\le 1$ the size $C_1(G^{\mathcal{V}}(n,\varkappa))$ stochastically dominates $C_1(G^{\mathcal{V}}(n,c\varkappa))$. Then we have by the previous case for any $0<c<1/\|T_\varkappa\|\le 1$,

$$\mathbf{P}\left\{\frac{C_1(G^{\mathcal{V}}(n,\varkappa))}{\log n}<\frac{1}{2\log r_{c\varkappa}}\right\} \le \mathbf{P}\left\{\frac{C_1(G^{\mathcal{V}}(n,c\varkappa))}{\log n}<\frac{1}{2\log r_{c\varkappa}}\right\}\to 0, \tag{3.86}$$

as $n\to\infty$. By Lemma 3.1 we have that $r_{c\varkappa}\to 1$ as $c\uparrow 1/\|T_\varkappa\|$. Therefore we derive from (3.86) that

$$\frac{C_1(G^{\mathcal{V}}(n,\varkappa))}{\log n} \xrightarrow{P} \infty = \frac{1}{\log r_\varkappa},$$

which finishes the proof of Theorem 1.7.

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